Relativistic Fermi Gas

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1. Grand potential

As the Fermi gas is compressed, the mean energy of the electrons increases (ε_F increases); when it becomes comparable with mc^2, relativistic effects begin to be important. Here we discuss a completely degenerate extreme relativistic electron gas, the energy of whose particles is large compared with mc^2.

Here we consider the equation of state of a relativistic completely degenerate electron gas. The electron energy and momentum is related by

$$\varepsilon_{\boldsymbol{p}} = \sqrt{m^2c^4 + c^2p^2}$$

We note that

$$\varepsilon_{\boldsymbol{p}} = \sqrt{m^2c^4 + c^2p^2} = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}}$$

$$u_{\boldsymbol{p}} = \frac{d\varepsilon_{\boldsymbol{p}}}{dp} = \frac{p}{m\sqrt{1 + \frac{p^2}{m^2c^2}}}$$

The grand potential Φ_G is given by

$$\begin{aligned}
\Phi_G &= -k_BT\sum_{\boldsymbol{p}} \ln[1 + e^{-\beta(\varepsilon_{\boldsymbol{p}} - \mu)}] \\
&= -k_BT\frac{gV}{(2\pi\hbar)^3}\int d^3\boldsymbol{p} \ln[1 + e^{-\beta(\varepsilon_{\boldsymbol{p}} - \mu)}] \\
&= -k_BT\frac{gV}{2\pi^2\hbar^3}\int_0^\infty p^2 dp \ln[1 + e^{-\beta(\varepsilon_{\boldsymbol{p}} - \mu)}]
\end{aligned}$$

where g = 2 (spin factor). The integration by part leads to

$$\int_0^\infty p^2 dp \ln[1+e^{-\beta(\varepsilon_p-\mu)}] = \int_0^\infty \frac{p^3}{3} dp \frac{e^{-\beta(\varepsilon_p-\mu)}}{1+e^{-\beta(\varepsilon_p-\mu)}} \beta \frac{\partial \varepsilon_p}{\partial p}$$

$$= \frac{1}{3}\beta \int_0^\infty p^3 dp \frac{1}{e^{\beta(\varepsilon_p-\mu)}+1} u_p$$

Then we get

$$\Phi_G = -k_B T \frac{gV}{2\pi^2\hbar^3} \frac{1}{3}\beta \int_0^\infty p^3 dp \frac{1}{e^{\beta(\varepsilon_p-\mu)}+1} u_p$$

$$= -\frac{gV}{6\pi^2\hbar^3} \int_0^\infty p^3 dp \overline{n}_p u_p$$

The pressure P is obtained as

$$P = -\left(\frac{\partial \Phi_G}{\partial V}\right)_{T,\mu} = -\frac{\Phi_G}{V} = \frac{g}{6\pi^2\hbar^3} \int_0^\infty p^3 dp \overline{n}_p u_p$$

The number N is given by

$$N = \sum_p \overline{n}_p$$

$$= \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 dp \overline{n}_p$$

$$= \frac{gV}{2\pi^2\hbar^3} \int_0^\infty p^2 dp \overline{n}_p$$

The internal energy E is given by

$$E = \sum_p \overline{n}_p \varepsilon_p$$

$$= \frac{gV}{(2\pi\hbar)^3} \int_0^\infty 4\pi p^2 dp \overline{n}_p \varepsilon_p$$

where

$$\overline{n}_p = \frac{1}{e^{\beta(\varepsilon_p-\mu)}+1} \qquad \text{(Fermi-Dirac function)}$$

