## Chemical potential of 1D, 2D. and 3D Fermion systems Masatsugu Sei Suzuki Department of Physics, SUNY at Binghamton (Date: November 09, 2018)

This part is already discussed in the section of the Sommerfeld formula. For convenience, I put this part again in the new section for convenience.

#### 1. Sommerfeld formula: 1D case

We consider the case of one-dimensional system.

$$N = \int_{0}^{\infty} d\varepsilon f(\varepsilon) D_{1}(\varepsilon) = \int_{0}^{\varepsilon_{F}} d\varepsilon D_{1}(\varepsilon) d\varepsilon$$

where the density of the state for the 1D system is

$$D_1(\varepsilon) = \frac{L}{\pi} \left(\frac{2m}{\hbar^2}\right)^{1/2} \varepsilon^{-1/2}$$

since

$$D_{1}(\varepsilon)d\varepsilon = 2\left(\frac{L}{2\pi}\right)2dk = \frac{L}{\pi}\left(\frac{2m}{\hbar^{2}}\right)^{1/2}\frac{d\varepsilon}{\sqrt{\varepsilon}}$$

The factor 2 of 2dk comes from the even function of the 1D energy dispersion relation. We use the Sommerfeld formula

$$N = \int_{0}^{\infty} d\varepsilon f(\varepsilon)\varphi(\varepsilon) = \int_{0}^{\mu} d\varepsilon\varphi(\varepsilon)d\varepsilon + \frac{\pi^{2}}{6}k_{B}^{2}T^{2}\varphi'(\mu)$$

where

$$\varphi(\varepsilon) = D_1(\varepsilon) = \frac{L}{\pi} \left(\frac{2m}{\hbar^2}\right)^{1/2} \varepsilon^{-1/2} = a_1 \varepsilon^{-1/2}$$
$$\frac{d\varphi(\varepsilon)}{d\varepsilon} = \frac{L}{\pi} \left(\frac{2m}{\hbar^2}\right)^{1/2} (-\frac{1}{2})\varepsilon^{-3/2} = -\frac{1}{2}a_1 \varepsilon^{-3/2}$$

Thus we have

$$N = a_1 \int_0^{\mu} d\varepsilon \varepsilon^{-1/2} d\varepsilon + \frac{\pi^2}{6} k_B^2 T^2 \left( -\frac{1}{2} a_1 \mu^{-3/2} \right)$$

$$= 2a_1 \mu^{1/2} - \frac{\pi^2}{12} k_B^2 T^2 a_1 \mu^{-3/2}$$
(1)

But we also have

$$N = \int_{0}^{\varepsilon_{F}} d\varepsilon D_{1}(\varepsilon) = a_{1} \int_{0}^{\varepsilon_{F}} \varepsilon^{-1/2} d\varepsilon = 2a_{1} \varepsilon_{F}^{1/2}$$
(2)

Combining Eqs.(1) and (2), we get

$$2a_{1}\varepsilon_{F}^{1/2} = 2a_{1}\mu^{1/2} - \frac{\pi^{2}}{12}k_{B}^{2}T^{2}a_{1}\mu^{-3/2}$$

or

$$\mu^{1/2} = \varepsilon_F^{1/2} + \frac{\pi^2}{24} k_B^2 T^2 \mu^{-3/2}$$

Then the chemical potential  $\mu$  is

$$\left(\frac{\mu}{\varepsilon_F}\right)^{1/2} = 1 + \frac{\pi^2}{24} \frac{k_B^2 T^2}{\varepsilon_F^{1/2}} \mu^{-3/2}$$
$$\approx 1 + \frac{\pi^2}{24} \frac{k_B^2 T^2}{\varepsilon_F^{2}}$$

or

$$\frac{\mu}{\varepsilon_F} = (1 + \frac{\pi^2}{24} \frac{k_B^2 T^2}{\varepsilon_F^2})^2 \approx 1 + \frac{\pi^2}{12} \frac{k_B^2 T^2}{\varepsilon_F^2}$$

We make a plot of the number distribution as a function of  $x = \varepsilon / \varepsilon_F$ 

$$D(\varepsilon)f(\varepsilon) \propto \frac{1}{\sqrt{x}} \frac{1}{\exp[\frac{x}{\alpha} - \frac{1}{\alpha}(1 + \frac{\pi^2}{12}\alpha^2)] + 1}$$

where  $\alpha = k_B T / \varepsilon_F$  is changed as a parameter. We choose  $\alpha = 0.3$ .



