

## **Bose-Einstein condensation (BEC) in alkali atoms**

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In 1925 Einstein predicted that at low temperatures particles in a gas could all reside in the same quantum state. This peculiar gaseous state, a Bose–Einstein condensate (BEC), was produced in the laboratory for the first time in 1995, using the powerful laser-cooling methods developed in recent years. These condensates exhibit quantum phenomena on a large scale, and investigating them has become one of the most active areas of research in contemporary physics.

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The first gaseous BEC was generated by Eric Cornell and Carl Wieman in 1995 at the University of Colorado at Boulder NIST-JILA lab, using a gas of Rb atoms cooled to **170 nK**. For their achievements Cornell, Wieman, and Wolfgang Ketterle at MIT received the 2001 Nobel Prize in Physics. In November 2010 the first photon BEC was observed.

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**Carl Wieman** was born in Corvallis, Oregon in the United States and graduated from Corvallis High School. Wieman earned his B.S. in 1973 from MIT and his Ph.D. from Stanford University in 1977; he was also awarded a Doctor of Science, *honoris causa* from the University of Chicago in 1997. He was awarded the Lorentz Medal in 1998. In 2001, he won the Nobel Prize in Physics, along with Eric Allin Cornell and Wolfgang Ketterle for fundamental studies of the Bose-Einstein condensate. In 2004, he was named United States Professor of the Year among all doctoral and research universities.

[http://en.wikipedia.org/wiki/Carl\\_Wieman](http://en.wikipedia.org/wiki/Carl_Wieman)

**Eric Allin Cornell** (born December 19, 1961) is an American physicist who, along with Carl E. Wieman, was able to synthesize the first Bose–Einstein condensate in 1995. For their efforts, Cornell, Wieman, and Wolfgang Ketterle shared the Nobel Prize in Physics in 2001.

[http://en.wikipedia.org/wiki/Eric\\_Allin\\_Cornell](http://en.wikipedia.org/wiki/Eric_Allin_Cornell)



Fig Carl Wieman (left) and Eric Cornell (right)

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**Wolfgang Ketterle** (born 21 October 1957) is a German physicist and professor of physics at the Massachusetts Institute of Technology (MIT). His research has focused on experiments that trap and cool atoms to temperatures close to absolute zero, and he led one of the first groups to realize Bose-Einstein condensation in these systems in 1995. For this achievement, as well as early fundamental studies of condensates, he was awarded the Nobel Prize in Physics in 2001, together with Eric Allin Cornell and Carl Wieman.



[http://en.wikipedia.org/wiki/Wolfgang\\_Ketterle](http://en.wikipedia.org/wiki/Wolfgang_Ketterle)

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### 1. Thermal de Broglie wave length

The thermal de Broglie wavelength  $\lambda$  is defined by

$$\lambda = \frac{h}{p} = \frac{h}{mv_{rms}} = \frac{h}{m\sqrt{\frac{3k_B T}{m}}} = \frac{h}{\sqrt{3mk_B T}},$$

where we use the root-mean squared speed  $v_{rms}$

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3k_B T}{m}}.$$

Note that

$$\frac{1}{2} m v_{rms}^2 = \frac{3}{2} k_B T.$$

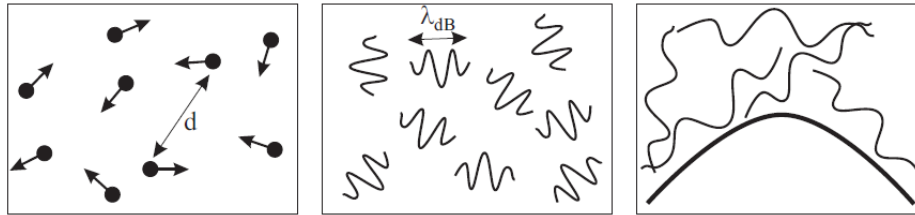
Suppose that there is one atom in a cube with the side  $d$ , where  $d$  is the average distance between atoms. The number density is given by

$$n = \frac{N}{V} = \frac{1}{d^3}.$$

At sufficiently high temperatures,  $\lambda$  is much shorter than  $d$ . The condition ( $\lambda \ll d$ ) can be expressed by

$$\lambda = \frac{h}{\sqrt{3k_B m T}} = \frac{2\pi\hbar}{\sqrt{3k_B m T}} \ll d.$$

The thermal de Broglie wavelength  $\lambda$  increases with decreasing temperature  $T$ . At very low temperatures,  $\lambda$  becomes larger than  $d$ .



**Fig.** Relation between the thermal de Broglie wavelength  $\lambda$  and the average distance  $d$  between atoms in the system. The thermal de Broglie wavelength increases with decreasing temperature. Very high temperature (left);  $\lambda \ll d$ . Intermediate temperature (middle):  $\lambda \approx d$ , and very low temperature (right):  $\lambda \gg d$ .

## 2. Quantum concentration $n_Q$

The quantum concentration  $n_Q$  is the particle concentration (i.e. the number of particles per unit volume) of a system where the interparticle distance is equal to the thermal de Broglie wavelength or equivalently when the wavefunctions of the particles are touching but not overlapping. Quantum effects become appreciable when the particle concentration is greater than or equal to the quantum concentration. The quantum concentration is defined by

$$n_Q \approx \frac{1}{\lambda^3}.$$

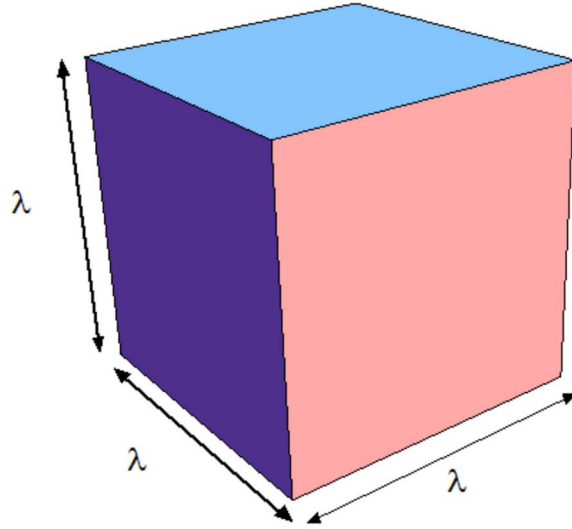
$n_Q$  is the concentration associated with one atom in a cube of side equal to the thermal de Broglie wavelength. The quantum effect (Bose-Einstein distribution) becomes prominent when  $n > n_Q$  (or  $\lambda > d$ ). The Maxwell-Boltzmann distribution is still valid for when  $n < n_Q$  (or  $\lambda \ll d$ ).

Using the thermal de Broglie wavelength given by

$$\lambda = \frac{2\pi\hbar}{\sqrt{3mk_B T}}.$$

$\frac{1}{\lambda^3}$  is evaluated as

$$\frac{1}{\lambda^3} = \frac{(3mk_B T)^{3/2}}{(2\pi\hbar)^3} = \left(\frac{3}{2\pi}\right)^{3/2} \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} = 0.3299 \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} \approx n_Q.$$



**Fig.** Definition of quantum concentration. The de Broglie wavelength is on the order of interatomic distance.

### 3. Bose-Einstein condensation (BEC) temperature

What is the temperature which  $n = n_Q$ ? The Bose-Einstein condensation (BEC) temperature  $T_E$  is defined by

$$n = n_Q = \left( \frac{mk_B T_E}{2\pi\hbar^2} \right)^{3/2},$$

or

$$k_B T_c = \frac{2\pi\hbar^2}{m} n^{2/3}.$$

Note that the BEC temperature is predicted by Einstein as

$$k_B T_E = \frac{2\pi\hbar^2}{m} \left( \frac{N/V}{2.61238} \right)^{2/3},$$

or

$$k_B T_E = \frac{2\pi\hbar^2}{m} \left( \frac{n}{2.61238} \right)^{2/3} = \frac{2\pi\hbar^2}{m} \frac{n^{2/3}}{2.61238^{2/3}} = 0.5272 \frac{2\pi\hbar^2}{m} n^{2/3}.$$

It is found that the expression of  $T_E$  derived from the condition  $n_Q = n$  is nearly the same as that of the  $T_E$  derived by Einstein.

**((Problem 9-52))**

In one experiment by Cornell and Wieman, a Bose-Einstein condensate contained 2000  $^{87}\text{Rb}$  atoms within a volume of about  $10^{-15} \text{ m}^3$ . Estimate the temperature at which Bose-Einstein condensation should have occurred. [Modern Physics for Scientists and Engineers, Third edition, Stephen T. Thornton and Andrew Rex (Brooks/Cole Cengage Learning). **Problem 9-52**]

**((Solution))**

For Rb atom

$$m = 85.4678 \text{ u},$$

$$N = 2000, \quad V = 10^{-15} \text{ m}^3.$$

Then  $T_E$  can be evaluated as

$$T_E = 29.8441 \text{ nK}.$$

**4. Scattering force**

How can we get such a very low temperature? In order to achieve the lowest temperature, we use the laser cooling techniques. The temperature of the atoms is linearly proportional to the kinetic energy of atoms. So we need to reduce the velocity of atoms.

**((Example))**

Rb atom

$$v_{\text{rms}} = 295.89 \text{ m/s} \quad \text{at } T = 300 \text{ K}$$

$$v_{\text{rms}} = 17.083 \text{ m/s} \quad \text{at } T = 1 \text{ K}$$

$$v_{\text{rms}} = 0.54 \text{ m/s} \quad \text{at } T = 1 \text{ mK}$$

$$v_{\text{rms}} = 0.017 \text{ m/s} \quad \text{at } T = 1 \mu\text{K}$$

The force that light could exert on matter are well understood. Maxwell's calculation of the momentum flux density of light, and the laboratory observation of light pressure on macroscopic object by Lebedev and by Nichols and Hull provided the first quantitative understanding of how light could exert forces on material object. Einstein pointed out the

quantum nature of this force: an atom that absorbs a photon of energy  $h\nu$  will receive a momentum

$$G = \frac{h\nu}{c},$$

along the direction of incoming photon. If the atom emits a photon with momentum, the atom will recoil in the opposite direction. Thus the atom experiences a net momentum change

$$\Delta \mathbf{p}_{atom} = \mathbf{P}_{in} - \mathbf{P}_{out},$$

due to the incoherent scattering process. Since the scattered photon has no preferred direction, the net effect is due to the absorbed photon, resulting in scattering force,

$$\mathbf{F}_{scatt} = N\mathbf{p}_{in},$$

where  $N$  is the number of photons scattered per second. Typical scattering rates for atoms excited by a laser tuned to a strong resonance line are on the order of  $10^7 - 10^8/\text{s}$ . The velocity of Na atom changes by 3cm/s.

$$m \frac{\Delta v}{\Delta t} = m \frac{\Delta v}{(1/N)} = p_{in} = \frac{h}{\lambda} \quad (\text{per one photon}).$$