2. Grand potential at $T = 0$ K

The grand potential at 0 K is given by

$$\begin{aligned}
\Phi_G &= -\frac{gV}{6\pi^2\hbar^3}\int_0^{p_F} dp(p^3 u_{\boldsymbol{p}}) \\
&= -\frac{gV}{6\pi^2 m\hbar^3}\int_0^{p_F} dp \frac{p^4}{\sqrt{1+\frac{p^2}{m^2c^2}}} \\
&= -\frac{gV}{6\pi^2 m\hbar^3}\frac{1}{8}[m^2c^2 p_F(2p_F{}^2 - 3m^2c^2)\sqrt{1+\frac{p_F{}^2}{m^2c^2}} + 3m^5c^5 \text{arcsinh}(\frac{p_F}{mc})]
\end{aligned}$$

where we use the Mathematica for the integral,

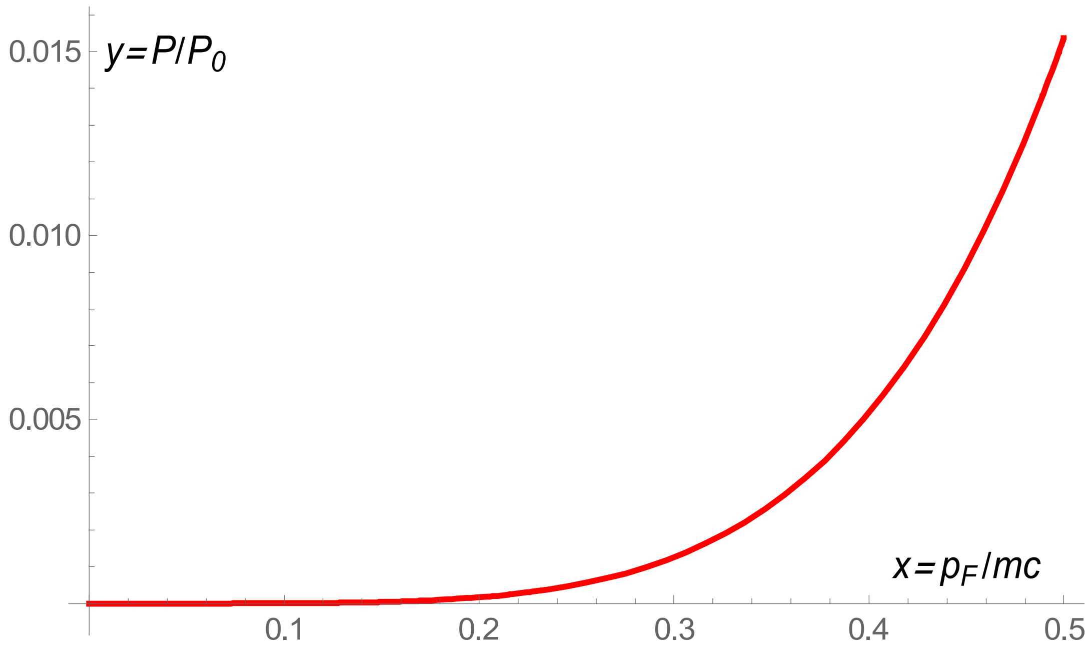
$$\int_0^{p_F} dp \frac{p^4}{\sqrt{1+\frac{p^2}{m^2c^2}}} = \frac{1}{8}[m^2c^2 p_F(2p_F{}^2 - 3m^2c^2)\sqrt{1+\frac{p_F{}^2}{m^2c^2}} + 3m^5c^5 \text{arcsinh}(\frac{p_F}{mc})]$$

3. Pressure at $T = 0$ K

When g = 2, we have

$$\begin{aligned}
P &= -\frac{\Phi_G}{V} \\
&= \frac{1}{3\pi^2 m\hbar^3}\frac{1}{8}[m^2c^2 p_F(2p_F{}^2 - 3m^2c^2)\sqrt{1+\frac{p_F{}^2}{m^2c^2}} + 3m^5c^5 \text{arcsinh}(\frac{p_F}{mc})] \\
&= \frac{m^4c^5}{8\pi^2\hbar^3}[x(\frac{2}{3}x^2 - 1)\sqrt{1+x^2} + \text{arcsinh}(x)]
\end{aligned}$$

where $x = \frac{p_F}{mc}$ and $P_0 = \frac{m^4c^5}{8\pi^2\hbar^3}$. We make a plot of $\frac{P}{P_0}$ as a function of $x = \frac{p_F}{mc}$



