

**Fermi liquid: Liquid  $^3\text{He}$**   
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**1. Heat capacity and entropy for Fermi gas**

The internal energy  $U_{FG}$  of the Fermi gas

$$\begin{aligned}\frac{U_{FG}}{N\varepsilon_F} &\approx \frac{3}{5} \left[ 1 - \frac{\pi^2}{12} \left( \frac{k_B T}{\varepsilon_F} \right)^2 \right]^{5/2} \left[ 1 + \frac{5\pi^2}{8} \left( \frac{k_B T}{\varepsilon_F} \right)^2 \right] \\ &\approx \frac{3}{5} \left[ 1 - \frac{5\pi^2}{24} \left( \frac{k_B T}{\varepsilon_F} \right)^2 \right] \left[ 1 + \frac{5\pi^2}{8} \left( \frac{k_B T}{\varepsilon_F} \right)^2 \right] \\ &= \frac{3}{5} \left[ 1 + \frac{5\pi^2}{12} \left( \frac{k_B T}{\varepsilon_F} \right)^2 \right]\end{aligned}$$

or

$$U_{FG} = \frac{3}{5} N \varepsilon_F \left[ 1 + \frac{5\pi^2}{12} \left( \frac{k_B T}{\varepsilon_F} \right)^2 \right]$$

The heat capacity:

$$C_{FG} = \frac{dU_{FG}}{dT} = \frac{3}{5} N \varepsilon_F \frac{5\pi^2}{12} \left( \frac{k_B}{\varepsilon_F} \right)^2 2T = \frac{N\pi^2 k_B^2}{2\varepsilon_F} T$$

or

The heat capacity  $C_{FG}$ :

$$\frac{C_{FG}}{Nk_B} = \frac{\pi^2 k_B T}{2\varepsilon_F}.$$

The entropy  $S_{FG}$ :

$$dS_{FG} = \frac{C_{FG}}{T} dT = \frac{N\pi^2 k_B^2}{2\varepsilon_F} dT$$

or

$$S_{FG} = \frac{N\pi^2 k_B^2}{2\varepsilon_F} T$$

$$\frac{S_{FG}}{Nk_B} = \frac{\pi^2 k_B}{2\varepsilon_F} T = \frac{\pi^2}{2} \frac{T}{T_F}$$

**((Note))**

In general case,

$$C_{FG} = \frac{\pi^2}{3} D(\varepsilon_F) k_B^2 T$$

where  $D(\varepsilon_F)$  is the density of states at the Fermi energy  $\varepsilon_F$ . Using the relation  $D(\varepsilon_F) = \frac{3N}{2\varepsilon_F}$ , we

have the specific heat for free electron Fermi gas

$$C_{FG} = \frac{\pi^2}{3} \frac{3N}{2\varepsilon_F} k_B^2 T = \frac{\pi^2 N k_B}{2} \frac{k_B T}{\varepsilon_F} = \frac{\pi^2 N k_B}{2} \frac{T}{T_F}$$

or

$$\frac{C_{FG}}{Nk_B} = \frac{\pi^2}{2} \frac{T}{T_F} = 0.998948 \frac{T}{T_F} \text{ (K)}$$

where  $T_F = 4.94 \text{ K}$  for liquid  $^3\text{He}$ .

## 2. Pauli paramagnetism for Fermi gas model

The Pauli paramagnetism is given by

$$\chi_{FG} = \mu^2 D(\varepsilon_F) = \frac{3}{2} \frac{N\mu^2}{\varepsilon_F} = \frac{3}{2} \frac{N\mu^2}{k_B T_F}$$

with

$$D(\varepsilon_F) = \frac{3N}{2\varepsilon_F}$$

The magnetic moment of liquid  $^3\text{He}$  is

$$\mu = \gamma \hbar I = \frac{1}{2} \gamma \hbar$$

where  $\mu = -2.1227 \mu_N$  and  $I = 1/2$ . The nuclear magneton is defined by

$$\mu_N = \frac{e \hbar}{2m_p c}$$

( $m_p$  is the mass of proton). The gyromagnetic ratio (the ratio of the magnetic moment to the angular momentum) is given by

$$\gamma = \frac{\mu}{\hbar I} = \frac{-2.1227 \mu_N}{\hbar / 2} = -4.2454 \frac{\mu_N}{\hbar} = -4.2454 \times 4789.43 = -2.033 \times 10^4 \text{ (rad/s G)}$$

with

$$\frac{\mu_N}{\hbar} = \frac{e}{2m_p c} = 4789.43$$

where

$$f = \frac{\omega}{2\pi} = \frac{\gamma B}{2\pi} = 32.361 \text{ (MHz/T)}$$

The Pauli susceptibility can be rewritten as

$$\chi_{FG} = n \frac{3\mu^2}{2\varepsilon_F} = n \frac{3 \left( \frac{\gamma \hbar}{2} \right)^2}{2k_B T_F} = n \frac{3\gamma^2 \hbar^2}{8k_B T_F}$$