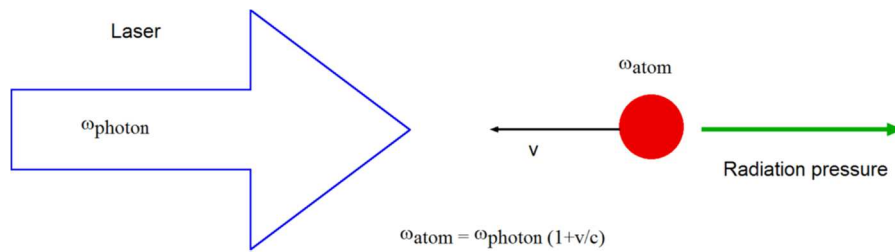
or

$$\Delta v = \frac{h}{Nm\lambda}.$$

After 1 sec, the velocity changes as

$$N\Delta v = \frac{h}{m\lambda} = 2.947 \text{ cm/s},$$

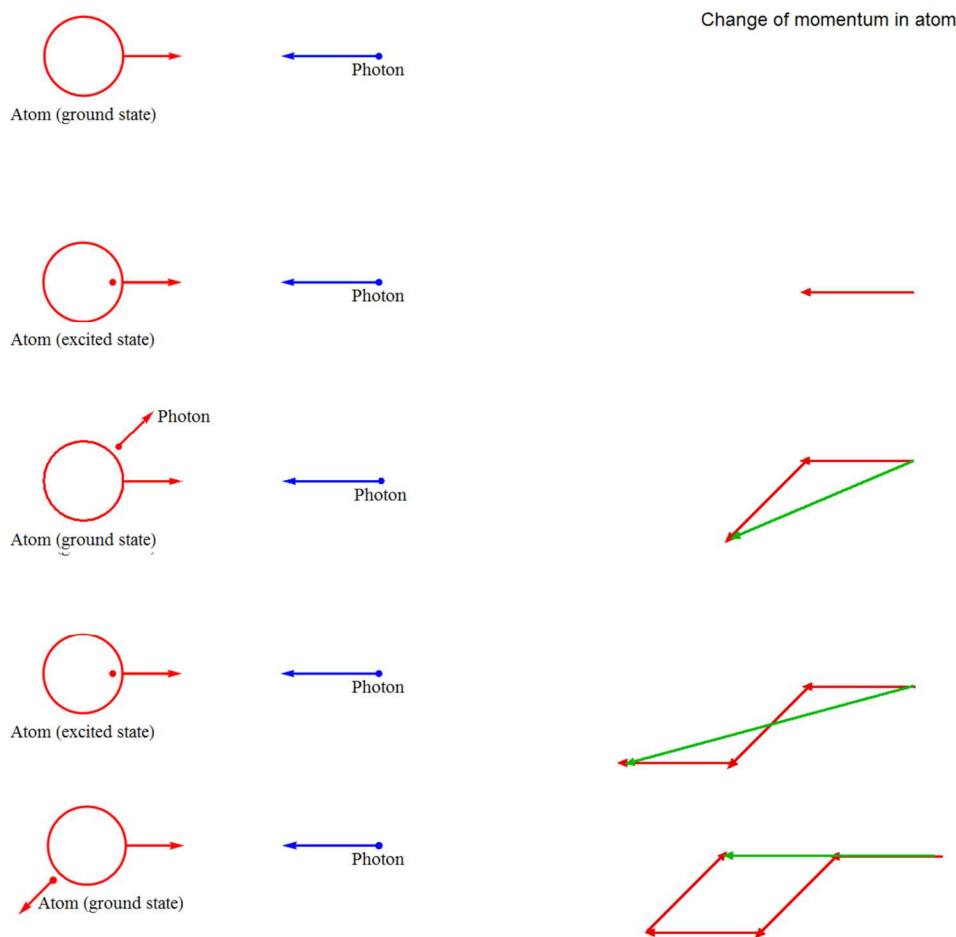
for the  $N$  photons, where  $m = 22.9897 \text{ u}$  for Na atom and  $\lambda = 589.0 \text{ nm}$  for the Na D line.



**Fig.** With the laser tuned to below the peak of atomic resonance. Due to the Doppler shift, atoms moving in the direction opposite the laser beam will scatter photons at a higher rate than those moving in the same direction as the beam. This leads to a larger force on the counter-propagating atoms.

**((Model)) Schematic explanation for the radiation pressure**

**(a), (b), (c) (d), (e)**





**Fig.** The change of linear momentum of atom due to the absorption and spontaneous emission of light.

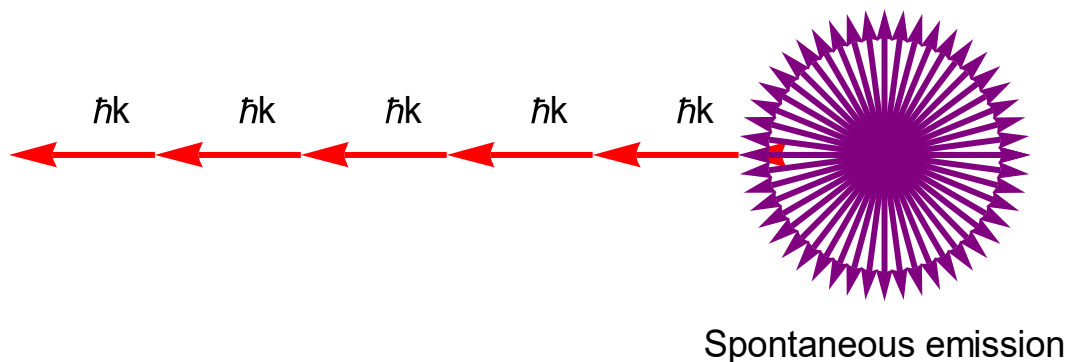
**Fig.(a)** Atom in the ground state. The laser beam comes in from the right side and is applied to the atom.

**Fig.(b)** The state of the atom changes from the ground state to the excited state due to the absorption of laser light. The atom gets a momentum  $\hbar k$  of photon.

**Fig.(c)** The atoms in the excited state return to the ground state due to the spontaneous emission. If the photon emits from the atom in the direction shown in Fig.(c), the atom receives momentum  $\hbar k$  in the opposite direction of photon. This behavior is called as recoil. The momentum is called recoiled momentum. Subsequently the spontaneous emission occurs. After the spontaneous emission, the atom returns to the ground state,

**Fig.(d)** Since the atom is again in the ground state, the atom absorbs the laser light. The atom receives momentum  $\hbar k$  of photon in the direction from right to left side.

**Fig.(e)** Due to the spontaneous emission, again the atom returns to the ground state. Note that the direction of the spontaneous emission may be different from that of the emission shown in Fig.(c). So in this case, the atom receives recoiled momentum. Again the atom returns to the ground state.

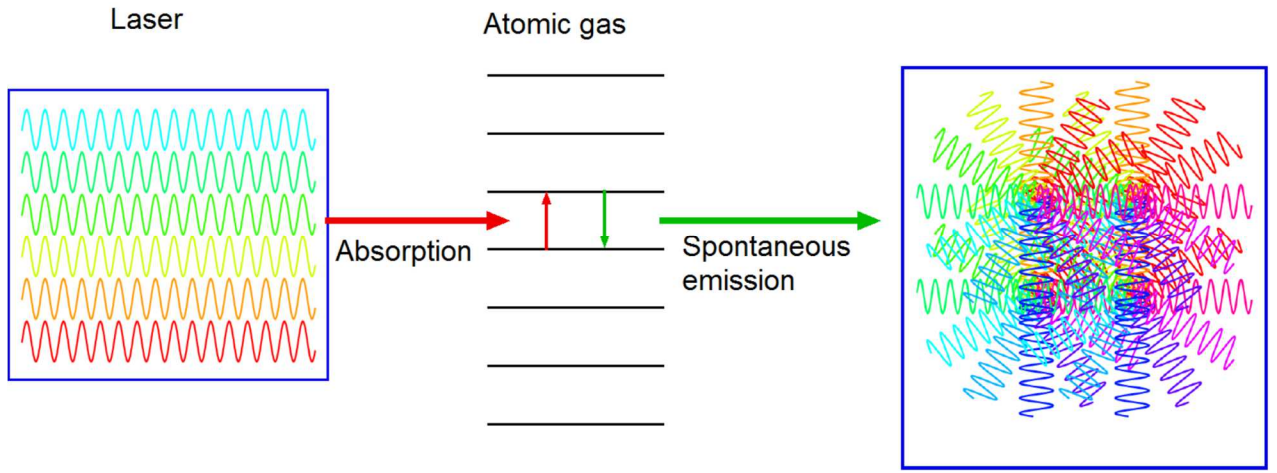


In a series of such processes, whenever the atom absorbs the laser light, the atom receives momentum  $\hbar k$ . During the spontaneous emission the atom receives isotropic recoiled

momentum, which become zero in momentum after many repeated spontaneous emission processes as shown in the above Fig. However, after each cycle of absorption and spontaneous emission, the atom receives linear momentum  $\hbar k$  in the direction of laser light.

When the angular frequency of the laser light is nearly equal to the energy difference between the ground state and the excited state for the atom, the atom receive momentum. Correspondingly, we define the radiation pressure as

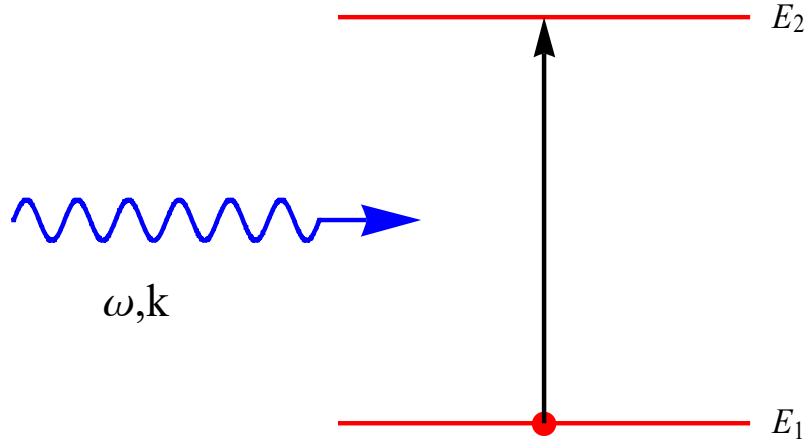
$$F = \frac{dp}{\Delta t} = \frac{\hbar \Delta k}{\Delta t}$$



**Fig.** Spontaneous emission and absorption.

## 5. Results from optical Bloch equations (just review from Quantum mechanics II)

Here we have a review on the solution of the optical Bloch equation for the two-level laser, where there is a ground state  $|1\rangle$  and an excited state  $|2\rangle$ ;  $E_2 > E_1$ .



The Rabi angular frequency is given by

$$\Omega = \frac{eE_0 X_{12}}{\hbar}.$$

The additional contribution to the line-width is known as power broadening, or saturation broadening. Note that  $X_{12}$  is the matrix element of the position operator of atom,

$$X_{12} = \langle 1 | \hat{x} | 2 \rangle$$

and  $E_0$  is the amplitude of electric field of photon. The diagonal matrix element  $\rho_{22}$  is given by

$$\rho_{22} = \frac{\frac{1}{4}\Omega^2}{(\omega_0 - \omega)^2 + \gamma^2 + \frac{1}{2}\Omega^2},$$

for monochromatic light, where  $\Gamma = 2\gamma$  is the line-width of the atomic transition and  $\rho_{22}$  is the probability for finding atoms in the excited state. This steady-state result is independent of the initial atomic population. The non-diagonal matrix element is

$$\rho_{12} = -e^{-i(\omega - \omega_0)t} \frac{\frac{1}{2}\Omega(\omega_0 - \omega - i\gamma)}{(\omega_0 - \omega)^2 + \gamma^2 + \frac{1}{2}\Omega^2},$$

where  $\rho_{21}^* = \rho_{12}$ .