**Fig.** The number distribution vs  $x = \varepsilon / \varepsilon_F$ .  $\alpha = k_B T / \varepsilon_F = 0.3$  for the 1D case. The chemical potential for the temperature with  $\alpha = k_B T / \varepsilon_F = 0.3$  is denoted by the dashed line. It shifts to the high energy side from the Fermi energy at T = 0 K (x = 1). The area for x > 1 (shaded region denoted by green) is the same as the area below x < 1.









$$N = \int_{0}^{\infty} d\varepsilon f(\varepsilon) D_{3}(\varepsilon) = \int_{0}^{\varepsilon_{F}} d\varepsilon D_{3}(\varepsilon) d\varepsilon$$

where the density of states for the 3D system is

$$D_3(\varepsilon) = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$$

We use the Sommerfeld formula

$$N = \int_{0}^{\infty} d\varepsilon f(\varepsilon)\varphi(\varepsilon) = \int_{0}^{\mu} \varphi(\varepsilon)d\varepsilon + \frac{\pi^{2}}{6}k_{B}^{2}T^{2}\varphi'(\mu)$$

Thus we have

$$N = a_3 \int_0^{\mu} \varepsilon^{1/2} d\varepsilon + \frac{\pi^2}{6} k_B^2 T^2 \frac{1}{2} a_3 \mu^{-1/2}$$
$$= \frac{2}{3} a_3 \mu^{3/2} + \frac{\pi^2}{12} k_B^2 T^2 a_3 \mu^{-1/2}$$

Combining Eqs.(1) and (2), we get

$$\frac{2}{3}a_3\varepsilon_F^{3/2} = \frac{2}{3}a_3\mu^{3/2} + \frac{\pi^2}{12}k_B^2T^2a_3\mu^{-1/2}$$

or

$$\varepsilon_F^{3/2} = \mu^{3/2} + \frac{\pi^2}{8} k_B^2 T^2 \mu^{-1/2}$$

Then the chemical potential is

$$\left(\frac{\mu}{\varepsilon_F}\right)^{3/2} = 1 - \frac{\pi^2}{8} k_B^2 T^2 \frac{\mu^{-1/2}}{\varepsilon_F^{3/2}}$$
$$\approx 1 - \frac{\pi^2}{8} \frac{k_B^2 T^2}{\varepsilon_F^2}$$

or

$$\left(\frac{\mu}{\varepsilon_F}\right) = \left(1 - \frac{\pi^2}{8} \frac{k_B^2 T^2}{\varepsilon_F^2}\right)^{2/3} \approx 1 - \frac{\pi^2}{12} \frac{k_B^2 T^2}{\varepsilon_F^2}$$

or



**Fig.** Chemical potential as a function of temperature for the ideal 3D Fermi gas and the 1D Fermi gas.









Fig. The number distribution vs  $x = \varepsilon / \varepsilon_F$ .  $\alpha = k_B T / \varepsilon_F = 0.1 - 0.6$  for the 3D case. The chemical potential for the temperature with  $\alpha = k_B T / \varepsilon_F$  is denoted by the dashed line. It shifts to the low energy side from the Fermi energy at T = 0 K (x = 1) with increasing temperature. The area for x > 1 (shaded region denoted by green) is the same as the area below x < 1.

