Magnons of ferromgnetism
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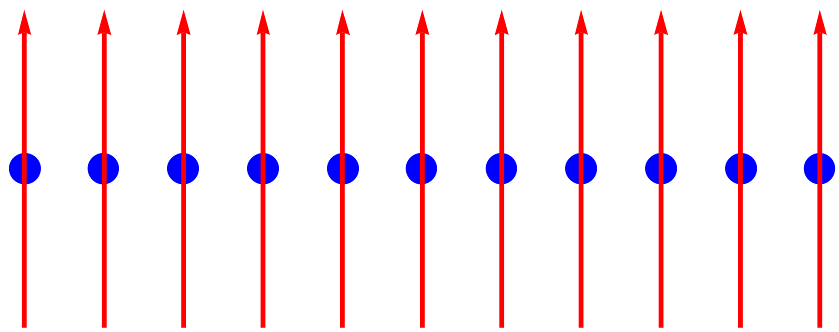
A magnon is quantized spin wave. We find the magnon dispersion relation of ω vs k.



1. Magnon

The ground state of a simple ferromagnet has all spins parallel. Consider N spins each of magnitude S on a line or a ring, with nearest neighbor spins coupled by the Heisenberg interaction,

$$H = -2J\sum_{p=1}^{N} \mathbf{S}_p \cdot \mathbf{S}_{p+1}$$

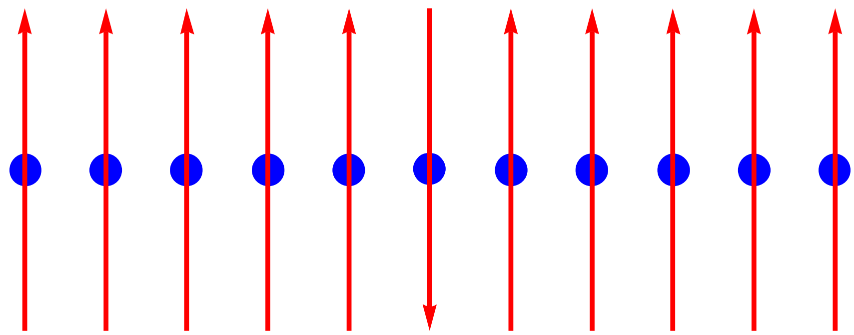
where J is the exchange integral, and $\hbar\mathbf{S}_p$ is the angular momentum of the spin at the point p. If we treat $\boldsymbol{S}_p$ as a classical vector, then in the ground state,

$$\mathbf{S}_p \cdot \mathbf{S}_{p+1} = S^2$$

and the exchange energy of the system is

$$E_0 = -2NJS^2$$

What is the energy of the first excited state? We consider an excited state with one particular spin reversed.



$$E_1 = -2(N-2)JS^2 + 2 \cdot 2JS^2 = -2NJS^2 + 8JS^2$$

We can form an excitation of much lower energy if we let all the spins share the reversal. The elementary excitations of a spin system have a wave-like form, and called magnons.

2. Derivation of the magnon dispersion relation

The term in H which involve the p-th spin are

$$-2J\mathbf{S}_p \cdot (\mathbf{S}_{p-1} + \mathbf{S}_{p+1}),$$

The magnetic moment at the site p is

$$\boldsymbol{\mu}_p = -g\mu_B \mathbf{S}_p .$$

Then we get

$$-2J\left(\frac{\boldsymbol{\mu}_p}{-g\mu_B}\right) \cdot (\boldsymbol{S}_{p-1} + \boldsymbol{S}_{p+1}) = -\boldsymbol{\mu}_p \cdot \boldsymbol{B}_p$$

where the effective magnetic field that acts on the p-th spin is

$$\boldsymbol{B}_p = -\frac{2J}{g\mu_B}(\boldsymbol{S}_{p-1} + \boldsymbol{S}_{p+1})$$