### 3. Fermi energy of liquid $^3\text{He}$

Liquid  $^3\text{He}$  as a Fermi gas

$$\text{spin } I = \frac{1}{2} \quad (\text{fermion})$$

$$\text{Density } \rho = 0.081 \text{ g/cm}^3$$

The Fermi energy:

$$\varepsilon_F = \frac{\hbar^2}{2m_0} (3\pi^2 n)^{2/3}$$

where  $m_0$  is the mass of  $^3\text{He}$  atom,

$$m_0 = \frac{3.016 \text{ g}}{6.022 \times 10^{23}} = 0.5 \times 10^{-23} = 5.0 \times 10^{-24} \text{ g}$$

The number density:

$$n = \frac{N}{V} = \frac{N}{M} \frac{M}{V} = \frac{\rho}{m_0} = \frac{0.081}{5.0 \times 10^{-24}} = 1.62 \times 10^{22} / \text{cm}^3 = 1.62 \times 10^{28} / \text{m}^3$$

Then the Fermi energy is

$$\varepsilon_F = 6.815 \times 10^{-16} \text{ erg} = 4.254 \times 10^{-4} \text{ eV}.$$

The Fermi temperature

$$T_F = \frac{\varepsilon_F}{k_B} = 4.94 \text{ K}.$$

That is only a little higher than the boiling point, 3.2 K.

#### 4. Heat capacity of fermi liquid

As predicted for the fermi gas mode, the heat capacity of liquid  $^3\text{He}$  should be

$$\frac{C_{FG}}{Nk_B T} = \frac{\pi^2}{2T_F} = 0.9989 \text{ K}^{-1}$$

So although the linear temperature dependence agrees with experiment, the predicted coefficient is too small by **almost a factor of 3**. According to the Fermi liquid theory, the heat capacity of the liquid  $^3\text{He}$  is predicted as

$$\frac{C_{FL}}{Nk_B T} = \frac{\pi^2}{2T_F} \left(1 + \frac{F_1^{(s)}}{3}\right) \approx \frac{\pi^2}{2T_F} 2.80$$

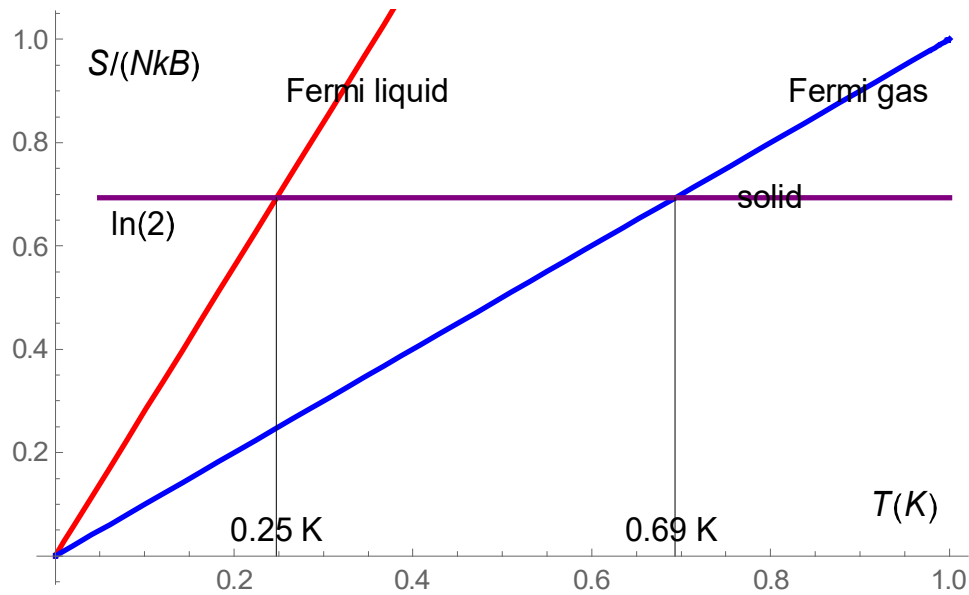
where  $F_1^{(s)} = 5.39$  for  $P$  (pressure) = 0. Similarly, the entropy  $S$  is predicted as

$$\frac{S_{FL}}{Nk_B T} = \frac{\pi^2}{2T_F} \left(1 + \frac{F_1^{(s)}}{3}\right) \approx \frac{\pi^2}{2T_F} 2.80$$