The atomic dipole moment is given by

$$d(t) = -e(\rho_{21}X_{12}e^{-i\omega_0t} + \rho_{12}X_{21}e^{i\omega_0t}).$$

The electric polarization is

$$p(t) = \frac{1}{2}\varepsilon_0 E_0 [\chi(\omega)e^{-i\omega t} + \chi(-\omega)e^{i\omega t}],$$

where

$$\chi(\omega) = \frac{e^2 D_{12}^2}{3\varepsilon_0 \hbar^2} \frac{\omega_0 - \omega + i\gamma}{(\omega_0 - \omega)^2 + \gamma^2 + \frac{1}{2}\Omega^2}.$$

This is no longer a linear susceptibility because  $E_0$  is contained in  $\Omega$ . The real part and the imaginary part of  $\chi(\omega)$  is

$$\begin{aligned} \chi'(\omega) &= \frac{e^2 D_{12}^2}{3\varepsilon_0 \hbar^2} \frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + \gamma^2 + \frac{1}{2}\Omega^2} \\ &= \frac{4\gamma}{\varepsilon_0 E_0^2} \frac{\Omega^2}{4} \frac{(\omega_0 - \omega)/\gamma}{(\omega_0 - \omega)^2 + \gamma^2 + \frac{1}{2}\Omega^2} \end{aligned}$$

or

$$\frac{1}{4}\varepsilon_0 E_0^2 \chi'(\omega) = \frac{\Omega^2}{4} \frac{(\omega_0 - \omega)}{(\omega_0 - \omega)^2 + \gamma^2 + \frac{1}{2}\Omega^2}$$

and

$$\begin{aligned}\chi''(\omega) &= \frac{\pi e^2 D_{12}^2}{3 \varepsilon_0 \hbar^2} \frac{\gamma / \pi}{(\omega_0 - \omega)^2 + \gamma^2 + \frac{1}{2} \Omega^2} \\ &= \frac{4\gamma}{\varepsilon_0 E_0^2} \frac{\Omega^2 / 4}{(\omega_0 - \omega)^2 + \gamma^2 + \frac{1}{2} \Omega^2} = \frac{4\gamma}{\varepsilon_0 E_0^2} \rho_{22}\end{aligned}$$

or

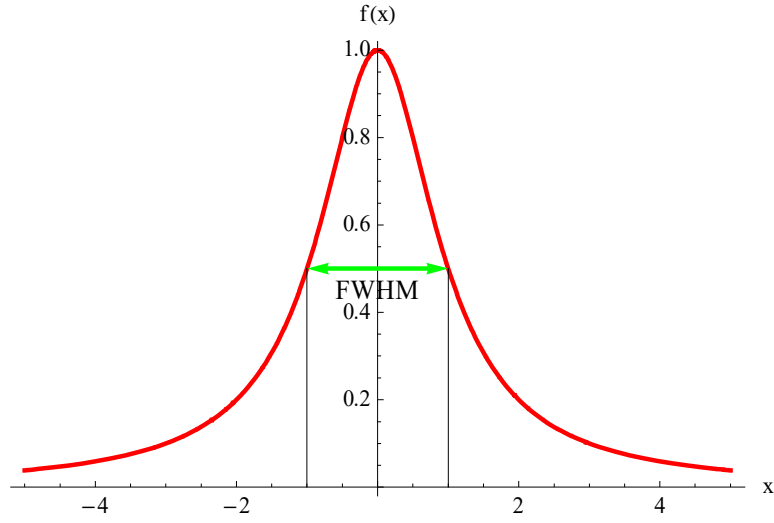
$$\frac{1}{4} \varepsilon_0 E_0^2 \chi''(\omega) = \gamma \rho_{22} = \frac{\Gamma}{2} \rho_{22}.$$

Here we make a plot of  $f(x)$  defined by

$$f(x) = \frac{\chi''(\omega)}{\frac{\pi e^2 D_{12}^2}{3 \varepsilon_0 \hbar^2}} \frac{\gamma^2 + \frac{1}{2} \Omega^2}{\frac{\gamma}{\pi}} = \frac{1}{x^2 + 1}, \quad (\text{Lorentian form})$$

as a function of  $x$  with

$$x = \frac{(\omega_0 - \omega)}{\sqrt{\gamma^2 + \frac{1}{2} \Omega^2}}$$



**Fig.** Lorentzian from  $f(x) [= 1/(1+x^2)]$ .

The line-width of the atomic transition is increased from  $2\gamma$  to

$$2\sqrt{\gamma^2 + \frac{1}{2}\Omega^2}.$$

- (i) In the limit of  $\Omega \gg \gamma$  (intense beam of incident light)

$$\rho_{22} = \frac{1}{2}.$$

- (ii) In the limit of  $\Omega \ll \gamma$  (weak beam of incident light)

$$\rho_{22} = \frac{\frac{1}{4}\Omega^2}{(\omega_0 - \omega)^2 + \gamma^2}$$

$$\int \rho_{22} d\omega = \int d\omega \frac{\frac{1}{4}\Omega^2}{(\omega_0 - \omega)^2 + \gamma^2} = \frac{\pi}{4\gamma} \Omega^2 = \frac{1}{2\gamma} \frac{\pi}{2} \Omega^2$$

and

$$A_{21}=\gamma \quad \Gamma=2\gamma$$

The radiative broadening produces a line-width equal to the spontaneous emission A coefficient for the transition.

$$N_2 = \int N \rho_{22} d\omega = N \frac{\pi}{4\gamma} \Omega^2 = N \frac{1}{2\gamma} \frac{\pi}{2} \Omega^2 = N \int \frac{B\overline{W}}{A} d\omega$$

((Note))

$X_{12}$  and  $D_{12}$  are the dipole matrix elements.

$$X_{12}^2 = \frac{1}{3} D_{12}^2,$$

$$\frac{1}{2} \epsilon_0 E_0^2 = \int \overline{W}(\omega) d\omega,$$

$$B_{12} = \frac{\pi e^2 D_{12}^2}{3 \epsilon_0 \hbar^2}.$$

Then we have

$$\Omega^2 = \frac{e^2 E_0^2}{\hbar^2} X_{12}^2 = \frac{e^2 E_0^2}{3 \hbar^2} D_{12}^2$$

$$\int B_{12} \overline{W}(\omega) d\omega = \frac{\pi e^2 D_{12}^2}{3 \epsilon_0 \hbar^2} \int \overline{W}(\omega) d\omega = \frac{\pi e^2 D_{12}^2}{3 \epsilon_0 \hbar^2} \frac{1}{2} \epsilon_0 E_0^2 = \frac{\pi}{2} \Omega^2$$

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## 6. Radiation pressure

We now consider the expression of the radiation pressure using the above expressions.

$$\rho_{22} = \frac{1}{2} \frac{\frac{1}{2} \Omega^2}{(\omega_0 - \omega)^2 + \gamma^2 + \frac{1}{2} \Omega^2} = \frac{1}{2} \frac{\frac{\Omega^2 / 2}{\delta^2 + \gamma^2}}{1 + \frac{\Omega^2 / 2}{\delta^2 + \gamma^2}} = \frac{1}{2} \frac{s}{1 + s}$$

where  $s$  is called the saturation parameter,

$$s = s(\delta) = \frac{\Omega^2 / 2}{\delta^2 + \gamma^2} = \frac{s_0}{\delta^2 + \gamma^2}.$$

with

$$s_0 = s(\delta = 0) = \frac{\Omega^2}{2\gamma^2} = \frac{I}{I_s},$$

where  $I$  is the intensity,

$$I = \frac{1}{2} \varepsilon_0 c E_0^2, \quad I_s = \frac{\gamma^2 \varepsilon_0 c}{\Omega^2} E_0^2.$$

The detuning parameter  $\delta$  is defined as

$$\delta = \omega - \omega_0.$$

Then the radiation force is derived as

$$F_{rad} = \frac{\Delta p}{\Delta t} = \hbar k (\gamma \rho_{22}) = \hbar k \frac{\Gamma}{2} \rho_{22} = \hbar k \frac{\Gamma}{2} \frac{s}{1 + s}.$$

from the impulse-momentum theorem. Note that

$\hbar k$ : the momentum change after one cycle of absorption and spontaneous emission.

$\frac{\Gamma}{2}$  The inverse of average time for one cycle of absorption and spontaneous emission.



$\rho_{22}$ : the probability for finding atoms in the excited state (absorption probability).

$R$ : the photon scattering rate;  $R = \frac{\Gamma}{2} \frac{s}{1+s}$

The acceleration  $a$ :

$$a = \frac{\mathbf{F}_{rad}}{M} = \frac{\hbar k \Gamma}{2M} \frac{s}{1+s}.$$

In the limit of  $s \rightarrow \infty$ ,

$$a = \frac{F_{rad}}{M} = \frac{\hbar k \Gamma}{2M}.$$

## 7. Electric dipole force

$$F_{dip} = -\frac{\delta}{2} \frac{\nabla s}{1+s} = -\nabla \left[ \frac{\delta}{2} \ln(1+s) \right] = -\nabla U(s).$$

$U(s)$  is the effective potential of dipole. This potential is called either light shift or AC Stark effect.

$$U_{dipole} = -\frac{1}{2} \mathbf{p} \cdot \mathbf{E} = -\frac{1}{4} \text{Re}(\tilde{p} \tilde{E}^*) = -\frac{1}{4} \chi'(\omega) \epsilon_0 E_0^2 = -\frac{\delta}{2} \frac{s}{1+s}$$

since

$$\frac{1}{4} \epsilon_0 E_0^2 \chi'(\omega) = \frac{\Omega^2}{4} \frac{(\omega_0 - \omega)}{(\omega_0 - \omega)^2 + \gamma^2 + \frac{1}{2} \Omega^2} = \frac{\delta}{2} \frac{s}{1+s}$$

where

$$s = s(\delta) = \frac{\Omega^2/2}{\delta^2 + \gamma^2} \quad \text{and} \quad \delta = \omega_0 - \omega$$

The electric dipole force is

$$F_{dipole} = -\nabla U_{dipole}$$

This force is a conserved force. This force will not be discussed further.

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## 8. Laser cooling

The primary force used in laser cooling and trapping is the recoil when the linear momentum is transferred from photons scattering off an atom. The momentum kick that the atom receives from each scattered photon is quite small; a typical velocity change is about 1 cm/s. However, by exciting a strong atomic transition, it is possible to scatter more than  $10^7$  photons per second and produce large accelerations. The radiation-pressure force is controlled in such a way that it brings the atoms in a sample to a velocity near zero ("cooling"), and holds them at a particular point in space ("trapping"). The cooling is achieved by making the photon scattering rate velocity-dependent using the Doppler effect.

The basic principle is illustrated below. If an atom is moving in a laser beam, it will see the laser angular frequency  $\omega_{photon}$  shifted by an amount  $\omega_{photon}(v/c)$  where  $v$  is the velocity of the atom along the opposite direction of the laser beam.

$$\omega_{atom} = \omega_{photon} \sqrt{\frac{1+\beta}{1-\beta}} \approx \omega_{photon} (1+\beta)^{1/2} (1-\beta)^{-1/2} \approx \omega_{photon} (1+\beta).$$

or

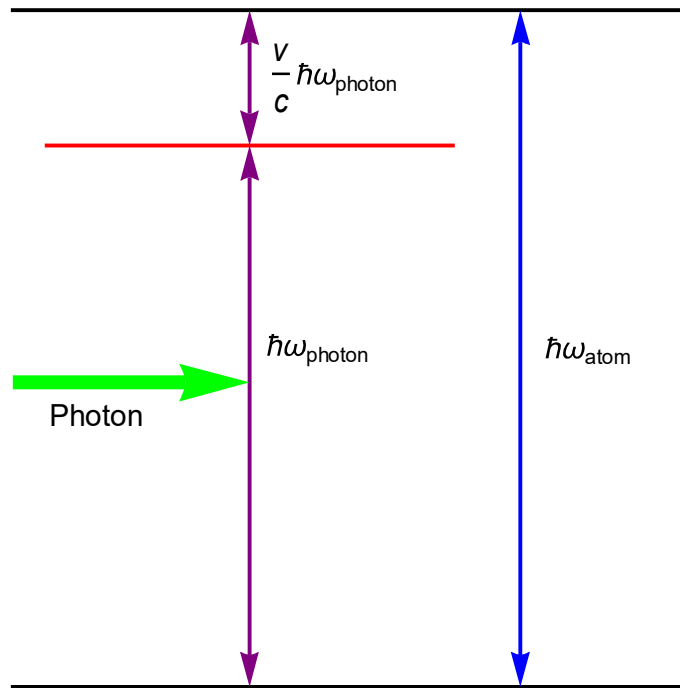
$$\omega_{atom} - \omega_{photon} = \omega_{photon} \beta = \omega_{photon} \frac{v}{c}$$

or

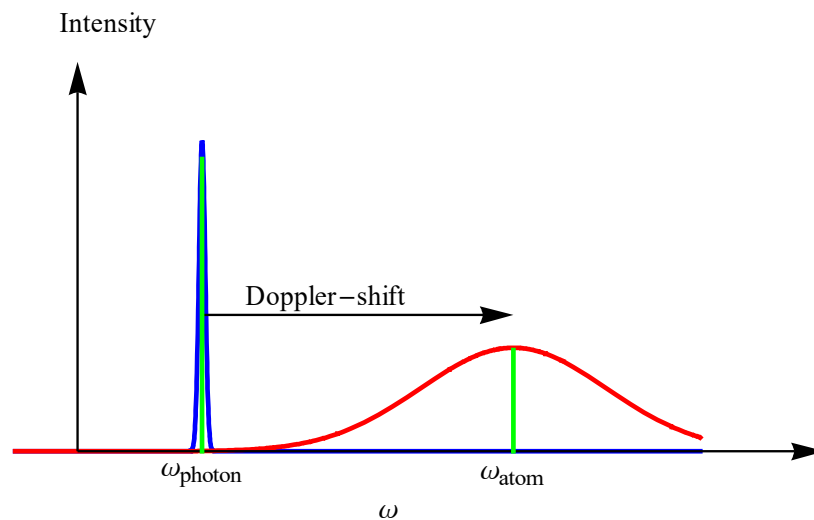
$$v = c \left( \frac{\omega_{atom} - \omega_{photon}}{\omega_{photon}} \right)$$