We know that

$$\frac{d(\hbar \boldsymbol{S}_p)}{dt} = \boldsymbol{\tau}_p = \boldsymbol{\mu}_p \times \boldsymbol{B}_p$$

or

$$\frac{d\boldsymbol{S}_p}{dt} = -\frac{g\mu_B}{\hbar}\boldsymbol{S}_p \times (-\frac{2J}{g\mu_B})(\boldsymbol{S}_{p-1} + \boldsymbol{S}_{p+1})$$
$$= \frac{2J}{\hbar}(\boldsymbol{S}_p \times \boldsymbol{S}_{p-1} + \boldsymbol{S}_p \times \boldsymbol{S}_{p+1})$$

In Cartesian components

$$\frac{dS_p{}^x}{dt} = \frac{2J}{\hbar}[S_p{}^y(S_{p-1}{}^z + S_{p+1}{}^z) - S_p{}^z(S_{p-1}{}^y + S_{p+1}{}^y)]$$

$$\frac{dS_p{}^y}{dt} = \frac{2J}{\hbar}[S_p{}^z(S_{p-1}{}^x + S_{p+1}{}^x) - S_p{}^x(S_{p-1}{}^z + S_{p+1}{}^z)]$$

$$\frac{dS_p{}^z}{dt} = \frac{2J}{\hbar}[S_p{}^x(S_{p-1}{}^y + S_{p+1}{}^y) - S_p{}^y(S_{p-1}{}^x + S_{p+1}{}^x)]$$

If the amplitude of the excitation is small ($|S_p^x| << S$, $|S_p^y| << S$). We may obtain an approximate set of linear equations by taking all $S_p^z = S$ and by neglecting terms in the product of S_x and S_y in $\frac{d\mathbf{S}_p}{dt}$,

$$\frac{dS_p{}^x}{dt} = \frac{2JS}{\hbar}(2S_p{}^y - S_{p-1}{}^y - S_{p+1}{}^y)$$

$$\frac{dS_p{}^y}{dt} = -\frac{2JS}{\hbar}(2S_p{}^x - S_{p-1}{}^x - S_{p+1}{}^x)$$

$$\frac{dS_p{}^z}{dt} = 0$$

We look for travelling wave solutions of the above equations of the form.

$$S_p{}^x = \mathrm{Re}[u\exp[i(pka - \omega t)], \qquad S_p{}^y = \mathrm{Re}[v\exp[i(pka - \omega t)]$$

where u and v are complex numbers, p is an integer, and a is the lattice constant. Then we get

$$-i\omega u = \frac{2JS}{\hbar}(2 - e^{-ika} - e^{-ika})v = \frac{4JS}{\hbar}[1-\cos(ka)]v$$

$$-i\omega v = -\frac{2JS}{\hbar}(2 - e^{-ika} - e^{-ika})u = -\frac{4JS}{\hbar}[1-\cos(ka)]u$$

These equations have a solution for u and v if the determinant of the coefficients is equal to zero.

$$\begin{vmatrix} i\omega & \frac{4JS}{\hbar}[1-\cos(ka)] \\ -\frac{4JS}{\hbar}[1-\cos(ka)] & i\omega \end{vmatrix} = 0$$

or

$$\hbar\omega = 4JS[(1-\cos(ka)]$$

With this solution we find that

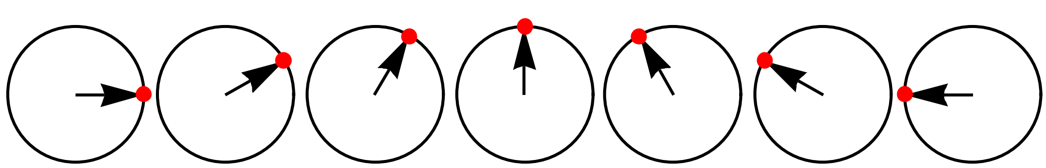
$$v = -iu$$

where u is assumed to be real. Then we have

$$\begin{aligned} S_p^{\,x} &= \mathrm{Re}[u\exp[i(pka-\omega t)] \\ &= u\cos(pka-\omega t) \end{aligned},$$

$$\begin{aligned} S_p^{\,y} &= \mathrm{Re}[ue^{-i\frac{\pi}{2}}\exp[i(pka-\omega t)] \\ &= u\sin(pka-\omega t) \end{aligned}$$

Circular precession:



3. Periodic boundary condition

The spin wave has the travelling wave form,

$$\psi = \exp[i(kx - \omega_k t)$$

We assume the periodic boundary condition,

$$\psi(0) = \psi(L)$$

where $L = Na$ and a is the lattice constant. Then we have

$$k = \frac{2\pi}{L} n$$

with n = 0, 1,2, 3,…., N/2 (the first Brillouin zone).

