

Anderson localization
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In 1958, Philip Anderson conceived the idea of electron localization. Beyond a critical amount of impurity scattering, the diffusive motion of the electron will come to a halt. Later work by Mott, Ioffe, and Regel predicted the transition to occur when the mean free path becomes shorter than the de Broglie wavelength. The localization has dramatic consequence for the conductivity. The system turns into an insulator. The concept of a minimum conductivity was devised by Nevill Mott. However, his idea conflicts with the scaling theory of localization (Abrahams, Licciardello, Ramakrishnan, and Anderson) at the end of 1970's. The concept of a minimum conductivity is incorrect. The metal-insulator transition is continuous.

Philip Anderson, Neville Mott, and John Van Vleck were awarded the Nobel Prize in Physics for his investigations into the electronic structure of magnetic and disordered systems.

Here we discuss the Anderson localization based on the scaling relation.

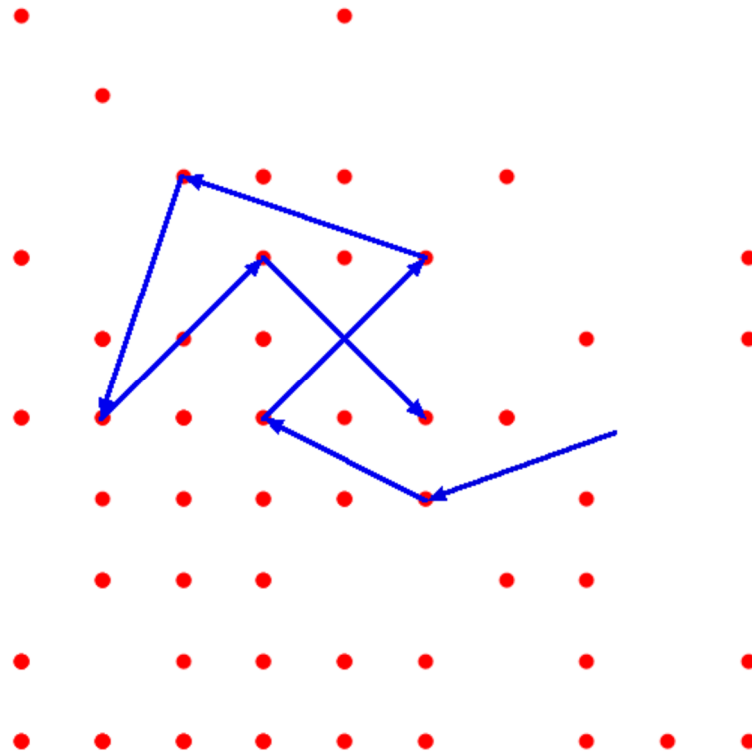


Fig. A non-interacting electron moving in random potential. The red dots denote impurities which are randomly distributed in the system. Anderson localization of electrons is the quantum interference of electron waves. A non-interacting electron in a random potential may be localized.

Sir Nevill Francis Mott, CH, FRS (30 September 1905 – 8 August 1996) was an English physicist. He won the Nobel Prize for Physics in 1977 for his work on the electronic structure of magnetic and disordered systems, especially amorphous semiconductors. The award was shared with Philip W. Anderson and J. H. Van Vleck, who had pursued independent research.



http://en.wikipedia.org/wiki/Nevill_Francis_Mott

Philip Warren Anderson (born December 13, 1923) is an American physicist and Nobel laureate. Anderson has made contributions to the theories of localization, antiferromagnetism and high-temperature superconductivity. Anderson was born in Indianapolis, Indiana and grew up in Urbana, Illinois. He graduated from University Laboratory High School in Urbana in 1940. Afterwards, he went to Harvard University for undergraduate and graduate work, with a wartime stint at the U.S. Naval Research Laboratory in-between. In graduate school he studied under John Hasbrouck van Vleck. From 1949 to 1984 he worked at Bell Laboratories in New Jersey, where he worked on a wide variety of problems in condensed matter physics. During this period he discovered the concept of localization, the idea that extended states can be localized by the presence of disorder in a system; the Anderson Hamiltonian, which describes electrons in a transition metal; he also claimed to have discovered the "Higgs" mechanism for generating mass in elementary particles; and the pseudospin approach to the BCS theory of superconductivity. He was elected a Fellow of the American Academy of Arts and Sciences in 1963.¹ From 1967 to 1975, Anderson was a professor of theoretical physics at Cambridge University. In 1977 Anderson was awarded the Nobel Prize in Physics for his investigations into the electronic structure of magnetic and disordered systems, which allowed for the development of electronic switching and memory devices in computers. Co-researchers Sir Nevill Francis Mott and John van Vleck shared the award with him. In 1982, he was awarded the National Medal of Science. He retired from Bell Labs in 1984 and is currently Joseph Henry Professor of Physics at Princeton University. Anderson's writings include Concepts of Solids, Basic Notions of