# 3. Exact solution for the chemical potential for the 2D case

The density of stares for the 2D system is

$$D_2(\varepsilon)d\varepsilon = \frac{2L^2}{(2\pi)^2} 2\pi k dk = \frac{2L^2}{(2\pi)^2} 2\pi \left(\frac{2m}{\hbar^2}\right) \frac{1}{2} d\varepsilon$$

or

$$D_2(\varepsilon)d\varepsilon = \frac{mL^2}{\pi\hbar^2}d\varepsilon$$
,  $D_2(\varepsilon) = \frac{mL^2}{\pi\hbar^2}$ 

The density of states for the 2D system is independent of  $\varepsilon$ . The chemical potential can be evaluated exactly as follows. In other words, we do not have to use the Sommerfeld approximation.

$$N_{2D} = \int_{0}^{\varepsilon_{F}} D_{2}(\varepsilon) d\varepsilon = \frac{mL^{2}}{\pi\hbar^{2}} \varepsilon_{F}$$

$$N_{2D} = \int_{0}^{\infty} \frac{D_{2}(\varepsilon)d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \frac{mL^{2}}{\pi\hbar^{2}} \int_{0}^{\infty} \frac{d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

Then we have

$$\varepsilon_F = \frac{\pi \hbar^2}{m} n_{2D}$$

with

$$n_{2D} = \frac{N_{2D}}{L^2}$$

We get

$$\varepsilon_F = \int_0^\infty \frac{d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1} = \int_0^\infty \frac{d\varepsilon}{\frac{1}{z}e^{\beta\varepsilon} + 1}$$

We put

$$x = e^{\beta \varepsilon}$$
  
$$dx = \beta e^{\beta \varepsilon} d\varepsilon = \beta x d\varepsilon$$
  
$$\int_{0}^{\infty} \frac{d\varepsilon}{\frac{1}{z}} e^{\beta \varepsilon} + 1 = \int_{1}^{\infty} \frac{dx}{\beta x \left(\frac{1}{z}x + 1\right)}$$
  
$$= \int_{1}^{\infty} \frac{z dx}{\beta x (x + z)}$$
  
$$= \frac{1}{\beta} \int_{1}^{\infty} dx \left(\frac{1}{x} - \frac{1}{x + z}\right)$$
  
$$= \frac{1}{\beta} [\ln(\frac{x}{x + z})]_{1}^{\infty}$$
  
$$= \frac{1}{\beta} \ln(1 + z)$$

 $\beta \varepsilon_F = \ln(1+z)$ 

or

$$e^{\beta \varepsilon_F} = 1 + z$$

Thus we have







Fig. Chemical potential of the 1D, 2D, and 3D systems. Plot of  $y = \mu / \varepsilon_F$  as a function of  $y = k_B T / \varepsilon_F$ 

#### 4. Problem and solution (Schroeder, Problem 7-28) D.V. Schroeder, An Introduction to Thermal Physics (Addison-Wesley, 2000)

Fermi gas in two dimension: Consider a free Fermi gas in two dimensions, confined to a square area  $A = L^2$ .

- (a) Find the Fermi energy (in terms of N and A), and show that the average energy of the particles is  $\varepsilon_F/2$ .
- (b) Derive a formula for the density of states. You should find that it is a constant, independent of  $\varepsilon$ .
- (c) Explain how the chemical potential of the system should behave as a function of temperature, both when  $k_BT \ll \varepsilon_F$  and when T is much higher.
- (d) Because  $D(\varepsilon)$  is a constant for this system, it is possible to carry out the integral for the number of particles analytically. Do so, and solve for  $\mu$  as a function of N. Show that the resulting formula has the expected qualitative behavior.
- (e) Show that in the high-temperature limit,  $k_B T >> \varepsilon_F$ , the chemical potential of this system is the same as that of an ordinary ideal gas.

### ((Solution))

(b)

$$D(\varepsilon)d\varepsilon = \frac{2A}{(2\pi)^2} 2\pi k dk = \frac{mA}{\pi\hbar^2} d\varepsilon$$

where

$$\varepsilon = \frac{\hbar^2}{2m}k^2, \qquad d\varepsilon = \frac{\hbar^2}{m}kdk$$

So the density of states for the 2D system is

$$D(\varepsilon) = \frac{mA}{\pi\hbar^2}$$

(a)

$$N = \int_{0}^{\varepsilon_{F}} D(\varepsilon) d\varepsilon = \frac{mA}{\pi\hbar^{2}} \int_{0}^{\varepsilon_{F}} d\varepsilon = \frac{mA}{\pi\hbar^{2}} \varepsilon_{F}$$