4. Magnon dispersion relation

$$\hbar\omega_k = 4JS[(1 - \cos(ka)]$$

For ka<<1,

$$\hbar\omega_k = 8JS\sin^2\frac{ka}{2} \approx 2JSa^2k^2 .$$

The frequency is proportional to k^2.

((Note))
We note that the dispersion relation for a ferromagnetic cubic lattice with nearest-neighbor interactions,

$$\hbar\omega_{\boldsymbol{k}} = 2S[J(0) - J(\boldsymbol{k})]$$

with

$$J(\mathbf{k}) = \sum_j J_{ij} e^{i\mathbf{k}\cdot\mathbf{r}_{ij}}$$

where the summation is over the δ vectors denoted by δ which join the central atom to its nearest neighbors.

5. Quantization of spin waves

Holstein-Primakoff

The z-component of S_j at the j-th takes the values

$$S_j^z = S - n_j$$

where n_j = 0, 1, 2, …, 2S (total number: 2S+1). Let $|n_j\rangle$ be the eigen state of n_j with $S_j^z = S - n_j$. Note that n_j is the spin deviation operator. Here we introduce the operators a_j and a_j^*,

$$a_j|n_j\rangle = \sqrt{n_j}|n_j - 1\rangle$$

$$a_j^*|n_j\rangle = \sqrt{n_j + 1}|n_j + 1\rangle$$

a_j is the annihilation operator and a_j^* is the creation operator.
The commutation relation is valid,

$$[a_j, a_j^*] = 1$$

and

$$a_j^* a_j|n_j\rangle = \sqrt{n_j}a_j^*|n_j - 1\rangle = n_j|n_j\rangle.$$

Spin operators satisfy the following relations,

$$(S_j^x + iS_j^y)|n_j\rangle = \sqrt{(S-M)(S+M+1)}|n_j - 1\rangle$$

$$(S_j^x - iS_j^y)|n_j\rangle = \sqrt{(S+M)(S-M+1)}|n_j + 1\rangle$$

with

$$M = S - n_j .$$

Then we have the Holstein-Primakoff transformation,

$$S_j^+ = S_j^x + iS_j^y = \sqrt{2S}(1 - \frac{a_j^* a_j}{2S})^{1/2} a_j$$

$$S_j^- = S_j^x - iS_j^y = \sqrt{2S} a_j^* (1 - \frac{a_j^* a_j}{2S})^{1/2} .$$

S_j^+ and S_j^- can be approximated as follows ($\frac{\langle a_j^* a_j \rangle}{S} << 1$).

$$S_j^+ = \sqrt{2S}(1 - \frac{a_j^* a_j}{4S} + ...) a_j$$

$$S_j^- = \sqrt{2S} a_j^* (1 - \frac{a_j^* a_j}{4S} + ...)$$

The Hamiltonian H is described as

$$H = -2\sum_{<i,j>} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j = -2\sum_{<i,j>} J_{ij} [S_i^z S_j^z + \frac{1}{2}(S_i^- S_j^+ + S_i^+ S_j^-)] ,$$

Here we note

$$S_j^z S_j^z = (S - n_i)(S - n_j) = S^2 - (n_i + n_j)S + n_i n_j ,$$

$$\begin{aligned} S_i^- S_j^+ &= 2S(a_i^* - \frac{a_i^* a_i^* a_i}{4S} + ...)(a_j - \frac{a_j^* a_j a_j}{4S} + ...) \\ &= 2S a_i^* a_j - \frac{1}{2} a_i^* a_j^* a_j a_j - \frac{1}{2} a_i^* a_i^* a_i a_j + ... \end{aligned}$$

$$S_i^+S_j^- = 2S(a_i - \frac{a_i^* a_i a_i}{4S} + ...)(a_j^* - \frac{a_j^* a_j^* a_j}{4S} + ...)$$
$$= 2Sa_i a_j^* - \frac{1}{2} a_i a_j^* a_j^* a_j - \frac{1}{2} a_i^* a_i a_i a_j^* + ...$$