If the laser frequency is below the atomic resonance frequency, the atom, as a result of this Doppler shift, will scatter photons at a higher rate if it is moving toward the laser beam than if it is moving away. If laser beams impinge on the atom from all six directions, the only

remaining force on the atom is the velocity-dependent part, which opposes the motion of the atoms. This provides strong damping of any atomic motion and cools the atomic vapor. This arrangement of laser fields is often known as "optical molasses". It will scatter photons at a higher rate than those moving in the same direction as the beam. This leads to a larger force on the counter propagating atoms.

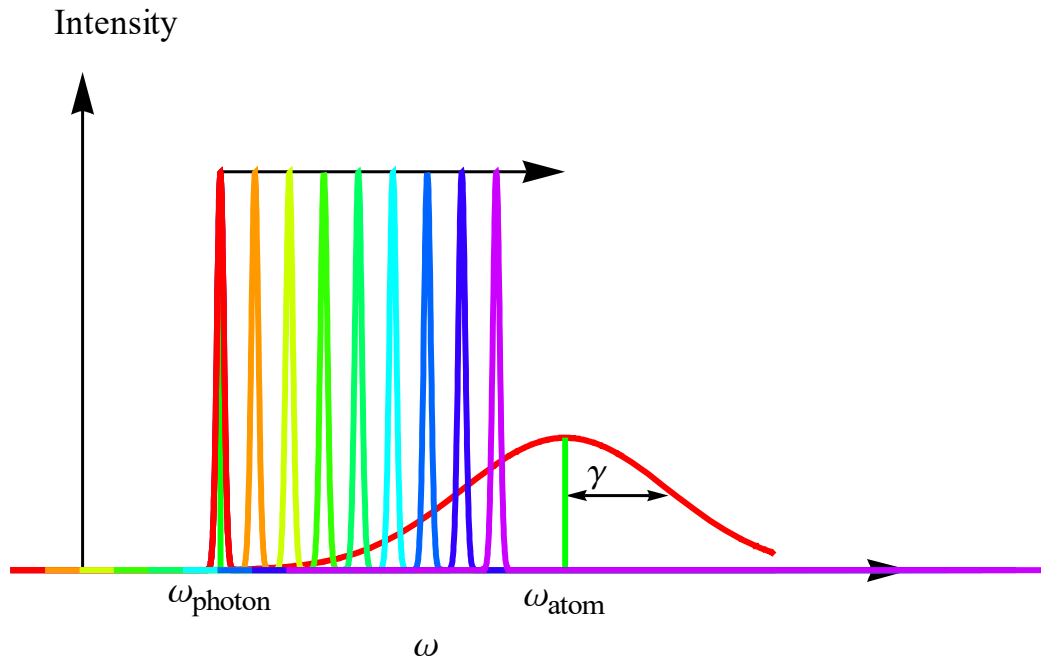


(a)



**Fig.** Doppler effect.  $\omega_{atom} - \omega_{photon} = \omega_{photon} \frac{v}{c}$ . The laser and the atom approach each other.

(b)



**Fig.** Doppler effect.  $\omega_{atom} - \omega_{photon} = \omega_{photon} \frac{v}{c}$ . The laser and the atom approach each other.

As the velocity of the atom decreases, one needs to increase  $\omega_{photon}$ .

**((Example))**

Rb atom

At 300 K,  $v_{rms}=295.89$  m/s

$$\lambda_{photon} = 780 \text{ nm.}$$

$$f_{photon} = 3.84349 \times 10^{14} \text{ Hz}$$

$$\Delta f = f_{atom} - f_{photon} = f_{photon} \frac{v}{c} = 309.0 \text{ MHz.}$$

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We define

$$\delta = \omega_{\text{photon}} - \omega_0 = \omega - \omega_0$$

where

$$\omega_{\text{photon}} = \omega$$

(i) Case-I (photon and atom approaching each other)

$$\omega_{\text{atom}} = \omega_{\text{photon}} \left(1 + \frac{v}{c}\right) = \omega_{\text{photon}} + \omega_{\text{photon}} \frac{v}{c} = \omega + kv, \quad \text{on approaching}$$

or

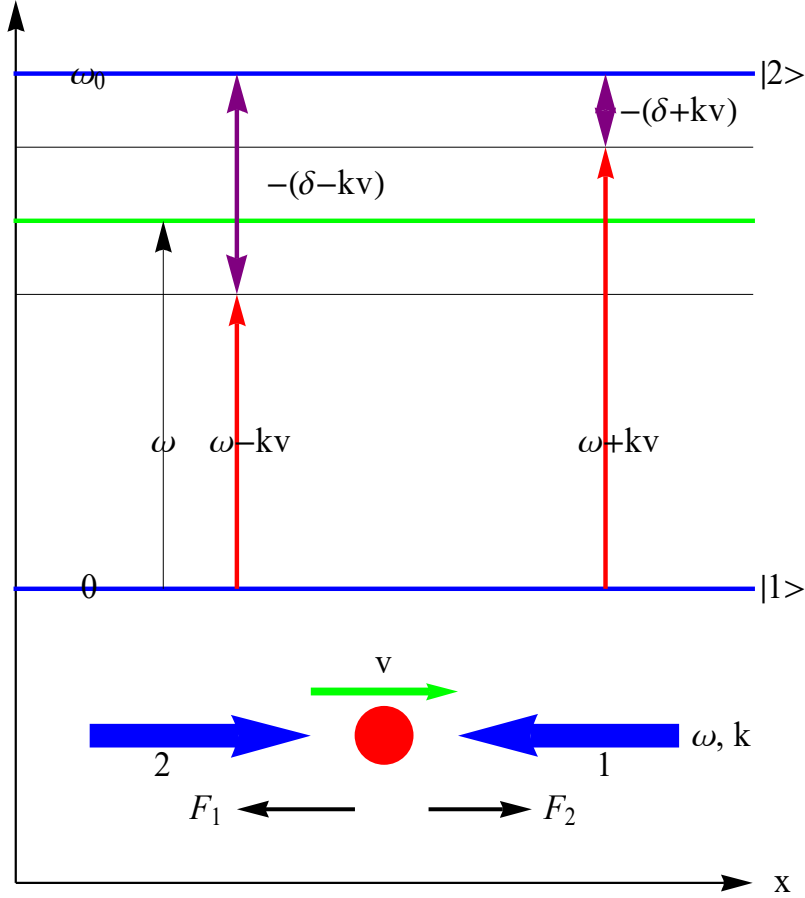
$$\omega_{\text{atom}} - \omega_0 = \omega + kv - \omega_0 = \delta + kv$$

(ii) Case-II (photon and atom leaving away each other)

$$\omega_{\text{atom}} = \omega_{\text{photon}} \left(1 - \frac{v}{c}\right) = \omega_{\text{photon}} - \omega_{\text{photon}} \frac{v}{c} = \omega - kv \quad \text{on leaving}$$

or

$$\omega_{\text{atom}} - \omega_0 = \omega - kv - \omega_0 = \delta - kv$$



**Fig.** The radiation force on the atom (mass  $M$ ) moving to the positive  $x$  direction (with the velocity  $v$ ), which is exerted from the laser (the photons with the angular frequency  $\omega$  and the wave number  $k$ ) propagating in both the negative  $x$  direction (the laser 1) and the positive  $x$  direction (the laser 2).  $\omega - \omega_0 = \delta$  ( $<0$  in this figure).

We now consider the radiation pressure from the photon beam-1 and beam-2 (see the above figure)

$$F_{molasses} = -F_1 + F_2 = \frac{\hbar k \Gamma}{2} \left[ -\frac{s(\delta + kv)}{1 + s(\delta + kv)} + \frac{s(\delta - kv)}{1 + s(\delta - kv)} \right] \approx -\alpha v$$

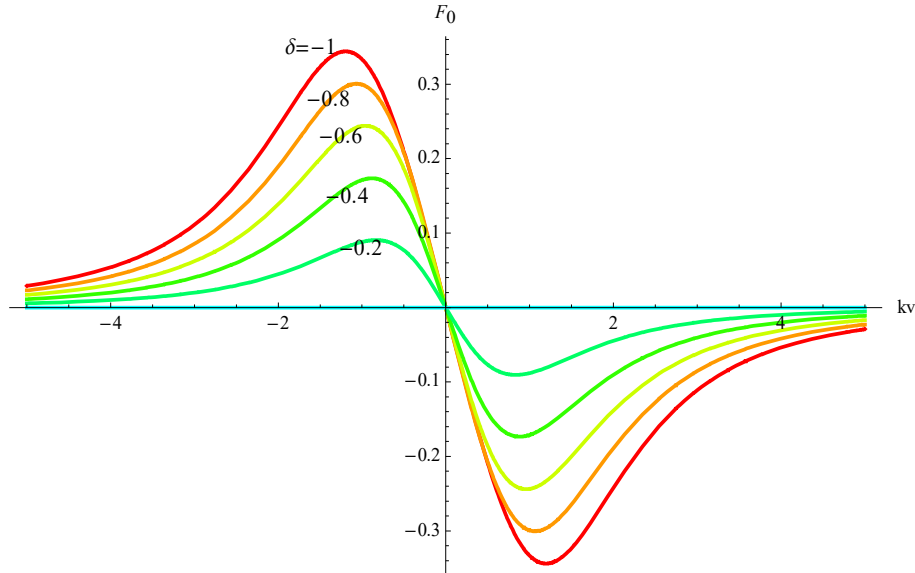
for  $|v| < |\delta/k|$ , where

$$\alpha = \hbar k^2 \frac{4s_0(-\delta/\gamma)}{[1 + s_0 + (\delta/\gamma)^2]}.$$

The light exerts a frictional or damping force on the atom just like that on a particle in viscous fluid (optical molasses). When  $\delta < 0$  ( $\omega < \omega_0$ ),  $\alpha$  becomes positive. Note that the velocity capture range is approximately

$$\Delta v_{cap} = \frac{\Gamma}{k}.$$

where  $F_{\text{molasses}}$  shows a local maximum and local minimum as shown in the plot of  $F_0$  vs  $kv$ .