The entropy of the solid, meanwhile, should be

$$k_B \ln 2^N = k_B N \ln 2$$

since each nucleus has two possible spin orientations. This constant value should apply down to very low temperatures, when the nuclear spins finally align and the entropy freezes out. Here is a sketch of both entropy functions. We make a plot of the entropy of the fermi gas and Fermi liquid as a function of  $T$  as well as the entropy of solid  $^3\text{He}$ .



For the Fermi gas, we have

$$T_{FG} = \frac{2T_F}{\pi^2} \ln 2 = 0.69 \text{ K}$$

where

$$\frac{S_{FG}}{Nk_B} = \frac{\pi^2}{2T_F} T_{FG} = \ln 2 .$$

For the Fermi liquid, we have

$$T_{FL} = \frac{2T_F}{\pi^2} \frac{\ln 2}{2.80} = 0.25 \text{ K}.$$

where

$$\frac{S_{FL}}{Nk_B} = \frac{\pi^2}{2T_F} 2.80T_{FL} = \ln 2 .$$

The temperature  $T_{FL}$  (=0.25 K) is much lower than the temperature  $T_{FG}$  (=0.69 K).

According to the Clausius-Clapeyron relation, the slope of the solid-liquid phase boundary on a graph of  $P$  vs  $T$  should be proportional to the entropy difference,  $S_{liquid} - S_{solid}$ . The above analysis predicts that the slope should be positive for  $T > 0.25$  K, and negative at lower temperatures. The experimental phase diagram shows just this behavior, with the transition from positive to negative slope at about 0.3 K, just slightly higher than the prediction 0.25 K. This discrepancy could be because of lattice vibrations giving the solid some additional entropy, and/or the entropy of the liquid no longer being quite linear at relatively high temperatures. At very low temperatures, where the entropy of the solid also goes to zero, the phase boundary becomes horizontal.