Then we get

$$H = -2\sum_{<i,j>} J_{ij}[S^2 - (n_i + n_j)S + n_i n_j + S(a_i^* a_j + a_i a_j^*)$$
$$-\frac{1}{4}(a_i^* a_j^* a_j a_j + a_i^* a_i^* a_i a_j + a_i a_j^* a_j^* a_j + a_i^* a_i a_i a_j^*) + ...]$$

We neglect the fourth term and higher order terms such as $a_i^* a_j^* a_j a_j$.

$$H = -2\sum_{<i,j>} J_{ij}[S^2 - (n_i + n_j)S + n_i n_j + S(a_i^* a_j + a_i a_j^*)]$$
$$= -2\sum_{<i,j>} J_{ij}[S^2 - (n_i + n_j)S + n_i n_j + S(a_i^* a_j + a_j^* a_i)]$$

because of $[a_i, a_j^*] = 0$ for $i \neq j$. We introduce the Fourier transform $a_{\mathbf{q}}^+$ and $a_{\mathbf{q}}$,

$$a_{\mathbf{q}} = \frac{1}{\sqrt{N}}\sum_j a_j e^{i\mathbf{q}\cdot\mathbf{r}_j}$$

$$a_{\mathbf{q}}^+ = \frac{1}{\sqrt{N}}\sum_j a_j^* e^{-i\mathbf{q}\cdot\mathbf{r}_j}$$

Commutation relation:

$$[a_{\mathbf{q}}, a_{\mathbf{q}}^+] = \frac{1}{N}\sum_{i,j}[a_i, a_j^*] e^{i\mathbf{q}\cdot(\mathbf{r}_j - \mathbf{r}_j)} = \frac{1}{N}\sum_{i,j}\delta_{ij} e^{i\mathbf{q}\cdot(\mathbf{r}_j - \mathbf{r}_j)} = 1$$

Similarly we get

$$a_i = \frac{1}{\sqrt{N}}\sum_{\mathbf{q}} a_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{r}_j}$$

We introduce

$$J(\mathbf{q}) = \sum_j J_{ij} e^{i\mathbf{q}\cdot(\mathbf{r}_i - \mathbf{r}_j)}.$$

Then the Hamiltonian can be expressed by

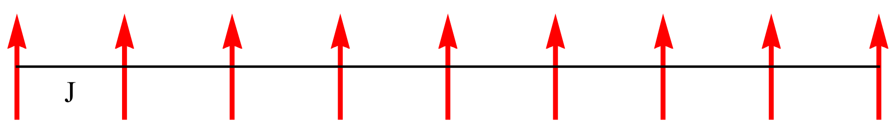
$$\begin{aligned} H &= -2\sum_{<i,j>} J_{ij}[S^2 - (n_i + n_j)S + n_i n_j + S(a_i^* a_j + a_j^* a_i)] \\ &= -2S^2 \sum_{<i,j>} J_{ij} + \sum_q 2S[J(0) - J(\mathbf{q})] a_\mathbf{q}^* a_\mathbf{q} \\ &= E_0 + \sum_\mathbf{q} \hbar\omega_\mathbf{q} a_\mathbf{q}^* a_\mathbf{q} \end{aligned}$$

where

$$\hbar\omega_\mathbf{q} = 2S[J(0) - J(\mathbf{q})]$$

is the magnon dispersion relation.