Condensed Matter Physics and The Theory of Superconductivity in the High-Tc Cuprates. Anderson currently serves on the board of advisors of Scientists and Engineers for America, an organization focused on promoting sound science in American government. He is a certified first degree-master of the Chinese board game Go. A 2006 statistical analysis of scientific research papers by José Soler, comparing number of references in a paper to the number of citations, declared Anderson to be the "most creative" physicist in the world.



1. Minimum conductivity (Mott)

In 1958, Anderson proposed a theory on electron localization. This theory did not attract wide attention immediately. In 1978, Mott suggested that when the concentration of impurities reaches a certain level, localized electron waves start overlapping with each other and form a continuous impurity band. The bottom of this band is still localized so that there exists a sharp boundary between the localized and extended states. Hence, the mobility of electrons is finite if the electron energy is above a critical value ε_c , and is zero below. When the energy is decreased toward the critical value, the conductivity will decrease until it reaches a minimum value σ_{\min} , below which it drops to zero. There is some discontinuity in change of the conductivity at ε_c .

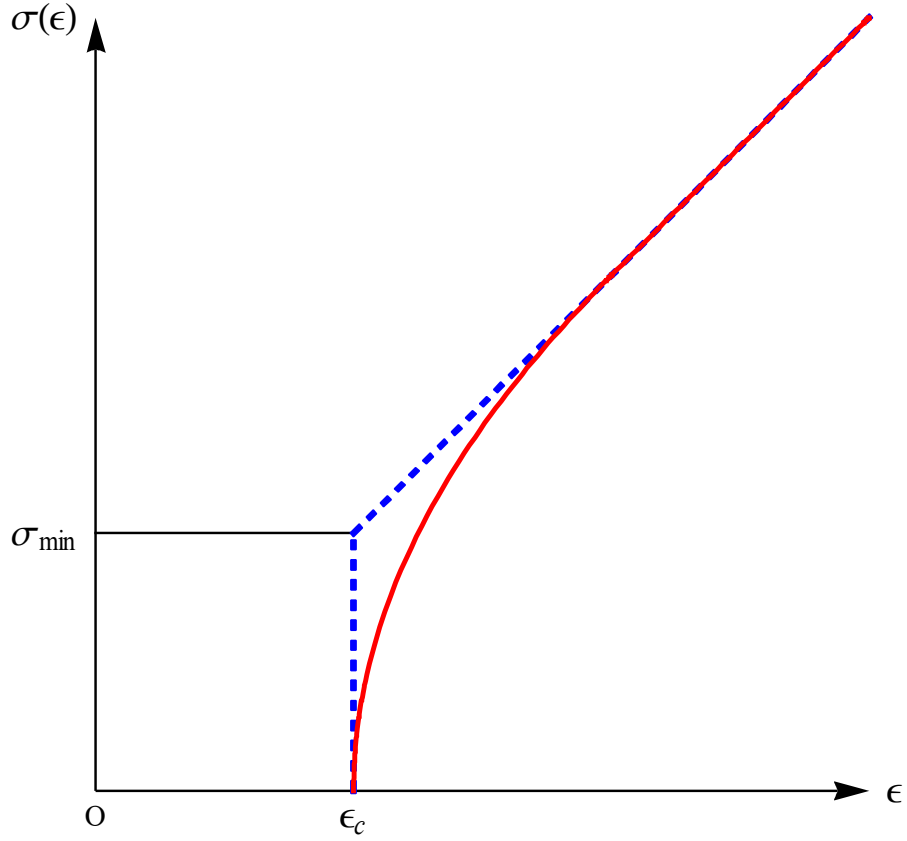


Fig. Conductivity vs electron energy for the 3D disordered system. The red line shows the continuous transition predicted from the scaling theory. The dashed line is prediction of the minimum conductivity (Mott).