4. Number at $T = 0$ K

$$N = \frac{V}{\pi^2\hbar^3}\int_0^{p_F} p^2 dp = \frac{V}{3\pi^2\hbar^3} p_F^{\ 3}$$

$$n = \frac{N}{V} = \frac{p_F^{\ 3}}{3\pi^2\hbar^3}, \qquad \text{or} \qquad p_F^{\ 3} = 3\pi^2\hbar^3\left(\frac{N}{V}\right) = 3\pi^2\hbar^3 n$$

Using $x = \frac{p_F}{mc}$, we have

$$\frac{N}{V} = \frac{m^3c^3}{3\pi^2\hbar^3}x^3$$

5. The internal energy E at $T = 0$ K

The internal energy E at T = 0 K is

$$E = \frac{Vc}{\pi^2\hbar^3}\int_0^{p_F} p^2 dp\sqrt{m^2c^2 + p^2}$$

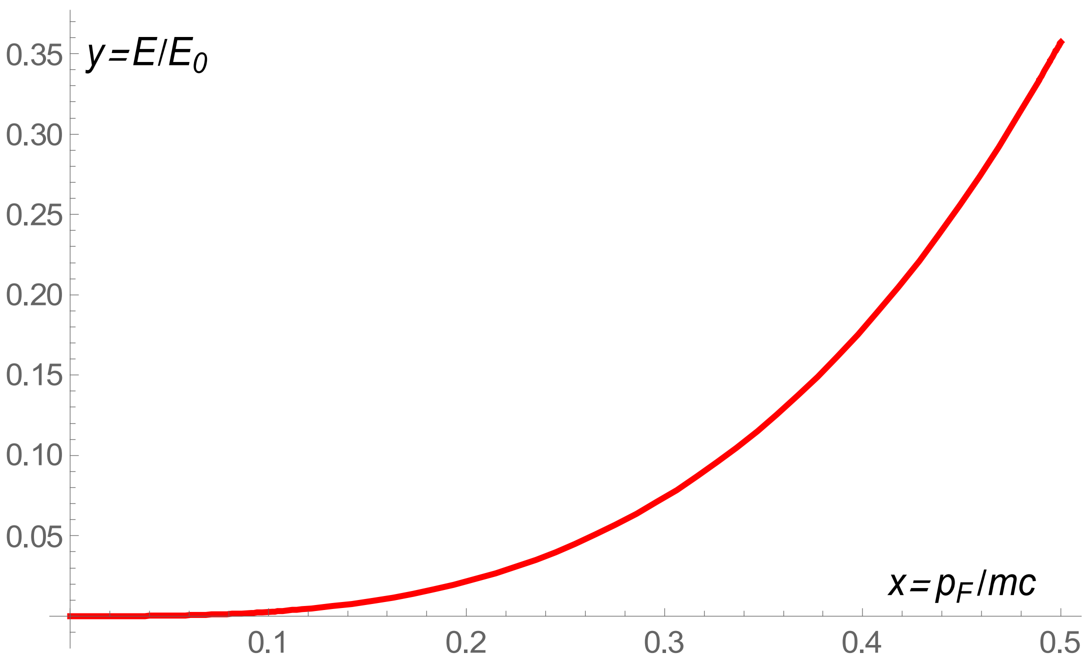
We use Mathematica for the evaluation of integral

$$\int_0^{p_F} dp p^2 \sqrt{1+\frac{p^2}{m^2c^2}} = \frac{1}{8}[p_F(2p_F{}^2 + m^2c^2)\sqrt{1+\frac{p_F{}^2}{m^2c^2}} - m^3c^3 \text{arcsinh}(\frac{p_F}{mc})]$$