((**Simple example**))



$J_{ij} = J$ only for the nearest- neighbor pairs.

$$J(\mathbf{q}) = \sum_j J_{ij} e^{i\mathbf{q}\cdot\mathbf{r}_{ij}} = J\sum_j e^{i\mathbf{q}\cdot\mathbf{r}_j} = J(e^{iqa} + e^{-iqa}) = 2J\cos(aq).$$

Then we have

$$\hbar\omega_\mathbf{q} = 2S[J(0) - J(\mathbf{q})] = 4JS(1 - \cos(qa)] = 2JS\sin^2(\frac{qa}{2})$$

which is the spin wave dispersion (magnon dispersion).

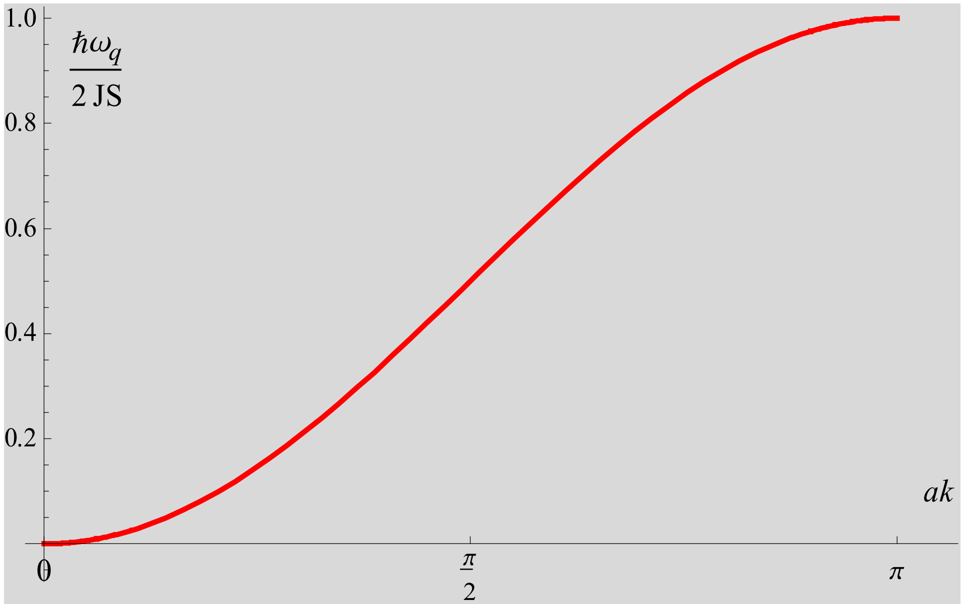


Fig. Dispersion curve of magnon.

6. Magnetization

The z-component of the magnetization is

$$M_z = g\mu_B \sum_j S_j^{\,z} = g\mu_B \sum_j \left(S - n_j\right) = g\mu_B (NS - \sum_q a_{\mathbf{q}}{}^* a_{\mathbf{q}})$$

The magnon is a boson.

$$M_s(0) - M_s(T) = \Delta M = g\mu_B \sum_q \left\langle a_{\mathbf{q}}{}^* a_{\mathbf{q}} \right\rangle = g\mu_B \sum_q \left\langle n_{\mathbf{q}} \right\rangle$$

The total number of magnons excited at a temperature T is

$$\sum_q \left\langle n_{\mathbf{q}} \right\rangle = \int d\omega D(\omega) < n(\omega) >$$

where $D(\omega)$ is the number of magnon modes per unit frequency range.

$$D(\omega)d\omega = \frac{V}{(2\pi)^3} 4\pi q^2 dq = \frac{V}{(2\pi)^3} 4\pi q^2 \frac{dq}{d\omega} d\omega$$

We use the magnon dispersion relation,

$$\hbar\omega_q = \hbar\omega = 2JSa^2q^2$$

$$\frac{d\omega}{dq} = \frac{4JSa^2}{\hbar} q$$

Then the density of states is given by

$$D(\omega) = \frac{V}{4\pi^2}\left(\frac{\hbar}{2JSa^2}\right)^{3/2} \omega^{1/2}$$

The integral is taken over the allowed range of $\boldsymbol{q}$, which is the first Brillouin zone. At $T = 0$ K, we may carry the integral between 0 and ∞, because $<n(\omega)> \rightarrow\infty$ exponentially as $\omega\rightarrow\infty$.

