Landauer formula

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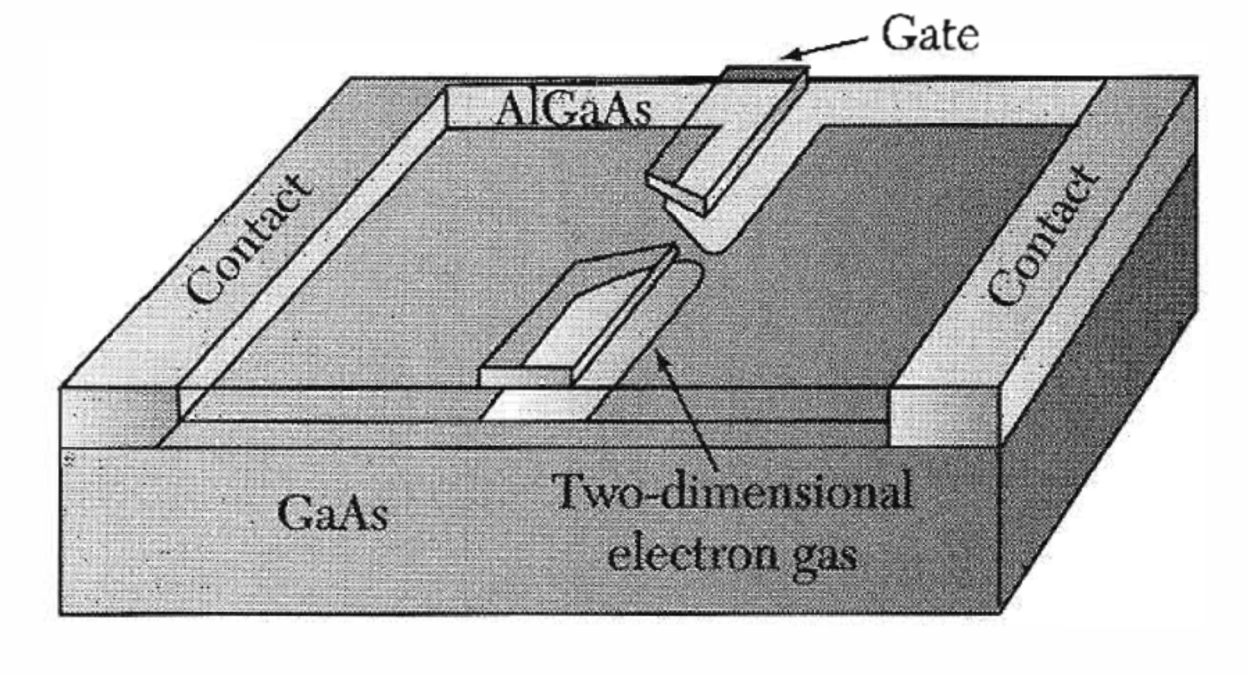
1. Introduction

The Landauer formula—named after Rolf Landauer, who first suggested its prototype in 1957 is a formula relating the electrical resistance of a quantum conductor to the scattering properties of the conductor. In the simplest case where the system only has two terminals, and the scattering matrix of the conductor does not depend on energy. The electrical conductance is

$$G = G_0 \sum_t \Im_n$$

where $G_0 = \frac{e^2}{\pi\hbar} = \frac{2e^2}{h} = 7.74809 \times 10^{-5}\Omega^{-1} = (12.906 k\Omega)^{-1}$ is the conductance quantum, $\Im_n$ is the transmission eigenvalues of the channels, and the sum runs over all transport channels in the conductor. $R_0 = G_0^{-1}$ is the resistance quantum.

A short quasi-1D channel is formed between two regions of a 2D electron gas in a GaAs/AlGaAs heterostructure. As the carrier density of the channel is increased, the conductance increases in discrete steps of height $\frac{2e^2}{h}$.