The idea of Mott for the minimum conductivity is as follows. For the system with a dimension d , the electron number density is on the order of

$$n \approx k_F^d .$$

According to the Drude model, the conductivity is given by

$$\sigma = \frac{ne^2\tau}{m} \approx \frac{e^2}{h} k_F^{d-1} l ,$$

since the mean free path l is given by

$$l = \frac{\hbar k_F \tau}{m} ,$$

where τ is the relaxation time of electron. The minimum conductivity is obtained as

$$\sigma_{\min} \approx \frac{e^2}{h} k_F^{d-1},$$

when $k_F l = 1$ (the mobility edge). This conductivity is called the minimum conductivity. The conductivity discontinuously jumps from this value to zero.

However, the scaling theory of localization proposed by Abrahams et al. showed that it was incorrect. The conductivity should decrease continuously to zero. Our understanding on in Anderson localization has been greatly enhanced by the scaling law.

2. Scaling theory (gang of four, 1979)

If the system is macroscopic, the conductance should be proportional to the cross-section area and inversely proportional to the length L . In d dimensions, the cross-section area is proportional to L^{d-1} . Then we get the conductance $G(L)$ as

$$G(L) = \sigma \frac{L^{d-1}}{L} = \sigma L^{d-2},$$

where σ is the conductivity, which is independent of L . We define the Thouless number g as

$$g(L) = \frac{G(L)}{\frac{e^2}{h}} = \frac{h\sigma}{e^2} L^{d-2},$$

(*Thouless number* is the dimensionless conductance), where the unit of e^2/h is Ω^{-1} .

We now consider the case when the length L of the system is changed into L' .

$$\frac{L'}{L} = \nu.$$

For an arbitrary length L' , the conductance is assumed to obey the scaling form

$$g(L') = F(g(L), \frac{L'}{L}),$$

or

$$g(\nu L) = F(g(L), \nu),$$

where F is a universal function which is independent of the microscopic structure of the system.

$$\frac{g(\nu L)}{g(L)} = \frac{F(g(L), \nu)}{g(L)} = f(g(L), \nu),$$

where f is also a universal function. Here we assume that ν is continuous variable. Then we get

$$\begin{aligned} \lim_{\nu \rightarrow 1} \frac{\ln\left[\frac{g(\nu L)}{g(L)}\right]}{\ln \nu} &= \lim_{\nu \rightarrow 1} \frac{\ln[g(\nu L)] - \ln[g(L)]}{\ln \nu} \\ &= \lim_{\Delta L \rightarrow 0} \frac{\ln[g(L + \Delta L)] - \ln[g(L)]}{\ln(L + \Delta L) - \ln L} \\ &= \left(\frac{d \ln[g(L)]}{d \ln L} \right) = \beta(g) \end{aligned}$$

We need to determine the form of $\beta(g)$, which is a universal function.

When $\beta(g) > 0$, $g(L)$ increases with increasing L .

When $\beta(g) < 0$, $g(L)$ decreases with increasing L .

3. The form of $\beta(g)$ in the metallic limit

In the ordinary metallic region where $g \gg 1$,

$$g = \frac{h\sigma}{e^2} L^{d-2}.$$

Then we have

$$\beta(g) = d - 2.$$

As the disorder increases, g will decrease and eventually vanish (insulator). For large systems,

$$g = 4e^{-2L/\xi},$$

(Pichard's formula). Then we have

$$\beta(g) = -\frac{2L}{\xi} = \ln g + \text{constant}$$

where $g \ll 1$.

If we assume that $\beta(g)$ is a smooth monotonic function of $\ln g$, we get the schematic diagram shown below. From this figure, we can conclude that in 3D $\beta(g)$ intersects the $\beta = 0$ axis at a certain $g = g_c$. Around the value of g_c , $g(L)$ continuously approaches zero with no finite minimum metallic conductivity.