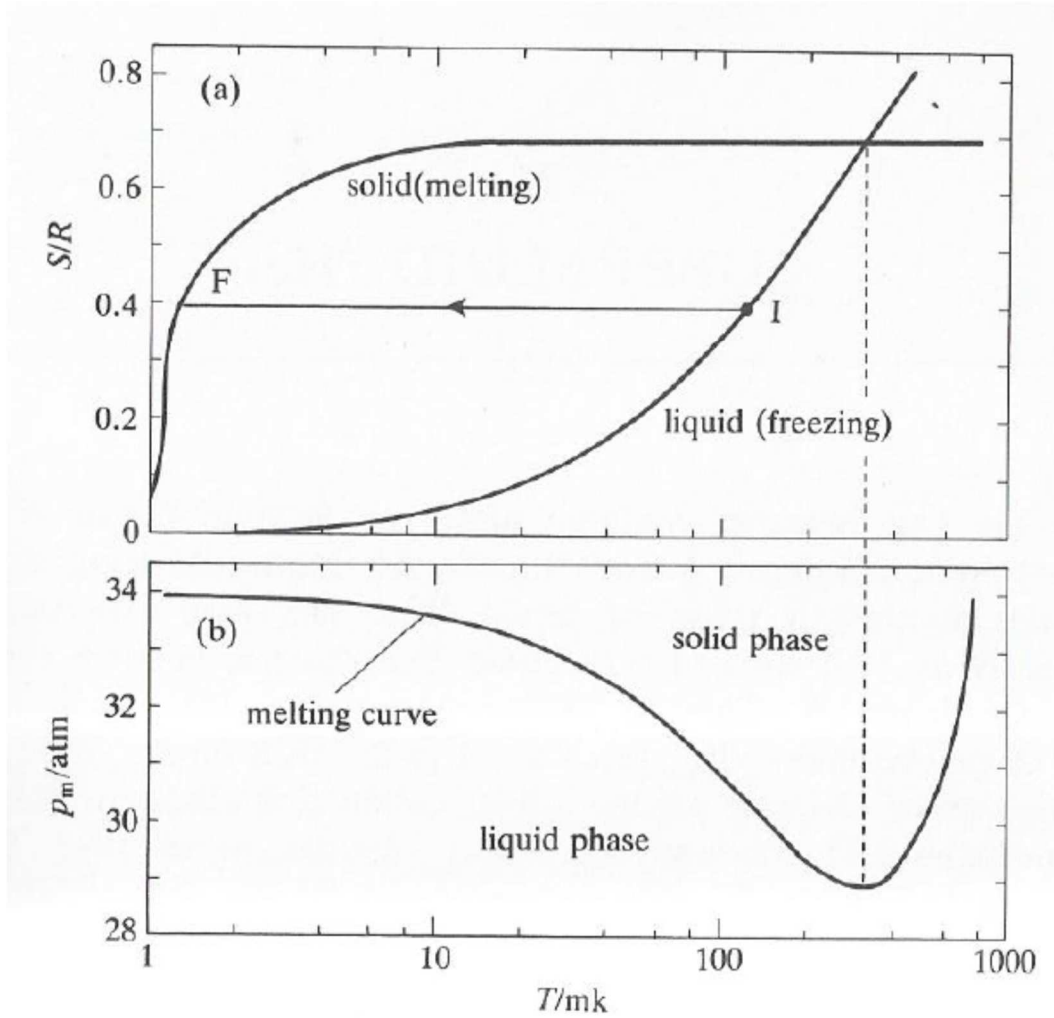
## 5. Entropy

The entropy is mainly due to the nuclear spin of the solid  $^3\text{He}$  around  $T = 0.3$  K. The  $^3\text{He}$  atoms form a lattice. Below 10 mK the nuclear spins starts to order because of the antiferromagnetic exchange interaction. The nuclear spins antiferromagnetically ordered below the Neel temperature  $T_N = 2$  mK. The antiferromagnetic exchange interaction is -0.85 mK.

$$V_{liq} > V_{Solid}$$

For example, at  $T = 0.3$  K, it is determined experimentally that

$$\frac{V_{liq} - V_{Solid}}{N} = 2.1 \times 10^{-24} \text{ cm}^3 / \text{atom}$$



## 6. Pomeranchuk cooling

In the  $P$ - $V$  phase diagram, the solid-liquid boundary below 0.3 K has a negative slope. This is very unusual. Note that the solid-liquid boundary for most materials (except water) has a positive slope. What are the entropies for the liquid phase and solid phase? The  $^3\text{He}$  atom is a fermion. The heat capacity is proportional to  $T$  at low temperatures

$$C_{\text{liquid}} = N_A \frac{\pi^2}{2} k_B \left( \frac{T}{T_F} \right) 2.80$$

The entropy is evaluated as

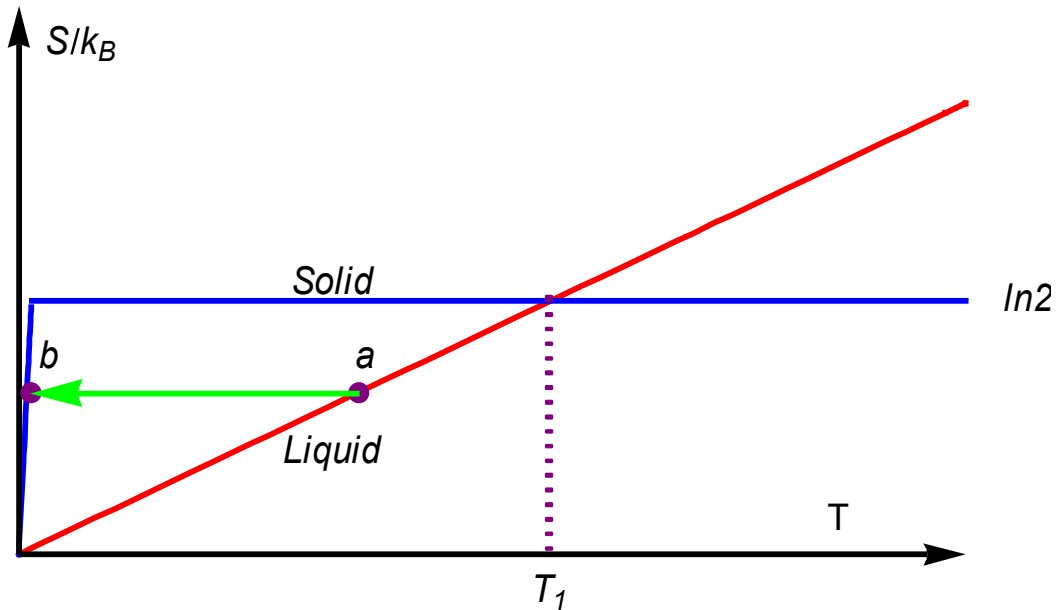
$$S_{\text{liquid}} = \int \frac{C_{\text{liquid}}}{T} dT = N_A \frac{\pi^2}{2} k_B \left( \frac{T}{T_F} \right) 2.80$$