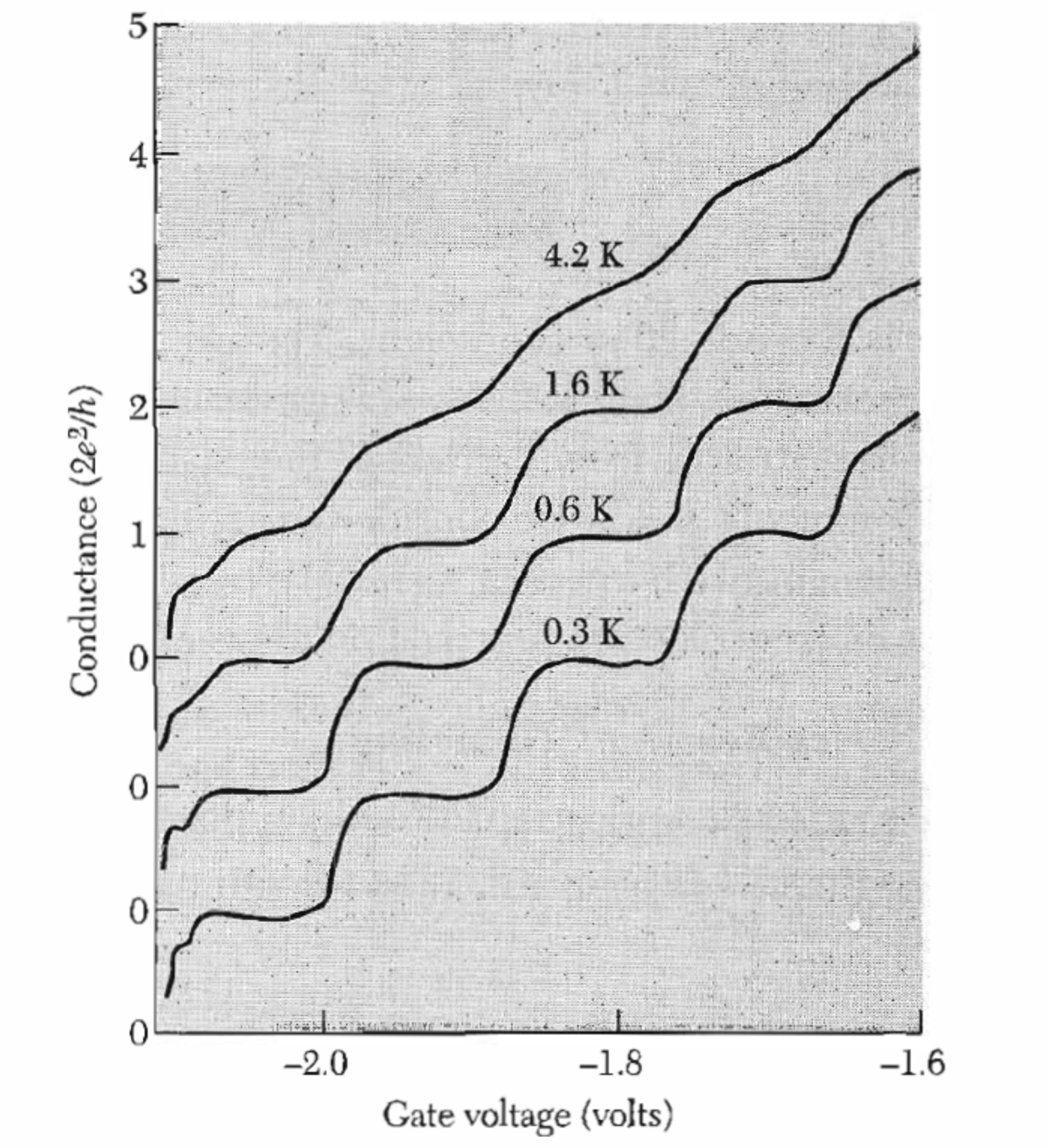


Fig. Conductance quantization in a short channel electrostatically defined in a GaAs/AlGaAs heterostructure at different temperatures. A negative gate voltage V_g applied to the metallic gate on the surface of the sample depletes the carriers in the underlying 2D electron gas. (Kittel, ISSP, 8-th edition).