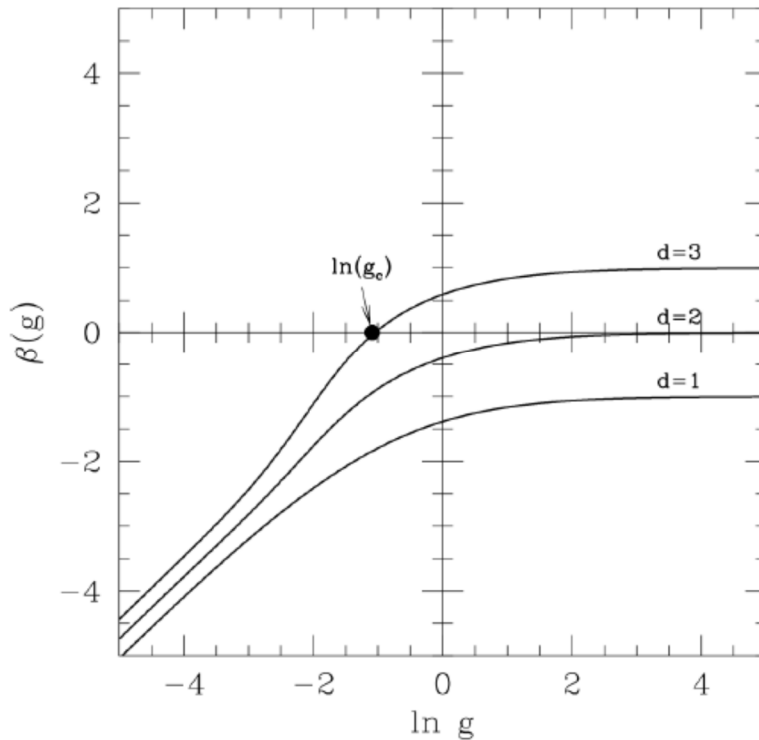


Fig. Schematic diagram of β as a function of g for $d = 1, 2, 3$. A critical point exists at $g = g_c$ for the $d = 3$ case.

((Conclusion)) Scaling theory

1. For $d = 3$, there exists a mobility edge at which the conductivity continuously reduces to zero. This implies that there is no minimum metallic conductivity.
2. For $d = 3$, both extended and localized states can exist.
3. For $d = 2$, there is no true metallic conduction in that, in the limit of infinite system size, all the states are localized no matter how small the randomness. The conductivity always decreases in the form of $-\ln(L)$ in two dimensions (weak localization regime).

4. $d = 1$ case

Note that in $d = 1$, we can calculate $\beta(g)$, using the Landauer formula,

$$g = \frac{T}{1-T}$$

with

$$T \propto \exp\left(-\frac{L}{\xi}\right).$$

From this, we get

$$\beta = -(1+g) \ln\left(1 + \frac{1}{g}\right)$$

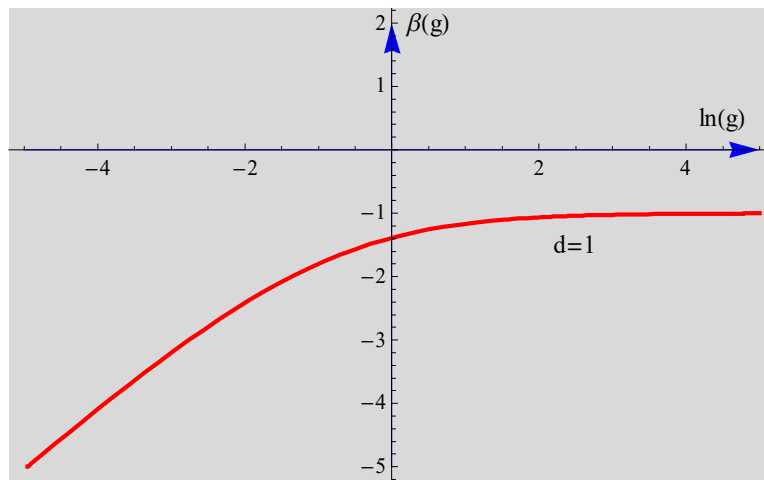


Fig. Exact solution for $\beta(g)$ vs $\ln g$ for $d = 1$.

5. Integrating the function $\beta(d = 2)$: weak localization regime

In $d = 2$, there is no fixed point. In the Ohmic limit ($g \gg 1$), we have

$$\beta(g) = \varepsilon - \frac{c}{g},$$

where c is a constant (Hikami 1980, 1981). Then we get

$$\ln\left(\frac{L}{L_0}\right) = \int_{\ln(g_0)}^{\ln(g)} \frac{d \ln g}{\beta(g)} = -\frac{1}{c}(g - g_0),$$

or

$$g = g_0 - c \ln\left(\frac{L}{L_0}\right).$$

When $L = l$, $g = k_F l$, which can be derived from the Drude model (see later section). So we choose

$$L_0 = l, \quad g_0 = k_F l.$$

or

$$g(L) = k_F l - c \ln\left(\frac{L}{l}\right)$$

We see that the localization length ξ is the value of L for which the correction term is on the same order as $k_F l$,

$$\xi = l \exp\left(\frac{k_F l}{c}\right)$$

A first principles treatment yields

$$c = \frac{2}{\pi}.$$

The metallic regime in $d = 2$ is called the **weak localization regime**.

$$g(L) = k_F l - \frac{2}{\pi} \ln\left(\frac{L}{l}\right).$$

The first term comes from the Drude model. The second term comes from the quantum interference. As the impurity concentration increases, the interference of electron waves going around the impurities becomes more and more important. Hence, in order to approach the regime of weak localization, it is necessary to improve the Drude model.