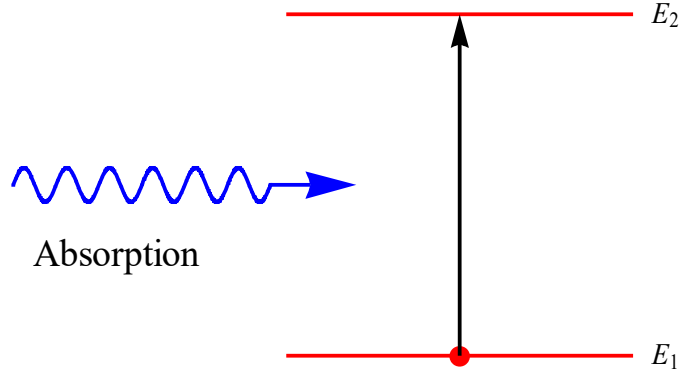


**Fig.** Radiation force  $F_{\text{molasses}}$  (normalized by  $\frac{\hbar k \Gamma}{2}$ ) vs  $kv$ .  $\delta = \omega - \omega_0$  ( $< 0$ ).  $\gamma = 1$ .

$\Omega_0^2/2 = 1$ .  $\delta = -1, -0.8, -0.6, -0.4, -0.2$ , and  $0$ . In the vicinity of the origin, the radiation force is linearly proportional to  $-\alpha v$ . The curve has a local maximum and a local minimum.

## 9. Limit of the Doppler cooling

Suppose that an atom with a momentum  $\mathbf{p}$  absorbs or emits a photon (energy  $\hbar\omega$ , momentum  $\hbar\mathbf{k}$ ). After that, the momentum of the atom changes from  $\mathbf{p}$  to  $\mathbf{p}_f$ .



Momentum conservation:

$$\mathbf{p} \pm \hbar \mathbf{k} = \mathbf{p}_f,$$

The initial kinetic energy  $E_i$  and the final kinetic energy  $E_f$  are given by

$$E_i = \frac{1}{2M} \mathbf{p}^2, \quad E_f = \frac{1}{2M} \mathbf{p}_f^2 = \frac{1}{2M} (\mathbf{p} \pm \hbar \mathbf{k})^2,$$

respectively. Then the change of the energy is

$$\Delta E = E_f - E_i = \frac{1}{2M} [(\mathbf{p} \pm \hbar \mathbf{k})^2 - \mathbf{p}^2] = \pm \frac{\hbar}{M} \mathbf{p} \cdot \mathbf{k} + \frac{1}{2M} (\hbar \mathbf{k})^2.$$

Here the recoil energy is defined as the second term of the above equation,

$$\hbar \omega_R = \frac{1}{2M} (\hbar \mathbf{k})^2.$$

The recoil velocity is defined by

$$v_R = \frac{\hbar k}{M}.$$

In the present case, there are two photon beam approaching the atom. When the emission occurs, subsequently, the spontaneous emission always occurs. So the increase of the kinetic energy of atom is



$$R_{heat} = 2\hbar\omega_R \frac{\Gamma}{2} \frac{s}{1+s}.$$

The energy lost due to the radiation pressure is

$$R_{cool} = Fv = \beta v^2.$$

In the steady state, the change in the total energy is zero,

$$R_{heat} = R_{cool}.$$

or

$$2\hbar\omega_R \frac{\Gamma}{2} \frac{s}{1+s} = \alpha v^2,$$

where

$$s(\delta) = \frac{\Omega^2 / 2}{\gamma^2 + \delta^2},$$

and

$$\alpha = \hbar k^2 \frac{4s_0(-\delta/\gamma)}{[1+s_0+(\delta/\gamma)^2]^2}.$$

Then we get

$$v^2 = \frac{2\hbar\omega_R}{\alpha} \frac{\Gamma}{2} \frac{s}{1+s} = \frac{2 \frac{\hbar^2 k^2}{2M}}{\hbar k^2 \frac{4s_0(-\delta/\gamma)}{[1+s_0+(\delta/\gamma)^2]^2}} \frac{\Gamma}{2} \frac{s}{1+s},$$

or

$$\begin{aligned}
\frac{1}{2} M v_{st}^2 &= \frac{\hbar \gamma}{2 s_0 (-\delta / \gamma)} \frac{s}{1+s} \\
&\quad \frac{[1+s_0+(\delta/\gamma)^2]^2}{[1+s_0+(\delta/\gamma)^2]^2} \\
&= \frac{\hbar \gamma [1+s_0+(\delta/\gamma)^2]^2}{2 s_0 (-\delta / \gamma)} \frac{s_0}{1+s_0+(\delta/\gamma)^2} \\
&= \frac{\hbar \gamma [1+s_0+(\delta/\gamma)^2]}{2 |\delta / \gamma|}
\end{aligned}$$

or

$$\frac{M v_{st}^2}{2} = \frac{\hbar \gamma}{2} \left[ \frac{\gamma(1+s_0)}{|\delta|} + \frac{|\delta|}{\gamma} \right] \geq \hbar \gamma \sqrt{1+s_0},$$

where

$$\Gamma = 2\gamma.$$

The minimum of  $\frac{M v_{st}^2}{2}$  occurs when

$$\gamma = \frac{|\delta|}{\sqrt{1+s_0}}.$$

For  $s_0 \ll 1$ , we have

$$\frac{M v_{st}^2}{2} = \hbar \gamma \sqrt{1+s_0} \geq \hbar \gamma = \frac{\hbar \Gamma}{2}.$$

Using the equi-partition law of the energy for the one dimensional direction,

$$\frac{M v_{st}^2}{2} = \frac{k_B T}{2}.$$

The above equation indicates that the temperature of atom is always higher than a characteristic temperature  $T_D$ ,

$$\frac{k_B T_D}{2} = \frac{\hbar \Gamma}{2},$$

$T_D$  is called the Doppler-limited temperature,

$$T_D = \frac{\hbar \Gamma}{k_B} = \frac{2\hbar \gamma}{k_B}.$$

This temperature is dependent only on  $\gamma$ . This gives the lowest temperature expected in the optical molasses technique. The lowest velocity is

$$v_{st} = \sqrt{\frac{k_B T_D}{M}} = \sqrt{\frac{\hbar \Gamma}{M}}.$$

**((Summary))**

- (i) The Doppler cooling works at low laser intensity (the saturation parameter  $s$  is much smaller than 1).
- (ii) The minimum temperature is equal to

$$\frac{k_B T_D}{2} = \frac{\hbar \Gamma}{2}$$

where  $\Gamma$  is the natural width of the excited state.

- (iii) The minimum is reached when

$$\delta = \omega_0 - \omega = -\gamma = -\frac{\Gamma}{2}$$

((Note))

We sometimes use the word, *red-detuning* for  $\delta < 0$  ( $\omega < \omega_0$ ). The wavelength of photon is

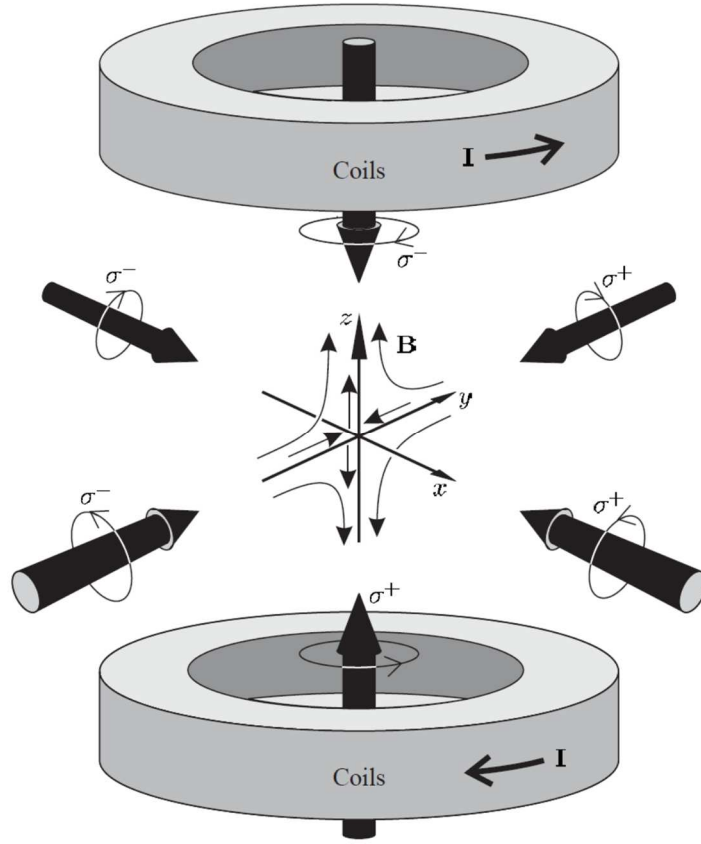
$$\lambda = \frac{2\pi c}{\omega} > \frac{2\pi c}{\omega_0}.$$

and is longer than the characteristic wavelength. The wavelength of the red light is much longer than that of the blue light.

## 10. Magneto-optical trap (MOT)

Although optical molasses will cool atoms, the atoms will still diffuse out of the region if there is no position dependence to the optical force. Position dependence can be introduced in "magneto-optical trap" (MOT)." The position-dependent force is created by using appropriately polarized laser beams and by applying an inhomogeneous magnetic field (quadrupole magnetic field) to the trapping region. Through Zeeman shifts of the atomic energy levels, the magnetic field regulates the rate at which an atom in a particular position scatters photons from the various beams and thereby causes the atoms to be pushed to a particular point in space. In addition to holding the atoms in place, this greatly increases the atomic density since many atoms are pushed to the same position.

Here we consider an atom with a  $F = 0$  ground state and a  $F = 1$  excited state, illuminated by circularly polarized beams of light coming from the left and the right. Because of its polarization, the beam from the left ( $\sigma^+$ ) can only excite transitions to the  $m = +1$  state, while the beam from the right ( $\sigma^+$ ) can only excite transitions to the  $m = -1$  state. The magnetic field is zero in the center, increases linearly in the positive  $x$  direction, and decreases linearly in the negative  $x$  direction. This field perturbs the energy levels so that the  $\Delta m = +1$  transition shifts to lower frequency if the atom moves to the left of the origin,

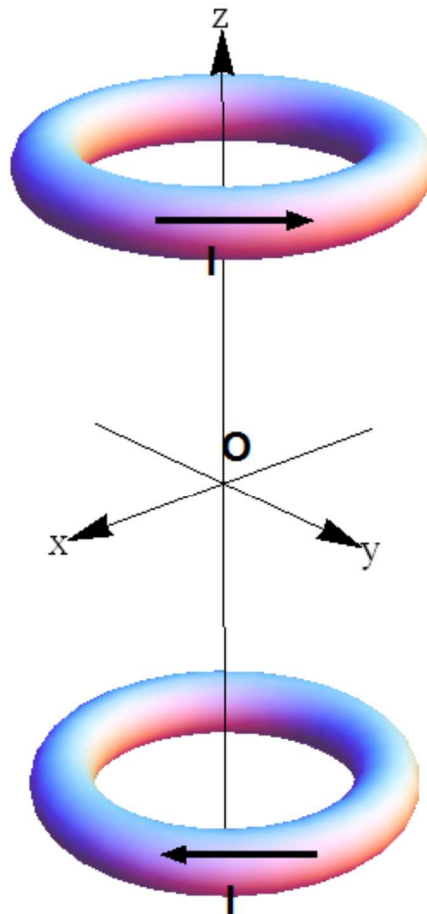


**Fig.** A MTO is formed from three orthogonal pairs of laser beams that have the requisite circular polarization states and intersect at the center of a pair coils with opposite currents. At the point in the middle of the coils the magnetic fields produced by the coils cancel out, so that  $B = 0$ . Close to this zero of the field there is a uniform field gradient that perturbs the atomic energy levels. The Zeeman effect causes the energy of the  $F = 1$  level to vary linearly with the atom's position. [C.J. Foot, Atomic Physics (Oxford University Press, 2005), p.192]