where  $T_F$  is the Fermi temperature. The entropy is proportional to  $T$ .

The entropy of solid  $^3\text{He}$  is dominated by the much larger contribution of the disordered nuclear spins (spin 1/2). Each the nuclear spin (spin 1/2) of  $^3\text{He}$  atom has a magnetic moment, just like a paramagnetic salts. In the paramagnetic state where the directions of spins are random, the solid entropy is given by

$$S_{\text{solid}} = N_A k_B \ln 2.$$

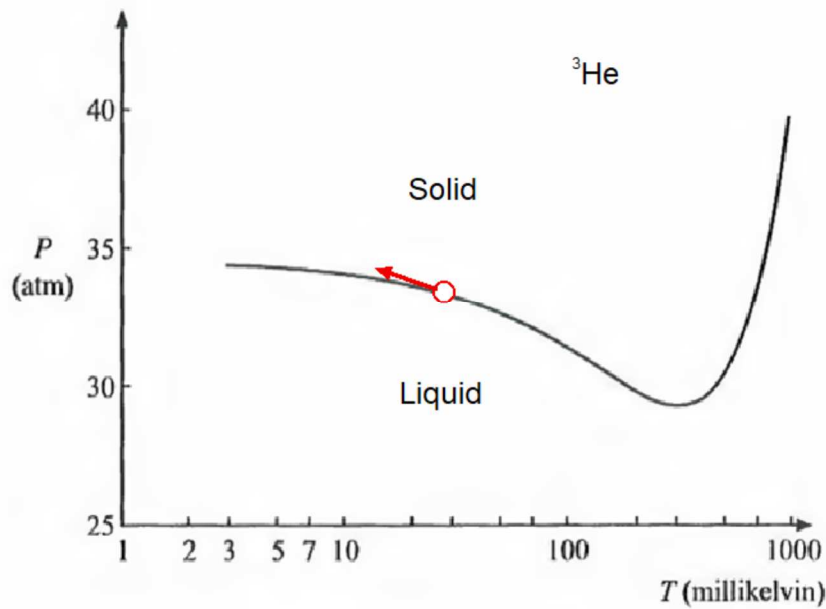
Note that these spins are antiferromagnetically ordered with a Neel temperature  $T_N \approx 1$  mK and the entropy drops rapidly to zero. The liquid entropy coincides with the solid entropy at a characteristic temperature  $T_1$  ( $=0.32$  K). Below  $T_1$ , the solid entropy is higher (more disordered) than the liquid entropy. Above  $T_1$  the liquid entropy (more disordered) is higher than the solid entropy. Such a cross-over of the liquid and solid entropies of  $^3\text{He}$  at about 0.32 K produces a pronounced minimum in the melting curve at about 2.93 MPa, followed by a rise in the melting pressure to 3.45 MPa at  $T = 0$  K. This negative slope is used to produce adiabatic compressional cooling along the melting curve, a technique called Pomeranchuk cooling after its proposer, and used by Osheroff *et al.* (1972a) in their discovery of the superfluid phases of  $^3\text{He}$ .



**Fig.** Entropy per atom in the coexisting solid and liquid phases of  $^3\text{He}$ . The entropy of the liquid phase is less than that of the solid phase below a characteristic temperature  $T_1 = \frac{2 \ln 2}{\pi^2} T_F$ .

The dotted line from  $a$  to  $b$  corresponds to slow adiabatic compression from pure liquid to pure solid.