Then we have

$$\begin{aligned} E &= \frac{Vc}{8\pi^2\hbar^3}[p_F(2p_F{}^2 + m^2c^2)\sqrt{1+\frac{p_F{}^2}{m^2c^2}} - m^3c^3 \text{arcsinh}(\frac{p_F}{mc})] \\ &= E_0[x(2x^2+1)\sqrt{x^2+1} - \text{arcsinh}(x)] \end{aligned}.$$

where $E_0 = \frac{Vm^3c^4}{8\pi^2\hbar^3}$. We make a plot of $\frac{E}{E_0}$ as a function of $x = \frac{p_F}{mc}$



6. Ultra-relativistic case: $\varepsilon = cp$

As the gas is compressed, the mean energy of the electrons increases. When it becomes comparable with mc^2, relativistic effects begin to be important. Here we discuss a completely degenerate extreme relativistic electron gas. The energy of the particles is large compared with mc^2.

In the ultra-relativistic case, the energy dispersion can be expressed by

$$\varepsilon_p = cp$$

The number N is given by

$$N = \frac{gV}{2\pi^2\hbar^3}\int_0^\infty p^2 dp\overline{n}_p = \frac{gV}{2\pi^2\hbar^3c^3}\int_0^\infty \frac{\varepsilon^2 d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

The internal energy U is given by

$$U = \frac{gV}{(2\pi\hbar)^3}\int_0^\infty 4\pi p^2 dp\overline{n}_p\varepsilon_p = \frac{gV}{2\pi^2\hbar^3c^3}\int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

where

$$\overline{n}_p = f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon_p-\mu)}+1} \qquad \text{(Fermi-Dirac function)}$$

Now we use the Sommerfeld formula;

$$\int_0^\infty g(\varepsilon)f(\varepsilon)d\varepsilon = \int_0^\mu g(\varepsilon)d\varepsilon + \frac{1}{6}(\pi k_B T)^2 g^{(1)}(\mu)$$
$$+\frac{7}{360}(\pi k_B T)^4 g^{(3)}(\mu) + ...$$

The number density:

$$N \approx \frac{gV}{2\pi^2\hbar^3c^3}[\frac{1}{3}\mu^3 + \frac{\pi^2}{6}(k_B T)^2(2\mu)]$$
$$= \frac{gV\mu^3}{6\pi^2\hbar^3c^3}[1+(\frac{\pi k_B T}{\mu})^2]$$

The internal energy:

$$U = \frac{gV}{2\pi^2\hbar^3c^3}\int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$
$$= \frac{gV}{2\pi^2\hbar^3c^3}[\frac{1}{4}\mu^4 + \frac{1}{2}(\pi k_B T)^2\mu^2 + \frac{7}{60}(\pi k_B T)^4]$$
$$= \frac{gV\mu^4}{8\pi^2\hbar^3c^3}[1+2(\frac{\pi k_B T}{\mu})^2 + \frac{7}{15}(\frac{\pi k_B T}{\mu})^4]$$

We note that

$$N = \frac{gV}{2\pi^2\hbar^3c^3}\int_0^{\varepsilon_F}\varepsilon^2 d\varepsilon = \frac{gV}{2\pi^2\hbar^3c^3}\frac{{\varepsilon_F}^3}{3}.$$

When $g = 2$,

$$\varepsilon_F = \hbar c(3\pi^2 n)^{1/3}$$

The chemical potential is obtained as

$$N = \frac{gV\mu^3}{6\pi^2\hbar^3c^3}[1+(\frac{\pi k_B T}{\mu})^2] = \frac{gV}{2\pi^2\hbar^3c^3}\frac{{\varepsilon_F}^3}{3}$$

or

$$\mu = \varepsilon_F[1-\frac{1}{3}(\frac{\pi k_B T}{\varepsilon_F})^2]$$

The internal energy:

$$\begin{aligned}\frac{U}{N} &= \frac{\frac{3\mu}{4}[1+2(\frac{\pi k_B T}{\mu})^2+\frac{7}{15}(\frac{\pi k_B T}{\mu})^4]}{[1+(\frac{\pi k_B T}{\mu})^2]}\\ &\approx \frac{3\mu}{4}[1+(\frac{\pi k_B T}{\mu})^2]\end{aligned}$$

Using the expression of μ, we get

$$\frac{U}{N} = \frac{3\varepsilon_F}{4}[1+\frac{2}{3}(\frac{\pi k_B T}{\varepsilon_F})^2]$$

The heat capacity:

$$C = \frac{dU}{dT} = \frac{\pi^2 {k_B}^2 T}{\varepsilon_F}$$

where $\varepsilon_F = \hbar c(3\pi^2 n)^{1/3}$.