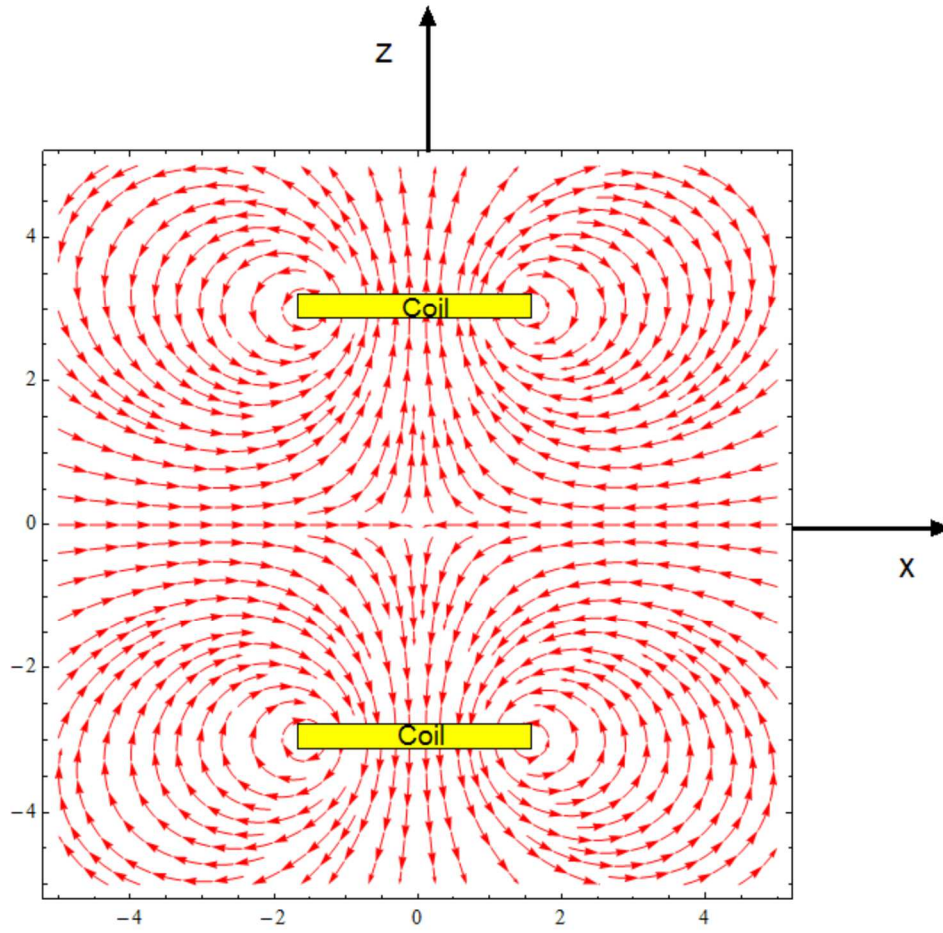
#### 11. Quadrupole magnetic field ("anti-Helmholtz" like configuration)

A quadrupole magnetic field is produced by the two coils with currents in opposite directions. This magnetic field causes an imbalance in the scattering forces of the laser beams. The magnetic field distribution in the  $(x, z)$  plane is presented below. The magnetic field  $B$  is equal to zero at the origin O. The magnetic field along the positive  $z$  axis near  $z = 0$  is directed along the positive  $z$  direction. It linearly increases with the distance  $z$ . The magnetic

field along the negative  $z$  axis near  $z = 0$  is directed along the negative  $z$  direction. It linearly varies with the distance  $|z|$ .



**Fig.** A pair of coils with currents in opposite directions produces a quadrupole magnetic field. The magnetic field is zero at the center of the coils and its magnitude increases linearly in every direction for small displacements from the origin  $O$ .



**Fig.** StreamPlot of the magnetic field distribution for the quadrupole magnet. Anti-Helmholtz configuration. Note that the length of the arrow is the same and is not proportional to the magnitude of the magnetic field.

## 12. Principle of MOT

The magnetic moment is given by

$$\boldsymbol{\mu} = -g_F \mu_B \mathbf{F}$$

The Zeeman energy is given by

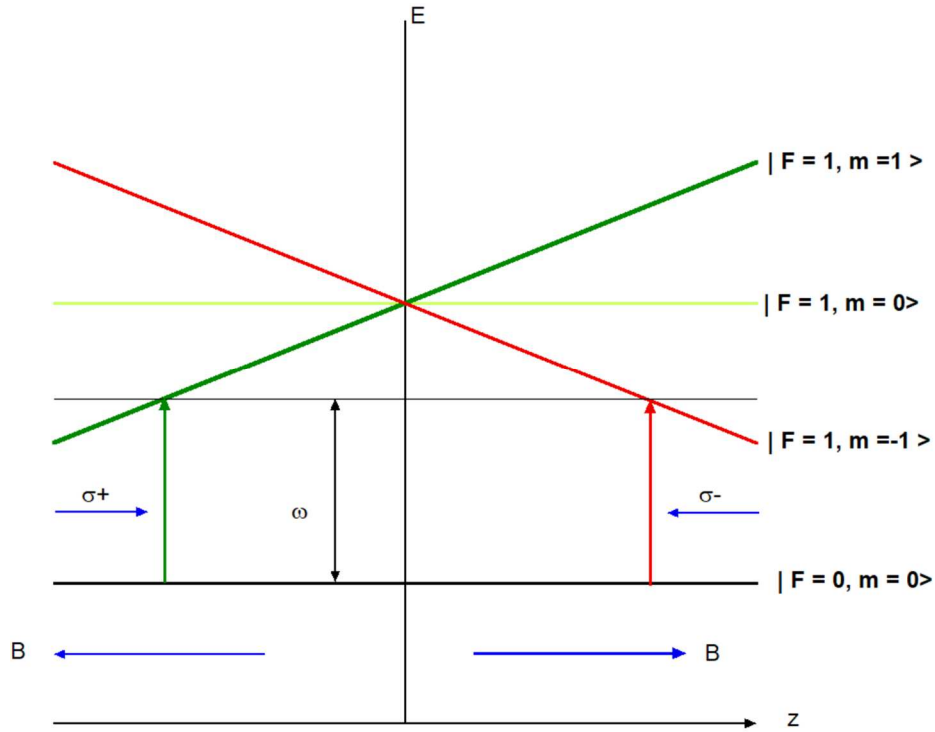
$$U_B = -\boldsymbol{\mu} \cdot \mathbf{B} = -(-g_J \mu_B \mathbf{J}) \cdot \mathbf{B} = g_J \mu_B m B_z$$

where  $m = F, F-1, \dots, -F$ . For simplicity we use the quantum number  $m$  instead of  $m_F$

We consider the two level system with  $F = 1$  (the excited state) and  $F = 0$  (the ground state). When the magnetic field is applied to this system, there occur the Zeeman splitting in the excited state with

$$F=1 \text{ (triplet)} \rightarrow |F=1, m\rangle \quad (m = 1, 0, -1).$$

Here we assume that the Landé  $g$ -factor is positive. We consider the energy diagram of the atoms near  $z = 0$  along the  $z$  axis. We note that the Zeeman splitting for  $z > 0$  is different from that for  $z < 0$ . For simplicity we discuss the case for  $z > 0$ .



**Fig.** Zeeman splitting of the  $F = 1$  level in the presence of a quadrupole magnetic field.

The Zeeman shift at the displacement  $z$  is denoted as

$$U_B(m = -1) = -g_F \mu_B \frac{dB_z}{dz} z = -\hbar \beta z,$$

$$U_B(m = 1) = g_F \mu_B \frac{dB_z}{dz} z = \hbar \beta z,$$



for  $z>0$ , where

$$\beta = \frac{g_F \mu_B}{\hbar} \frac{dB_z}{dz}.$$

(i)  $\sigma^+$  radiation ( $\Delta m = 1$ )

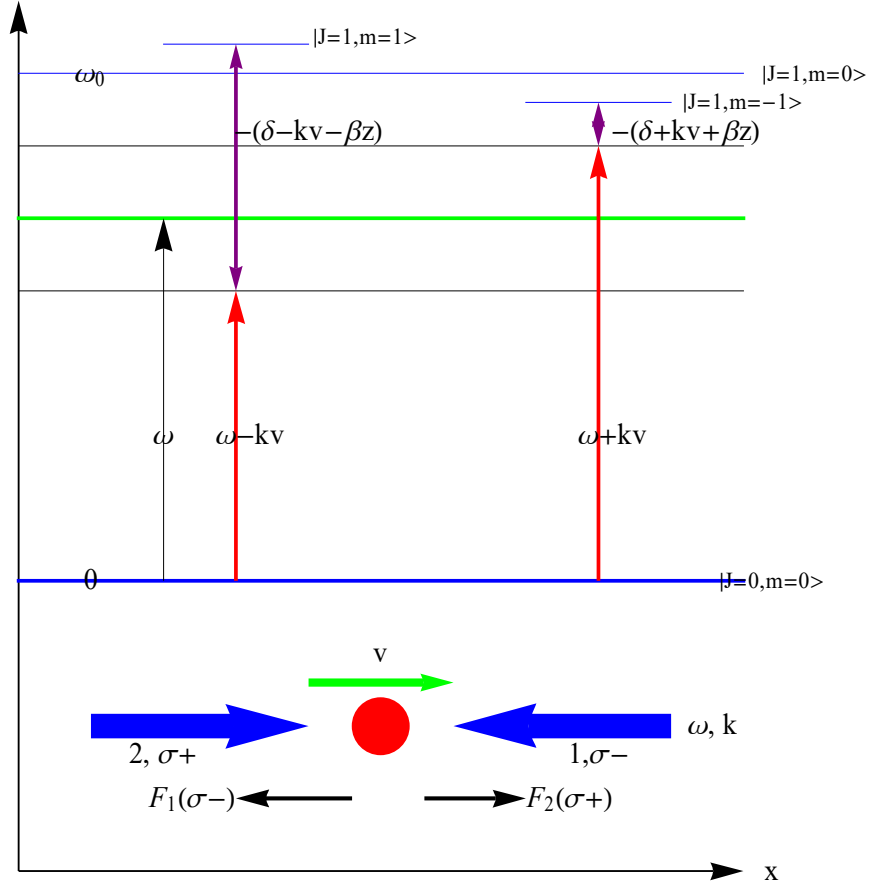
$$\omega_{atom} - \beta z = \omega + k\nu$$

$$\omega_{atom} - \omega_0 = \omega + k\nu + \beta z - \omega_0 = \delta + (k\nu + \beta z)$$

(ii)  $\sigma^-$  radiation ( $\Delta m = -1$ )

$$\omega_{atom} + \beta z = \omega - k\nu$$

$$\omega_{atom} - \omega_0 = \omega - k\nu - \beta z - \omega_0 = \delta - (k\nu + \beta z)$$



**Fig.** Schematic diagram for the radiation pressure on the atom (moving in the positive  $x$  direction with the velocity  $v$ ) from the photon beam-1 ( $\sigma^-$  photon,  $\omega$ ,  $k$ ,  $\Delta m = -1$  selection) and the photon beam-2 ( $\sigma^+$  photon,  $\omega$ ,  $k$ ,  $\Delta m = +1$  selection).  $z > 0$ .

We now consider the radiation pressure from the photon beam-1 ( $\sigma^-$  photon,  $\Delta m = -1$  selection) and the photon beam-2 ( $\sigma^+$  photon,  $\Delta m = +1$  selection)

$$\begin{aligned}
 F_{\text{molasses}} &= -F_1 + F_2 \\
 &= \frac{\hbar k \Gamma}{2} \left[ -\frac{s(\delta + kv + \beta z)}{1 + s(\delta + kv - \beta z)} + \frac{s(\delta - kv - \beta z)}{1 + s(\delta - kv - \beta z)} \right] \\
 &\approx -\alpha \left( v + \frac{\beta z}{k} \right)
 \end{aligned}$$

This means that the motion of the molasses is well described by an over-damped harmonics. From the Newton's second law, we have

$$M \frac{d^2 z}{dt^2} + \alpha \frac{dz}{dt} + \frac{\alpha \beta}{k} z = 0,$$

where  $\alpha > 0$  and  $\beta > 0$ . The condition for the overdamped harmonic oscillation is that

$$\beta > \frac{k\alpha}{4M}.$$

with

$$\alpha = \hbar k^2 \frac{4s_0(-\delta/\gamma)}{[1 + s_0 + (\delta/\gamma)^2]^2}.$$

When atoms enter the trap region, the velocity of atoms is decreased by the radiation pressure of the laser beam. The position dependent force pushes the cold atoms to the trap center. Because of the combination of slowing down and trapping, the MOT is very widely used in the laser cooling experiment.