The phase boundary is expressed by the Clausius-Clapeyron equation

$$\frac{dP}{dT} = \frac{S_{liq} - S_{solid}}{V_{liq} - V_{solid}}$$

In the phase diagram of  $^3\text{He}$  around 0.3 K,

$$\frac{dP}{dT} > 0 \quad \text{for } T > 0.3 \text{ K} \quad \text{corresponding to } S_{liq} > S_{solid}$$

$$\frac{dP}{dT} < 0 \quad \text{for } T < 0.3 \text{ K} \quad \text{corresponding to } S_{liq} < S_{solid}$$

since  $V_{liq} > V_{solid}$  in the vicinity of 0.3 K.

Consider the liquid-solid boundary (denoted by red open circle) below 0.32 K, where  $S_{liq} < S_{solid}$ . When temperature is increased, the entropy increases. The only way this can happen here is to solid which has higher entropy. So the liquid freezes when the temperature rises. This explains the negative slope below 0.32 K.

In order for entropy to increase, heat must be absorbed from the surrounding (like when a normal solid melts). This gives rise to a cooling effect. Higher pressure forces the liquid to become

solid. This gives a cooling effect. This "compressional cooling" method was proposed by Issak Pomeranchuk in 1950.

Pomeranchuk predicted that adiabatic compression of coexisting solid and liquid would cool  $^3\text{He}$ . If the initial temperature is below 0.32 K, then an increase in the external pressure moves the helium along the melting curve up and to the left: toward higher pressure and lower temperatures. Typically, the temperature decreases from 0.3 K to 1 mK. The solid-liquid boundary (arrow) below 0.3 K has a negative slope.

## 7. Problem: Clausius-Clapeyron equation

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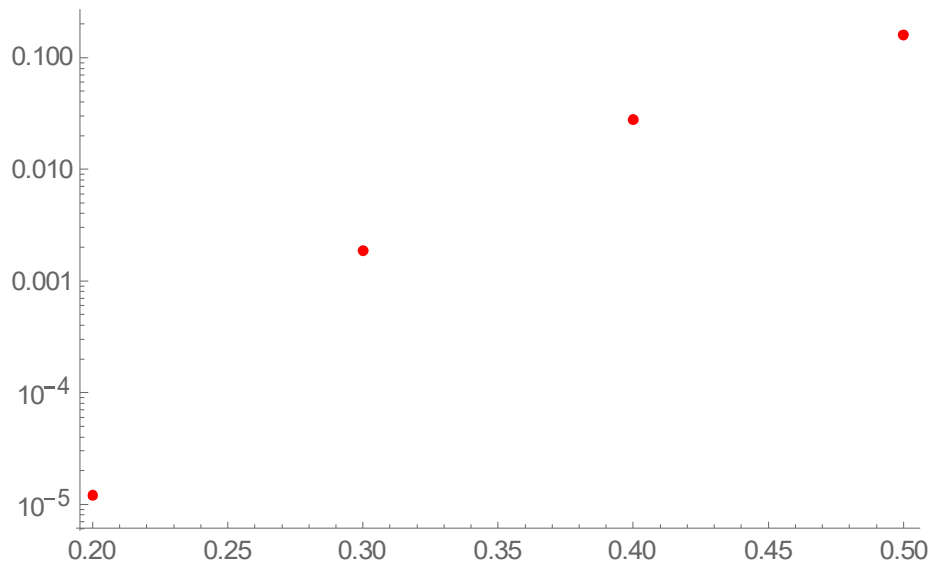
9. For liquid  $^3\text{He}$ , the temperature and associated vapor pressure are given by the following pairs of numbers:

$(T, P)$ :  $(0.2, 1.21 \times 10^{-5})$ ,  $(0.3, 1.88 \times 10^{-3})$ ,  $(0.4, 2.81 \times 10^{-2})$ ,  $(0.5, 1.59 \times 10^{-1})$ .

The temperature is in kelvin; the pressure is in millimeters of mercury. Atmospheric pressure corresponds to 760 millimeters of mercury (in an old fashioned mercury manometer).

- What is the latent heat of vaporization,  $L_{\text{vap}}$ , in the temperature range  $0.2 \leq T \leq 0.5$  K?
- Determine the vapor pressure at  $T = 0.1$  K.