$$\begin{aligned}\Delta M &= g\mu_B \frac{V}{4\pi^2}\left(\frac{\hbar}{2JSa^2}\right)^{3/2} \int_0^\infty \frac{\omega^{1/2} d\omega}{e^{\beta\hbar\omega} - 1} \\ &= g\mu_B \frac{V}{4\pi^2}\left(\frac{k_B T}{2JSa^2}\right)^{3/2} \int_0^\infty \frac{x^{1/2} dx}{e^x - 1}\end{aligned}$$

Since

$$\int_0^\infty \frac{x^{1/2} dx}{e^x - 1} = \frac{\sqrt{\pi}}{2} \varsigma(\frac{3}{2}) = 0.0587(4\pi^2)$$

we have

$$\frac{\Delta M}{M_s(0)} = \frac{V}{NSa^3}\left(\frac{k_B T}{2JS}\right)^{3/2} 0.0587 = \frac{1}{S}\left(\frac{k_B T}{2JS}\right)^{3/2} 0.0587$$

wherec $M_s(0) = Ng\mu_B S$ and $V = Na^3$. This result is called the **Bloch $\boldsymbol{T^{3/2}}$ law**. The magnetization

$$M(T) = M_s(0)[1 - \frac{1}{S}\left(\frac{k_B T}{2JS}\right)^{3/2} 0.0587]$$

7. Heat capacity

The heat capacity is given by

$$C = \frac{dE}{dT} = \frac{\hbar^2}{k_B T^2} \int d\omega D(\omega) \frac{\omega^2}{(e^{\beta\hbar\omega} - 1)(1 - e^{-\beta\hbar\omega})}$$

where $\beta = 1/(k_B T)$,

$$E = \sum_q \hbar\omega_q \langle n_q \rangle = \int d\omega D(\omega)\hbar\omega \frac{1}{e^{\beta\hbar\omega} - 1}$$

$$\frac{d}{dT}\frac{1}{e^{\beta\hbar\omega} - 1} = \frac{\hbar\omega}{k_B T^2}\frac{1}{(e^{\beta\hbar\omega} - 1)(1 - e^{-\beta\hbar\omega})}$$

Then we get

$$\begin{aligned} C &= \frac{1}{k_B T^2}\frac{\hbar^2 V}{4\pi^2}\left(\frac{\hbar}{2JSa^2}\right)^{3/2} \int d\omega \frac{\omega^{5/2}}{(e^{\beta\hbar\omega} - 1)(1 - e^{-\beta\hbar\omega})} \\ &= \frac{1}{k_B T^2}\frac{\hbar^2 V}{4\pi^2}\left(\frac{\hbar}{2JSa^2}\right)^{3/2}\left(\frac{k_B T}{\hbar}\right)^{7/2} \int_0^\infty \frac{x^{5/2} dx}{(e^x - 1)(1 - e^{-x})} \\ &= k_B \frac{V}{4\pi^2 a^3}\left(\frac{k_B T}{2JS}\right)^{3/2} \int_0^\infty \frac{x^{5/2} dx}{(e^x - 1)(1 - e^{-x})} \end{aligned}$$

Here we note that

$$\int_0^\infty \frac{x^{5/2} dx}{(e^x - 1)(1 - e^{-x})} = 4.45823$$

and $V = Na^3$. Then we have

$$C = k_B \frac{N}{4\pi^2}\left(\frac{k_B T}{2JS}\right)^{3/2} 4.45823$$

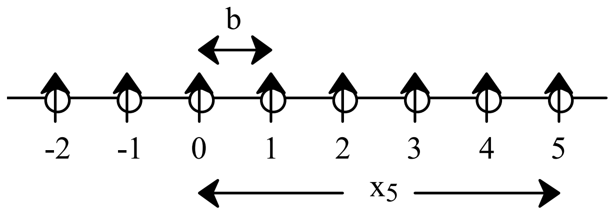
The heat capacity is proportional to $T^{3/2}$.