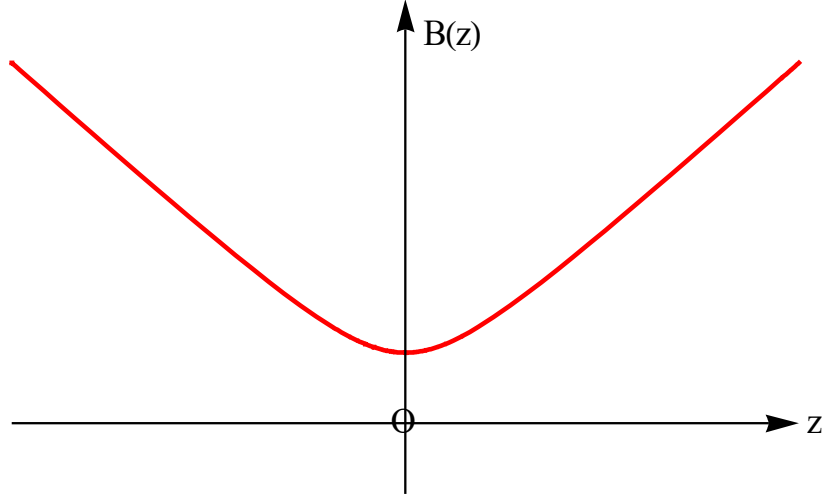
### 13. Magnetic trap and evaporative cooling

The temperature reached by laser cooling are impressively low, but they are not low enough to give rise to the BEC in gases at the densities that are realized experimentally. The BEC of alkali gases can be achieved by using evaporative cooling. When atoms escaping from a system have an energy higher than the average energy of atoms, the remaining atoms are cooled.

Suppose that we have a magnetic field  $\mathbf{B}$  along the  $z$  axis;

$$B(z) = \beta \sqrt{z^2 + z_0^2}.$$

which is symmetric with respect to  $z = 0$ , where  $\beta > 0$  and  $z_0 > 0$ .



**Fig.** Magnetic field as a function of  $z$ .  $z = 0$  is the center of the trap.

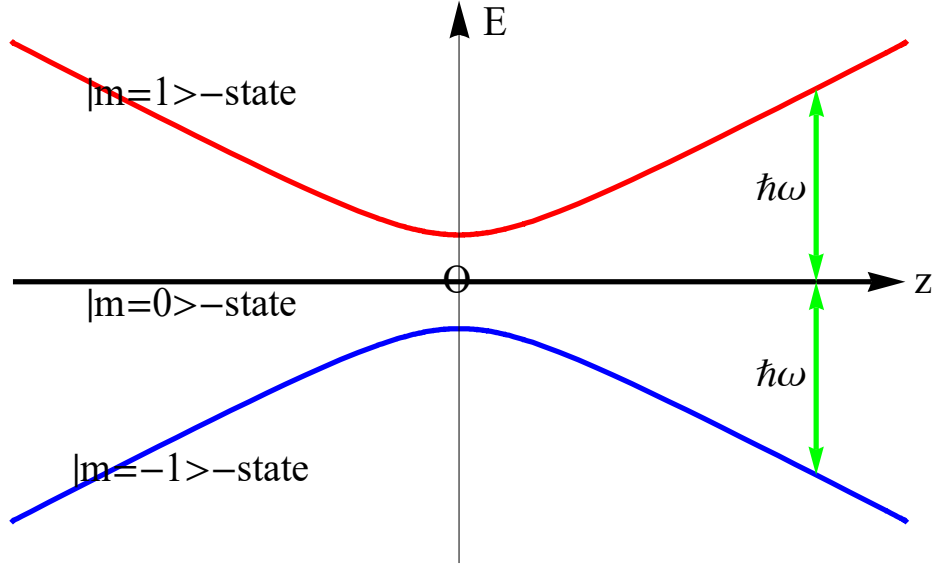
In the presence of such a magnetic field, the Zeeman energy is given by

$$U = -(-g_F \mu_B \mathbf{F}) \cdot \mathbf{B} = g_F \mu_B m B(z) = g_F \mu_B m \beta z_0 \sqrt{\frac{z^2}{z_0^2} + 1}.$$

Here we assume that  $F = 1$ , where  $m = 1, 0$ , and  $-1$ . The upper potential energy is given by

$$U = g_F \mu_B \beta z_0 \sqrt{\frac{z^2}{z_0^2} + 1},$$

for  $m = 1$ . The state of the upper energy level is denoted by  $|F = 1, m = 1\rangle$ . The plot of the potential energy (scaled by  $g_F \mu_B \beta z_0$ ) is shown below. There are three energy levels with  $m = 1, 0$ , and  $-1$ .

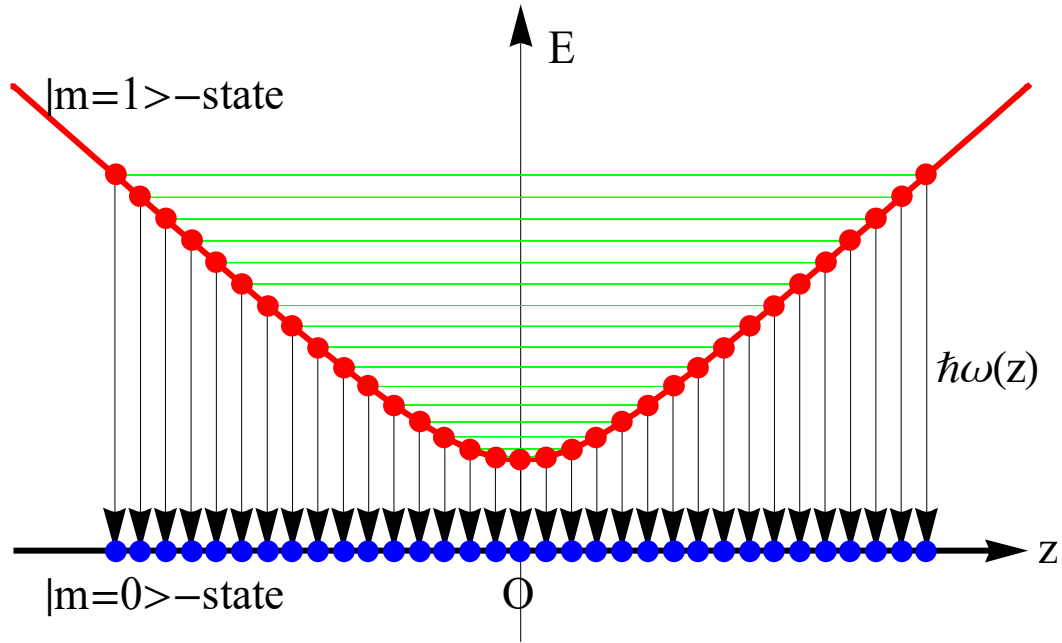


**Fig.** The energy diagram for the Zeeman splitting levels in the case of  $F = 1$ .  $m = 1, 0, -1$ .

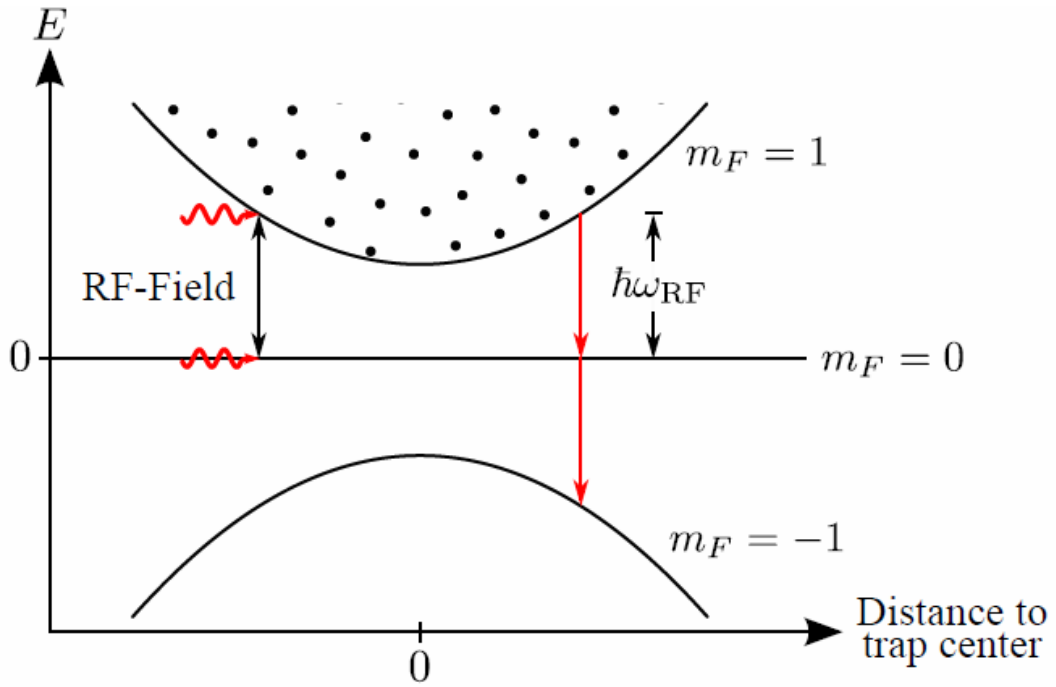
The atoms with higher energy ( $F = 1$ ,  $m = 1$ ) are far from the center of the trap, while the atoms with lower energy are round the center of the trap. When the rf electromagnetic wave ( $\hbar\omega$ ) is applied to the system, for example, there occurs the transition from  $m = 1$  to  $m = 0$ . where

$$E(m = 1, z) - E(m = 0, z) = \hbar\omega(z) .$$

where  $z$  is the position of atoms from the trap center. As  $z$  decreases,  $\omega(z)$  also need to decrease. Then the total energy of atoms in the trap is reduced, leading to the lowering temperature of atoms. This process is repeated for getting the lowest temperatures.



**Fig.** Transition between  $m = 1$  state to  $m = 0$  state for  $J = 1$  system.  
 $\hbar\omega(z) = U(z, m=1) - U(z, m=0)$ .



rf radiation flips atomic spin. Attractive trapping force turns into repulsive force and expels atoms from trap.

#### 14. **Boson or fermion?**

The BEC was observed in Rb atoms. Rb has a electron configuration of

$$(1s)^2(2s)^2(2p)^6(3s)^2(3p)^6(3d)^{10}(4s)^2(4p)^6(5s)^1.$$

There is one electron in the outer shell (5s). So the system obeys a Fermi-Dirac statistics. In other words, it seems that Rb atom behaves like a fermion. However, in fact Rb atoms behave like a boson. What is the reason why Rb atom is regarded as a boson, but not a fermion? In this system, as a result of the combination of the spin-orbit coupling and hyperfine interaction,  $F$  is the good quantum number and  $F$  is integers;  $F = 1, 2, 3$ . This means that Rb atom behaves like a boson.

$^{87}\text{Rb}$

$(Z = 37, N = 50)$

Proton number = 37

Atomic mass = 85.4678

Nuclear spin  $I = 3/2$

For  $n=5$ , we have  $l_{\max} = n - 1 = 5 - 1 = 4$

$l = 4 (g), 3(f), 2(d), 1(p)$  and  $0(s)$ .