((Solution))



**Fig.** Phase diagram.  $P$  (mmHg=torr) vs  $T$  (K) of liquid  $\text{He}^3$ .

$$\frac{dP}{dT} = \frac{l_v}{T(v_g - v_l)} \approx \frac{l_v}{Tv_g} = \frac{Pl_v}{TPv_g} = \frac{Pl_v}{k_B T^2}$$

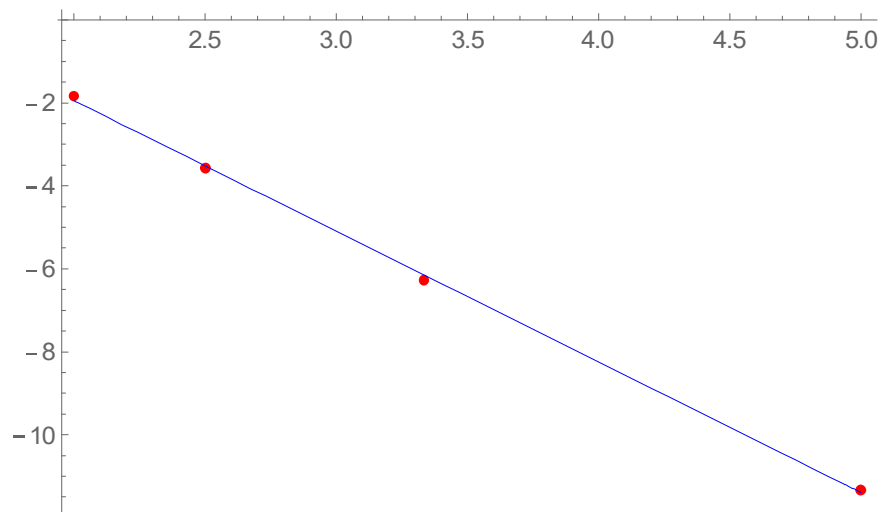
or

$$\frac{1}{P} dP = \frac{l_v}{k_B T^2} dT$$

or

$$\ln P = \ln P_0 - \frac{l_v}{k_B T}$$

We make a plot of  $\ln P$  vs  $1/T$  for  $^3\text{He}$ .



**Fig.** Least squares fit to the straight line; Plot of  $\ln P$  vs  $1/T$ .

(a) The least-squares fit of the data ( $\ln P$  vs  $1/T$ ) to a straight line yields

$$\ln P = 4.34001 - 3.14569(1/T)$$

$$l_v = 3.14569 k_B \text{ (latent heat of evaporation per particle)}$$

or

$$I_v = 2.7108 \times 10^{-4} \text{ (eV/particle)}$$

(b)  $T = 0.1 \text{ K.}$

$$\ln P = 4.34001 - 3.14569(1/0.1) = -27.1169$$

$$P = 1.672 \times 10^{-12} \text{ torr.}$$

## 8. Problem: Phonon

Ralph Baierlein

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10. For solid  $^3\text{He}$ , calculate the entropy per atom that the lattice vibrations contribute at temperatures below 1 K. The Debye temperature is  $\theta_D = 16 \text{ K}$ . Compare this contribution to  $s_{\text{sol}}$  with that from the nuclear spin.

((Solution))

$$C_V = \frac{12\pi^4}{5} R \left( \frac{T}{\Theta_D} \right)^3$$

The entropy:

$$dS = \frac{C_V}{T} dT = \frac{12\pi^4}{5} R \left( \frac{1}{\Theta_D} \right)^3 T^2$$

$$S_{\text{solid}} = \frac{4\pi^4}{5} R \left( \frac{T}{\Theta_D} \right)^3$$

with

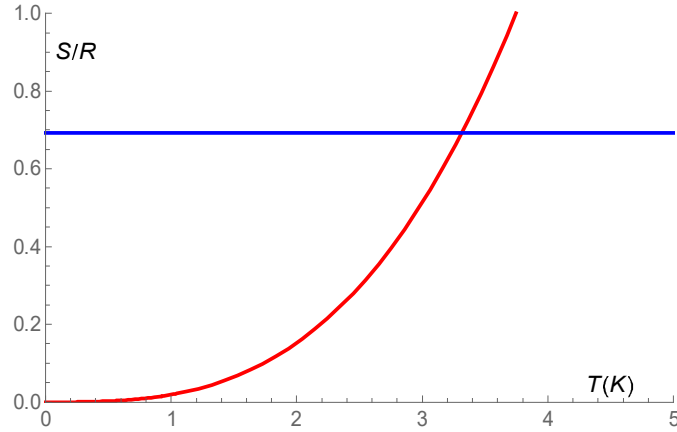
$$\Theta = 16 \text{ K.}$$

Nuclear spin of  $^3\text{He}$

The entropy of solid  $^3\text{He}$  is dominated by the much larger contribution of the disordered nuclear spins (spin 1/2). Each the nuclear spin (spin 1/2) of  $^3\text{He}$  atom has a magnetic moment, just like a