2. Derivation of the Landauer formula for the single channels

Here we discuss the derivation of Landauer formula,

$$I = e\sum_{k>0} v_k n_k = \frac{e}{L}\sum_{k>0} v_k f_k$$

where $k > 0.$ We use the energy dispersion for the 2D system,

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$$

$$\sum_{k>0} \rightarrow \frac{2L}{2\pi} dk = \frac{L}{\pi} dk\,,$$

Note that for the density of states for the 1D system we need to use the factor 2 for the even parity of the energy dispersion $\varepsilon_{-k} = \varepsilon_k$. This is not the case since $k > 0$. Only the spin degeneracy (the factor 2) is taken into account. The group velocity is

$$v_k = \frac{1}{\hbar}\frac{\partial \varepsilon_k}{\partial k} = \frac{\hbar k}{m}$$

The current is given by

$$\begin{aligned} I &= \frac{e}{L}\frac{L}{\pi}\int dk \frac{\hbar k}{m} f_k \\ &= \frac{e}{\pi}\frac{\hbar}{m}\frac{m}{\hbar^2}\int_0^{\mu} f(\varepsilon) d\varepsilon \\ &= \frac{e}{\pi\hbar}\int_0^{\mu} f(\varepsilon) d\varepsilon \end{aligned}$$

since

$$d\varepsilon = \frac{\hbar^2 k}{m} dk$$

At $T = 0$ K, we have the current between two elecrodes (having chemical potentials, μ_1 and μ_2)