#### **(a) Spin-orbit coupling**

$n = 5$

$(5p)^1 \quad l = 1 \text{ and } s = 1/2$

$$D_1 \times D_{1/2} = D_{3/2} + D_{1/2}$$

leading to  $j = 3/2$  and  $j = 1/2$  (half integers)

(a)  $j = 3/2$

$$D_{3/2} (j = 3/2, m = 3/2, 1/2, -1/2, -3/2)$$

or

$$^{2s+1=2}P(l=1)_{j=3/2} = ^2P_{3/2} \rightarrow 5^2P_{3/2}$$

$$(b) \quad j = 1/2$$

$$D_{1/2} (j = 1/2, m = 1/2, -1/2)$$

or

$$^{2s+1=2}P(l=1)_{j=1/2} = ^2P_{1/2} \rightarrow 5^2P_{1/2}$$

$$(5s)^1 \quad l = 0 \text{ and } s = 1/2:$$

$$D_0 \times D_{1/2} = D_{1/2},$$

leading to the state with  $j = 1/2$ .

$$D_{1/2} (j = 1/2, m = 1/2, -1/2)$$

$$^{2s+1=2}S(l=0)_{j=1/2} = ^2S_{1/2}$$

### **(b) Hyperfine interaction**

Nuclear spin  $I = 3/2$  for  $^{87}\text{Rb}$

$$(1) \quad j = 3/2 (5^2P_{3/2}) \text{ and } I=3/2$$

$$D_{3/2} \times D_{3/2} = D_3 + D_2 + D_1 + D_0$$

leading to the magnetic sub-states with  $F$  integers

$$F = 3 (m_f = -3, -2, -1, 0, 1, 2, 3), \quad F = 2 (m_f = -2, -1, 0, 1, 2)$$

$$F = 1 (m_f = -1, 0, 1), \quad F = 0 (m_f = 0).$$



(2)  $j = 1/2$ . ( $5^2P_{1/2}$ ) and  $I=3/2$

$$D_{3/2} \times D_{1/2} = D_2 + D_1$$

leading to the magnetic sub-states

$$F = 2 \ (m_f = 2, 1, 0, -1, -2),$$

$$F = 1 \ (m_f = 1, 0, -1).$$

(3)  $j = 1/2$  ( $5^2S_{1/2}$ ) and  $I=3/2$ .

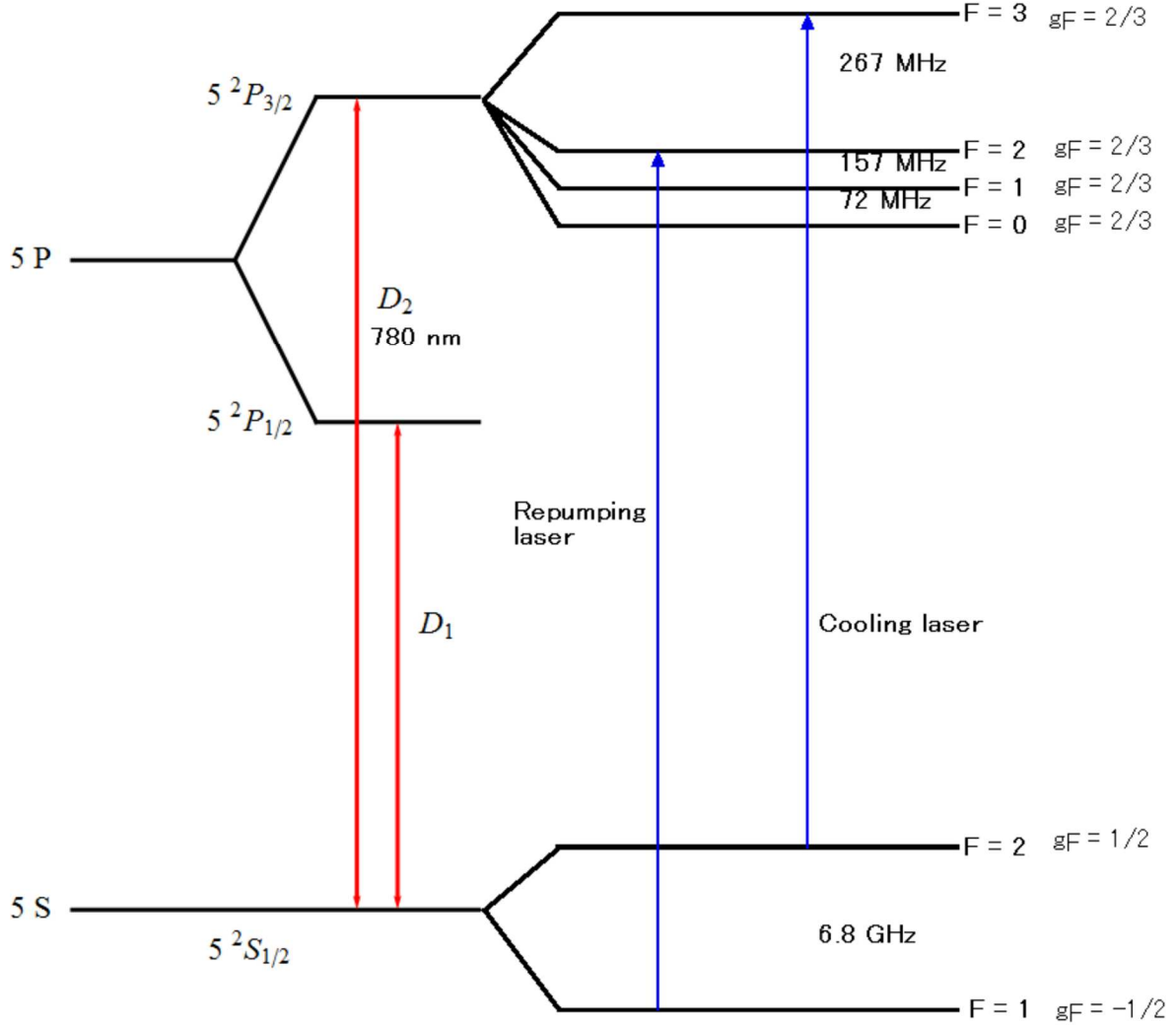
$$D_{3/2} \times D_{1/2} = D_2 + D_1$$

leading to the magnetic substates

$$F = 2 \ (m_f = 2, 1, 0, -1, -2),$$

$$F = 1 \ (m_f = 1, 0, -1).$$

The energy level diagrams of  $^{87}\text{Rb}$  is schematically shown below.



**Fig.** Level diagram of  $^{87}\text{Rb}$  with the nuclear spin  $I = 3/2$ . The splittings are not to scale.  $D_1$  line: ( $5^2S_{1/2} \rightarrow 5^2P_{1/2}$ ).  $\lambda = 794.978851156(23)$  nm.  $D_2$  line: ( $5^2S_{1/2} \rightarrow 5^2P_{3/2}$ ).  $\lambda = 780.241209686(13)$  nm.  $D_1$ : 377.107463380 THz (794.978851156 nm). Zeeman splitting are also shown.  $g_F = 1/2$  for  $5^2S_{1/2}$   $F = 2$ .  $g_F = 2/3$  for  $5^2P_{3/2}$   $F = 3$ .

In the MTO for  $^{87}\text{Rb}$  atom, all the trapping and cooling is done by one laser which is tuned slightly (1-3 natural linewidths) to the low frequency side of the  $5^2S_{1/2}$   $F = 2 \rightarrow 5^2P_{3/2}$   $F = 3$  transition of  $^{87}\text{Rb}$ . Unfortunately, about one excitation out of 1000 will cause the atom to decay to the  $F=1$  state instead of the  $F=2$  state. This takes the atom out of resonance with the trapping laser. Another laser (called the "hyperfine pumping laser") is used to excite the atom

from the  $5^2S_{1/2} F = 1$  to the  $5^2P_{3/2} F = 1$  or  $2$  state, from which it can decay back to the  $5^2S_{1/2} F = 2$  state where it will again be excited by the trapping laser.

The Landé  $g$ -factors are given as

$$g_F = \frac{1}{2}$$

for  $5^2S_{1/2} F = 2$  ( $l = 0, s = 1/2, j = 1/2, F = 2, I = 3/2$ )

$$g_F = \frac{2}{3}$$

for  $5^2P_{3/2} F = 3$  ( $l = 1, s = 1/2, j = 3/2, F = 3, I = 3/2$ ).

We use the formula

$$g_F = g_J \left[ \frac{j(j+1) - I(I+1) + F(F+1)}{2F(F+1)} \right].$$

with

$$g_J = \frac{3}{2} + \frac{s(s+1) - l(l+1)}{2j(j+1)}$$

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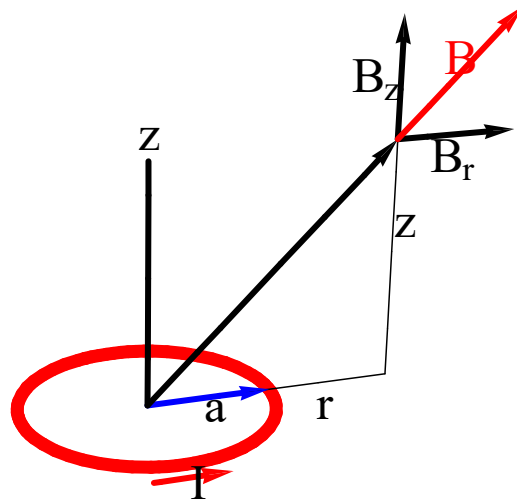
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## APPENDIX

### A Magnetic field lines around a current loop

We consider the magnetic field lines around a loop current. The magnetic field  $\mathbf{B}$  at the point  $(x, y, z)$  can be calculated using the vector potential  $\mathbf{A}$ . The magnetic field  $\mathbf{B}$  is calculated as  $\mathbf{B} = \nabla \times \mathbf{A}$ . The results for the expression of  $\mathbf{B}$  is presented below.



**Fig.** Magnetic field  $\mathbf{B}$  due to a current loop (radius  $a$ ). the current  $I$  flows along the ring. The magnetic field  $\mathbf{B}$  has a  $B_r$  component and a  $B_z$  component. Note that  $B_z$  and  $B_r$  can be expressed in terms of the elliptic integrals.

- $B_z$ : magnetic field component which is aligned along the  $z$  axis (the coil axis).
- $B_r$ : magnetic field component in a radial direction.
- $I$ : current in the wire
- $a$ : radius of the current loop
- $z$ : distance, on axis, from the center of the current loop to the field measurement point.
- $r$ : radial distance from the axis of the current loop to the field measurement point.

$$B_z(r, z) = \frac{B_0}{\pi\sqrt{Q}} \left[ \left( \frac{1 - \alpha^2 - \beta^2}{Q - 4\alpha} \right) \text{EllipticE}[k] + \text{EllipticK}[k] \right]$$

$$B_r(r, z) = \frac{B_0\gamma}{\pi\sqrt{Q}} \left[ \left( \frac{1 + \alpha^2 + \beta^2}{Q - 4\alpha} \right) \text{EllipticE}[k] - \text{EllipticK}[k] \right]$$

where

$$\alpha = \frac{r}{a}, \quad \beta = \frac{z}{a}, \quad \gamma = \frac{z}{r},$$

$$Q = (1 + \alpha)^2 + \beta^2,$$

$$k = \frac{4\alpha}{Q}.$$

Note that the definition of the Elliptic integral in the *Mathematica* is a little different from the standard definition.

$$\text{EllipticE}[k] = \int_0^{\pi/2} (1 - k \sin^2 \theta)^{1/2} d\theta$$

$$\text{EllipticK}[k] = \int_0^{\pi/2} (1 - k \sin^2 \theta)^{-1/2} d\theta$$

EllipticE[k]: the complete elliptic integral of the first kind.

EllipticK[k]: the complete elliptic integral of the second kind

where

$$B_0 = \frac{\mu_0 I}{2a}$$