6. Weak localization regime

(i) Temperature dependence of conductivity

At finite temperatures, inelastic scatterings are present and introduce a finite inelastic scattering length

$$L_\varepsilon = \sqrt{D\tau_\varepsilon},$$

where τ_ε is the inelastic scattering time and D is the diffusion constant. At low temperatures τ_ε is expressed by

$$\tau_\varepsilon = \alpha T^{-p},$$

where the exponent p is between 1 and 2, and α is a constant. Then we get

$$\begin{aligned} g(L_\varepsilon) &= k_F l - \frac{2}{\pi} \ln\left(\frac{L_\varepsilon}{l}\right) \\ &= k_F l - \frac{2}{\pi} \left[\ln\left(\frac{\sqrt{D\alpha} T^{-p/2}}{l}\right) \right] \\ &= k_F l - \frac{2}{\pi} \left[\ln\left(\frac{1}{l} \sqrt{D\alpha}\right) - \frac{p}{2} \ln T \right] \\ &= k_F l - \frac{2}{\pi} \ln\left(\frac{1}{l} \sqrt{D\alpha}\right) + \frac{p}{\pi} \ln T \end{aligned}$$

The logarithmic term ($\ln T$) appears as a result of quantum interference of electron waves. Such a behavior was first observed experimentally in thin metal films (Dolan and Osheroff 1979).

(ii) Negative magnetoresistance

The phase of a wavefunction changes appreciably over the magnetic length scale

$$L_H = \sqrt{\frac{c\hbar}{eH}}.$$

When $L = L_H$,

$$\begin{aligned}
g(L_H) &= k_F l - \frac{2}{\pi} \ln\left(\frac{L_H}{l}\right) \\
&= k_F l - \frac{2}{\pi} \left[\ln\left(\frac{\sqrt{c\hbar}}{l}\right) \right] \\
&= k_F l - \frac{2}{\pi} \left[\ln\left(\frac{1}{l} \sqrt{\frac{c\hbar}{e}}\right) - \frac{1}{2} \ln H \right] \\
&= k_F l - \frac{2}{\pi} \ln\left(\frac{1}{l} \sqrt{\frac{c\hbar}{e}}\right) + \frac{1}{\pi} \ln H
\end{aligned}$$

Then the conductance increases with increasing H through a logarithmic term. In turn, the resistance decreases with increasing H (*negative resistance*).

7. Drude model for $d = 2$

According to the Drude model, the sheet resistance, i.e., the resistance of a square segment of the 2D electron gas, can be written as

$$R_s = \frac{\hbar}{e^2} \frac{1}{k_F l},$$

where $l = v_F \tau$ is the mean free path. The sheet density n_s is defined by

$$n_s = \frac{N}{L^2} = \frac{1}{L^2} 2 \frac{L^2}{(2\pi)^2} \pi k_F^2 = \frac{k_F^2}{2\pi}$$

where N is the total number of electrons below the Fermi energy ε_F ,

$$N = 2 \frac{L^2}{(2\pi)^2} \pi k_F^2$$

The sheet resistance is defined by

$$R_s = \frac{m}{n_s e^2 \tau}$$

Using the mean free path $l = v_F \tau$, we get

$$R_s = \frac{m}{\frac{k_F^2}{2\pi} e^2 \tau} = \frac{h}{e^2} \frac{1}{k_F l},$$

since

$$mv_F = \hbar k_F.$$

The conductance G is

$$G = \frac{1}{R_s} = \frac{e^2}{h} k_F l$$

The corresponding Thouless number is

$$g = k_F l$$

((Note)) For $d = 2$,

$$g = \frac{h\sigma}{e^2} L^{d-2} = \frac{h\sigma}{e^2}$$

8. Integrating the function $\beta(d = 3)$

For $d = 3$, we do not know the precise form of $b(g)$. So we approximate it piecewise:

$$\beta(g) = 1, \quad \text{if } g \geq g_+$$

$$\beta(g) = \alpha \ln \left(\frac{g}{g_c} \right), \quad \text{if } g_- < g < g_+$$

$$\beta(g) = \ln g, \quad \text{if } g < g_-$$

where $\beta(g) = 0$ at $g = g_c$. This point is called a fixed point. From the condition of continuity, we have

$$\ln g_+ = \ln g_c + \frac{1}{\alpha}, \quad \ln g_- = \frac{\varepsilon}{\alpha - 1} \ln g_c$$