The number density is

$$n = \frac{N}{A} = \frac{m\varepsilon_F}{\pi\hbar^2}$$

The internal energy:

$$U = \int_{0}^{\varepsilon_{F}} \varepsilon D(\varepsilon) d\varepsilon = \frac{mA}{\pi\hbar^{2}} \int_{0}^{\varepsilon_{F}} \varepsilon d\varepsilon = \frac{mA}{\pi\hbar^{2}} \frac{\varepsilon_{F}^{2}}{2}$$

The ratio:

$$\frac{U}{N} = \frac{\frac{mA}{\pi\hbar^2} \frac{\varepsilon_F^2}{2}}{\frac{mA}{\pi\hbar^2} \varepsilon_F} = \frac{1}{2} \varepsilon_F$$

The average energy of the particles is  $\frac{1}{2}\varepsilon_F$ 

$$N = \frac{mA}{\pi\hbar^2} \varepsilon_F$$
$$= \int_0^\infty D(\varepsilon) f(\varepsilon) d\varepsilon$$
$$= \frac{mA}{\pi\hbar^2} \int_0^\infty \frac{1}{\frac{1}{z} e^{\beta\varepsilon} + 1} d\varepsilon$$

or

$$\varepsilon_F = \int_0^\infty \frac{1}{\frac{1}{z}e^{\beta\varepsilon} + 1} d\varepsilon$$

We get

$$\varepsilon_F = \int_{0}^{\infty} \frac{d\varepsilon}{\frac{1}{z}e^{\beta\varepsilon} + 1}$$

We put

$$x = e^{\beta\varepsilon}$$
$$dx = \beta e^{\beta\varepsilon} d\varepsilon = \beta x d\varepsilon$$
$$\int_{0}^{\infty} \frac{d\varepsilon}{\frac{1}{z} e^{\beta\varepsilon} + 1} = \int_{1}^{\infty} \frac{dx}{\beta x \left(\frac{1}{z} x + 1\right)}$$
$$= \int_{1}^{\infty} \frac{z dx}{\beta x (x + z)}$$
$$= \frac{1}{\beta} \int_{1}^{\infty} dx \left(\frac{1}{x} - \frac{1}{x + z}\right)$$
$$= \frac{1}{\beta} [\ln(\frac{x}{x + z})]_{1}^{\infty}$$
$$= \frac{1}{\beta} \ln(1 + z)$$

Thus we have

$$\beta \varepsilon_F = \ln(1+z)$$

or

$$e^{\beta\varepsilon_F} = 1 + z = 1 + e^{\beta\mu}$$

Thus we have

$$\mu = k_B T \ln(e^{\beta \varepsilon_F} - 1)$$



Fig. Chemical potential of the 2D system. Plot of  $y = \mu / \varepsilon_F$  as a function of  $y = k_B T / \varepsilon_F$ 

(d), (e)

For  $\beta \varepsilon_F >> 1$ 

$$\mu = k_B T \ln(e^{\beta \varepsilon_F} - 1) \approx k_B T \beta \varepsilon_F = \varepsilon_F$$

For  $\beta \varepsilon_{F} \ll 1$ 

$$\mu = k_B T \ln(e^{\beta \varepsilon_F} - 1)$$
  

$$\approx k_B T \ln(1 + \beta \varepsilon_F - 1)$$
  

$$= k_B T \ln(\beta \varepsilon_F)$$

or

$$\mu = k_B T \ln(\frac{\pi \hbar^2 n}{m k_B T}) = k_B T \ln[\frac{1}{2} \frac{n}{(n_{2D})_Q}]$$

which is the same as that of ideal 2D gas. We note that the quantum c0oncentration for the 2D system is

$$\left(n_{2D}\right)_{Q} = \frac{mk_{B}T}{2\pi\hbar^{2}}$$

The factor 2 in the denominator of ln term arises from the spin degeneracy.

Note that

$$n=\frac{N}{A}=\frac{m\varepsilon_F}{\pi\hbar^2}.$$

# **REFERENCES** D.V. Schroeder, An Introduction to Thermal Physics (Addison-Wesley, 2000).