8. Spin wave theory (Feynman) S = 1/2.

We consider a spin wave theory which is described by Feynman. The Hamiltonian is

$$\hat{H} = -\frac{J}{2}\sum_{n} \hat{\boldsymbol{\sigma}}_n \cdot \hat{\boldsymbol{\sigma}}_{n+1}$$

where s is the Pauli operator. With this Hamiltonian we have a complete description of the ferromagnet.



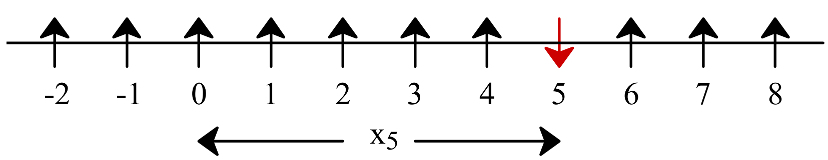
$$\hat{\boldsymbol{\sigma}}_n \cdot \hat{\boldsymbol{\sigma}}_{n+1} = 2\hat{P}_{n,n+1} - 1$$

where $\hat{\boldsymbol{\sigma}}_n \cdot \hat{\boldsymbol{\sigma}}_{n+1}$ interchanges the spins of the n-th and (n+1)-th electrons.

For the ground state all spins are up ($|+\rangle$, so if you exchange a particular pair of spins, one can get back the original state. The ground state is a stationary state: -J/2 for each pair of spins. That is, the energy of the system in the ground state is -J/2 per spin.

It is convenient to measure the energies with respect to the ground state. Our new Hamiltonian is

$$\hat{H} = -J\sum_{n} (\hat{P}_{n,n+1} - 1)$$



With this Hamiltonian, the energy of the ground state is zero. Here we define the state $|x_{.n}\rangle$ where all the spins except for the one on the spin at x_n.

$$\hat{H}|x_5\rangle = -J\sum_{n} (\hat{P}_{n,n+1} - 1)|x_5\rangle = -J(\hat{P}_{5,6} - 1)|x_5\rangle - J(\hat{P}_{4,5} - 1)|x_5\rangle$$
$$= -J(|x_6\rangle - 2|x_5\rangle + |x_4\rangle)$$

where

$$\hat{P}_{45}|x_5\rangle = |x_4\rangle,\ \hat{P}_{56}|x_5\rangle = |x_6\rangle,\ \hat{P}_{78}|x_5\rangle = |x_5\rangle, \text{ and . } \hat{P}_{34}|x_5\rangle = |x_5\rangle.$$

Similarly,

$$\hat{H}|x_n\rangle = -J(|x_{n+1}\rangle - 2|x_n\rangle + |x_{n-1}\rangle$$

$$\hat{H}|x_{n+1}\rangle = -J(|x_{n+2}\rangle - 2|x_{n+1}\rangle + |x_n\rangle$$

Here we consider

$$|\psi\rangle = \sum_n C_n |x_n\rangle$$

Eigenvalue problem

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

or

$$\sum_n C_n \hat{H}|x_n\rangle = E\sum_n C_n |x_n\rangle$$

or

$$\sum_n C_n \hat{H}|x_n\rangle = E\sum_n C_n |x_n\rangle$$

or

$$\sum_n (-J)(C_n|x_{n+1}\rangle - 2C_n|x_n\rangle + C_n|x_{n-1}\rangle = E\sum_n C_n |x_n\rangle$$

or

$$\sum_n (-J)(C_{n-1} - 2C_n + C_{n+1})|x_n\rangle = E\sum_n C_n |x_n\rangle$$

or

$$(-J)(C_{n-1} - 2C_n + C_{n+1}) = EC_n$$

Let us take as a trial function

$$C_n = e^{ikx_n} \qquad \text{(Bloch state)}$$

$$(-J)(e^{ik(x_n - a)} - 2e^{ikx_n} + e^{ik(x_n + a)}) = Ee^{ikx_n}$$

$$E = 2J[1 - \cos(ka)] \qquad \text{(energy dispersion)}$$

The difference energy solutions correspond to "waves" of down spin-called "spin waves."

For ka<<1, E is approximated by

$$E = 2A\frac{k^2 a^2}{2} = Ak^2 a^2$$