9. Mobility edge

On the metallic side of the transition near the fixed point, a linear variation may be assumed;

$$\beta(g) = a(g - g_c).$$

where a is a constant. In order to incorporate the asymptotic property that

$$\beta(g) = 1. \quad (g \rightarrow \infty)$$

Let us adopt a simple form

$$\beta(g) = \frac{a(g - g_c)}{1 + a(g - g_c)}.$$

Although this form specifies a special approach to the asymptotic value, the final result we will obtain is independent of this special form.

$$\left(\frac{L dg}{g dL} \right) = \beta(g) = \frac{a(g - g_c)}{1 + a(g - g_c)},$$

or

$$\begin{aligned} \int_L^{L_0} \frac{dL}{L} &= \ln\left(\frac{L_0}{L}\right) = \frac{1}{a} \int_{g(L)}^{g(L_0)} \frac{1 + a(g - g_c)}{g(g - g_c)} dg \\ &= \frac{1}{ag_c} \ln\left[\frac{g(L_0) - g_c}{g(L) - g_c} \cdot \frac{g(L)}{g(L_0)} \right] + \ln\left[\frac{g(L_0)}{g(L)} \right] \end{aligned}$$

where a macroscopic length L_0 is much larger than L . We define the exponent ν as

$$\nu = \frac{1}{ag_c} = \frac{1}{g_c \beta'(g_c)}.$$

The result of integration is

$$\left(\frac{g(L) - g_c}{g(L_0) - g_c} \cdot \frac{g(L_0)}{g(L)} \right)^\nu \frac{g(L)}{g(L_0)} = \frac{L}{L_0}.$$

For the macroscopic L_0 , we have

$$g(L_0) = \frac{h\sigma}{e^2} L_0.$$

We choose L such that $g(L)$ is sufficient close to g_c . Since $g(L_0)$ is much larger than g_c , we get

$$\left(\frac{g(L) - g_c}{g(L_0) - g_c} \cdot \frac{g(L_0)}{g(L)} \right)^\nu \frac{g(L)}{g(L_0)} = \frac{L}{L_0},$$

or

$$\sigma = \frac{e^2}{h} \frac{g(L_0)}{L_0} = \frac{e^2}{h} \frac{g(L)^{1-\nu}}{L} \left(\frac{g(L_0)}{g(L_0) - g_c} \right)^\nu (g(L) - g_c)^\nu = \sigma_c (g - g_c)^\nu.$$

Under the assumption that the conductance is proportional to the electron energy, one can rewrite this equation in the form

$$\sigma = \sigma_c (\varepsilon - \varepsilon_c)^\nu$$

where ε_c represents a mobility edge. At this edge, the conductivity is expected to vanish rather than to approach a constant. The critical exponent ν is predicted to be equal to 1. Experimentally $\nu = 1/2$ was reported.

10. Drude model for $d = 3$

The electron number density n is defined by

$$n = \frac{N}{L^3} = \frac{1}{L^3} 2 \frac{L^3}{(2\pi)^3} \frac{4\pi}{3} k_F^3 = \frac{k_F^3}{3\pi^2},$$

where N is the total number of electrons below the Fermi energy ε_F ,

$$N = 2 \frac{L^3}{(2\pi)^3} \frac{4\pi}{3} k_F^3.$$

The conductivity is defined by

$$\sigma_{3D} = \frac{ne^2\tau}{m}.$$

Using the mean free path $l = v_F \tau$, we get

$$\sigma_{3D} = \frac{ne^2\tau}{m} = \frac{e^2}{h} \frac{2}{3\pi} (k_F l) k_F = \frac{e^2}{h} \frac{1}{3\pi^2} (k_F l) k_F,$$

since

$$mv_F = \hbar k_F.$$

Since

$$G(L) = \sigma_{3D} L,$$

we get

$$g(L) = \frac{G(L)}{\frac{e^2}{h}} = \frac{\sigma_{3D} L}{\frac{e^2}{h}} = \frac{2}{3\pi} (k_F^2 l L).$$

Since $\varepsilon = \frac{\hbar^2}{2m} k_F^2$,

$$g(L) = \frac{4m}{3\pi\hbar^2} (\varepsilon L)$$

Then $g(L)$ is proportional to the energy of electron ε .

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