In general case, the pressure P is obtained as

$$P = -\left(\frac{\partial \Phi_G}{\partial V}\right)_{T,\mu} = -\frac{\Phi_G}{V} = \frac{g}{6\pi^2\hbar^3}\int_0^\infty p^3 dp \overline{n}_p u_p$$

We consider the case of $u_p = c$ and $\varepsilon = cp$

$$P = -\frac{\Phi_G}{V} = \frac{cg}{6\pi^2\hbar^3}\int_0^\infty p^3 dp \overline{n}_p$$

The internal energy is

$$\frac{U}{V} = \frac{cg}{2\pi^2\hbar^3}\int_0^\infty p^3 dp \overline{n}_p = 3P$$

or

$$PV = \frac{1}{3}U$$

At $T = 0$ K, the pressure P is

$$PV = \frac{1}{3}U = \frac{\varepsilon_F}{4}N = \frac{\hbar c}{4}(3\pi^2 n)^{1/3} N$$

or

$$P = \frac{1}{4}(3\pi^2)^{1/3}\hbar c n^{4/3}$$

6. Scaling relation of the grand potential

When $u_p = c$

$$\Phi_G = -\frac{gV}{6\pi^2\hbar^3}\int_0^\infty p^3 dp\overline{n}_{\boldsymbol{p}} u_{\boldsymbol{p}}$$

$$= -\frac{cgV}{6\pi^2\hbar^3}\int_0^\infty p^3 dp\overline{n}_{\boldsymbol{p}}$$

$$= -\frac{gV}{6\pi^2\hbar^3 c^3}\int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

or

$$\Phi_G = -\frac{gVk_B{}^4T^4}{6\pi^2\hbar^3c^3}\int_0^\infty \frac{x^3dx}{e^{x-\frac{\mu}{k_BT}}+1} = VT^4F(\frac{\mu}{k_BT}) = VT^4 f(\frac{\mu}{T})$$

The entropy:

$$S = -\left(\frac{\partial \Phi_G}{\partial T}\right)_{V,\mu}$$

$$= -4VT^3 f(\frac{\mu}{T}) + VT^3\frac{\mu}{T} f'(\frac{\mu}{T})$$

$$= VT^3[-4f(\frac{\mu}{T}) + \frac{\mu}{T} f'(\frac{\mu}{T})]$$

$$= VT^3 f_S(\frac{\mu}{T})$$

$$N = -\left(\frac{\partial \Phi_G}{\partial \mu}\right)_{T,V}$$

$$= -VT^3 f'(\frac{\mu}{T})$$

$$= VT^3 f_N(\frac{\mu}{T})$$

$$P = -\left(\frac{\partial \Phi_G}{\partial V}\right)_{T,\mu}$$

$$= -T^4 f(\frac{\mu}{T})$$

Thus

$$\frac{S}{N} = \frac{f_S\left(\frac{\mu}{T}\right)}{f_N\left(\frac{\mu}{T}\right)} = \varphi\left(\frac{\mu}{T}\right)$$

is independent of V. It means that if we keep entropy constant and change the volume, namely, for adiabatic change, μ / T is kept constant. Therefore, for adiabatic change, we have

$$\frac{P}{T^4} = \text{const}, \qquad VT^3 = \text{const}, \qquad PV^{4/3} = \text{const}$$

indicating that $\gamma = \frac{4}{3}$.

((Note))

$$\frac{\Phi_G}{N} = -\frac{1}{4}\varepsilon_F [1 + \frac{2}{3}\left(\frac{\pi k_B T}{\varepsilon_F}\right)^2]$$

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