$$I = \frac{e}{\pi\hbar}\int_{\mu_1}^{\mu_2} f(\varepsilon)d\varepsilon$$

$$= \frac{e}{\pi\hbar}\int_{\mu_1}^{\mu_2} d\varepsilon$$

$$= \frac{e}{\pi\hbar}(\mu_2 - \mu_1)$$

$$= \frac{2e^2}{h}(V_2 - V_1)$$

or

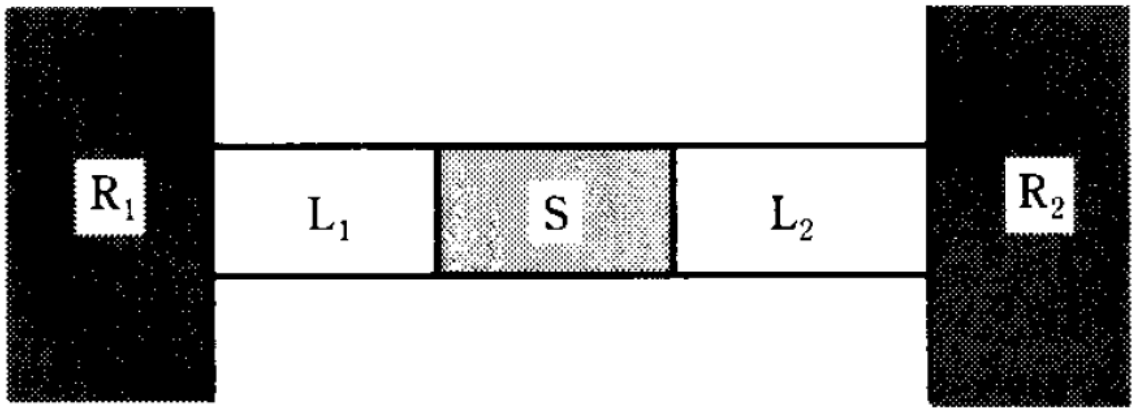
$$I = \frac{2e^2}{h}V$$

where $V = V_1 - V_2$

$$\frac{h}{e^2} = 25.812807 \text{ k}\Omega$$

and

$$eV = \mu_2 - \mu_1$$



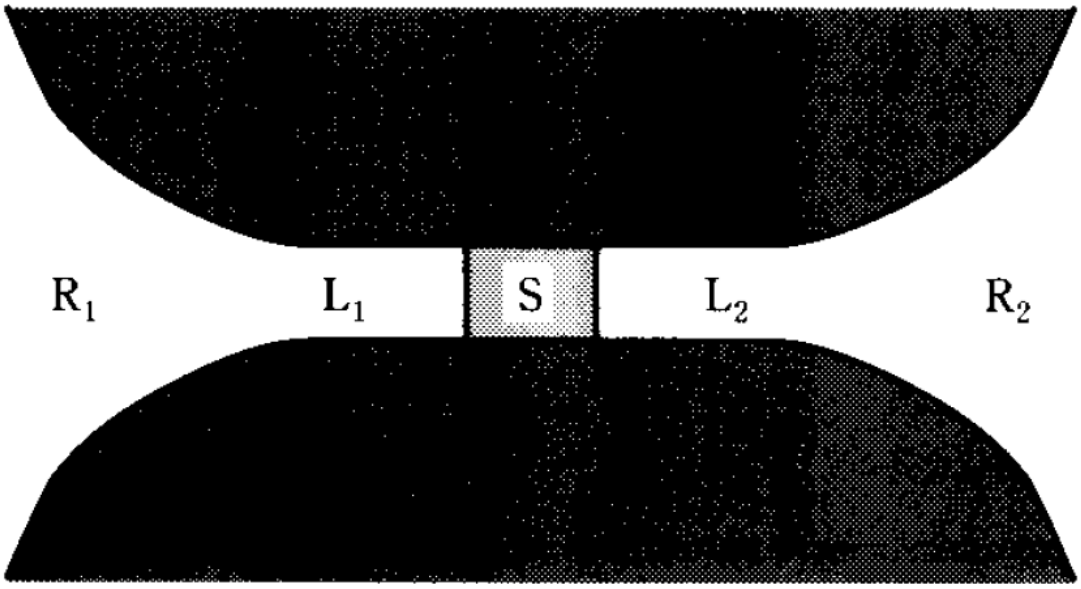


Fig. Model of electric conduction for the derivation of Landauer formula. The chemical potential of the reserver R_1 is μ_1 and the chemical potential of the reserver R_2 is μ_2. Both L_1 and L_2 are ideal perfect conductors where electrons are not scattered.

((**Note**))
Suppose we have one electron with charge q. The current is flowing under the perturbation field

$$I = \frac{\Delta Q}{\Delta t} = \frac{q}{T/2} = \frac{q\nu}{2}$$

where T is the period of the wave as a duality of electron and ν $(=1/T)$ is the frequency of the wave. $T/2$ is the time taken for electron to move from the one electrode to the other electrode. The relation $I = q\nu$ is used to evaluate the current for electron rotating around a proton in the Bohr hydrogen atom model. Note that ν is the characteristic frequency

$$h\nu = qV$$

from the energy conservation. Then we have

$$I = \frac{q^2}{2h}V = \frac{e^2}{2h}V$$

Note that for V=1μV, we have $\nu = 483.598$ MHz.

This relation is independent of the nature of material used in the 1D channel.

2. The case of many channels

If the channel is not perfectly conducting, the overall conductance is the quantum of conductance times the probability $\Im(\varepsilon_F)$ for electron transmission through the channel,

$$G(\varepsilon_F) = G_0 \Im(\varepsilon_F) \qquad \text{(Landauer formula)}$$

For a quasi-1D system with multiple channels, we sum over the contributions of each channel, since conductances in parallel add:

$$\Im(\varepsilon_F) = \sum_{i,j} \Im_{ij}(\varepsilon_F)$$

where $\Im_{ij}$ is the probability of finding electron in the output channel i, which enters from the channel j.

3. Resistance quantum for the single channel

The resistance quantum for the simple channel is

$$\begin{aligned} R &= \frac{1}{G} \\ &= \frac{1}{G_0}\frac{1}{\Im} \\ &= \frac{h}{2e^2}\left(\frac{1-\Im+\Im}{\Im}\right) \\ &= \frac{h}{2e^2} + \frac{h}{2e^2}\left(\frac{1-\Im}{\Im}\right) \\ &= \frac{h}{2e^2} + \frac{h}{2e^2}\left(\frac{\Re}{\Im}\right) \end{aligned}$$

where $\Re = 1 - \Im$ is the reflection coefficient. The first term is the resistance quantum. The second term is due to scattering from barriers in the channel, and it is zero for a perfect conductor.

APPENDIX

The Fermi momentum in the 2D system

$$N = \frac{2V}{(2\pi)^2}\int 2\pi k dk$$

$$\varepsilon = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \qquad\qquad d\varepsilon = \frac{\hbar^2 k dk}{m}$$

$$N = \frac{2V}{(2\pi)^2} 2\pi \frac{m}{\hbar^2}\int_0^{\varepsilon_F} d\varepsilon = \frac{V}{\pi}\frac{m}{\hbar^2}\varepsilon_F$$

The number density n is

$$n = \frac{N}{V} = \frac{m\varepsilon_F}{\pi\hbar^2}$$

Thus the Fermi momentum is

$$n = \frac{m\varepsilon_F}{\pi\hbar^2}\frac{{p_F}^2}{2m} = \frac{{p_F}^2}{2\pi\hbar^2}, \qquad p_F = \hbar\sqrt{2\pi n}.$$