paramagnetic salts. In the paramagnetic state where the directions of spins are random, the solid entropy is given by

$$S_{\text{nuclear}} = R \ln 2$$



**Fig.** Entropy of lattice vibration (red) and nuclear spin (blue) for Helium three.

#### APPENDIX-I Fermi liquid theory

The life time a quasi-particle excitation dressed by interactions is rather long owing to the surrounding degenerate Fermi sea. Thus the physical properties at  $T \ll T_F$  may be well described by a theory based on quasi-particle excitations, the so-called Landau Fermi liquid theory. Since the critical temperature  $T_c$  of this superfluid is of the order of  $T_F/1000$ , similar to that for superconductivity in a metal, one can naturally study the system on the basis of this theory.

Table 6.1. *The molar volume, the Landau parameters at  $T < T_c$  and the critical temperature of liquid  $^3\text{He}$*

$P$ (bar)	$V$ (cm <sup>3</sup> )	$n \times 10^{21}$ (cm <sup>-3</sup> )	$k_F \times 10^7$ (cm <sup>-1</sup> )	$m^*/m$	$F_1^{(s)}$	$F_0^{(s)}$	$F_0^{(a)}$	$T_c$ (mK)	$T_{AB}$ (mK)
0	36.84	16.3	7.84	2.80	5.39	9.30	-0.6951	0.929	—
15	28.89	20.8	8.49	4.28	9.85	41.73	-0.753	2.067	—
34.4	25.50	23.6	8.87	5.85	14.56	88.47	-0.753	2.491	1.933

Fermi liquid relations between the Landau parameters and experimentally measured quantities

(a) Effective mass:

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^{(s)}$$

(b) Specific heat:

$$\frac{C_N}{C_g} = \frac{m^*}{m} = 1 + \frac{1}{3} F_1^{(s)}$$

where

$$C_g = \frac{1}{2} \pi^2 N k_B \frac{T}{T_F}$$

(c) Spin susceptibility:

$$\frac{\chi_N}{\chi_g} = \frac{m^*}{m} \frac{1}{1 + F_0^{(a)}} = \frac{1 + \frac{1}{3} F_1^{(s)}}{1 + F_0^{(a)}}$$

where

$$\chi_g = \mu_B^2 \frac{3N}{2\varepsilon_F} = \frac{3\mu_B^2 N}{2k_B} \frac{1}{T_F}$$

(d) Compressibility:

$$\frac{\kappa_N}{\kappa_g} = \frac{m^*}{m} \frac{1}{1 + F_0^{(s)}} = \frac{1 + \frac{1}{3} F_1^{(s)}}{1 + F_0^{(s)}}$$

where

$$\kappa_g = \frac{3}{2} \frac{1}{n \varepsilon_F}$$

(e) The sound velocity:

$$\frac{c^2}{c_g} = \frac{1 + F_0^{(s)}}{1 + \frac{1}{3}F_1^{(s)}}$$

where

$$T_F = \frac{\hbar^2}{2mk_B}(3\pi^2n), \quad T_F^* = \frac{\hbar^2}{2m^*k_B}(3\pi^2n)$$

$$p_F = \hbar(3\pi^2n)^{2/3}$$

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## APPENDIX II      Proton; magnetic moment

Nuclear magnetic moment  $\mu = 2.792\mu_N$  and  $I = 1/2$  for proton.

where

$$\mu_N = \frac{e\hbar}{2m_p c}$$

The gyromagnetic ratio:

$$\gamma = \frac{\mu}{\hbar I} = \frac{2.792\mu_N}{\hbar / 2} = 5.584 \frac{\mu_N}{\hbar} = 5.584 \times 4789.43 = 2.67442 \times 10^4 \text{ (rad/s G)}$$

$$\frac{\mu_N}{\hbar} = \frac{e}{2m_p c} = 4789.43$$

$$f = \frac{\omega}{2\pi} = \frac{\gamma B}{2\pi} = 42.5647 \text{ (MHz/T)}$$


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