**Problem and solution**
**Huang Problem 18.10**
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**Possible photon condensation**

If the mechanism for photon absorption or emission can be neglected, as may happen in some cosmological settings, the number of photons would be conserved. Can a photon gas undergo Bose-Einstein condensation under these circumstances? If so, give the critical photon density at temperature $T$.

((**Solution**))
For a gas of $N$ photons with number conservation, the fugacity $z$ is determined by

$$N = \frac{2V}{(2\pi)^3}\int 4\pi k^2 dk\bar{n}_k = \frac{V}{\pi^2}\int \frac{k^2 dk}{\frac{1}{z}e^{\beta\varepsilon_k} - 1}$$

The zero-momentum state is ignored in the continuum approximation used. The energy dispersion for photon;

$$\varepsilon = \hbar ck$$

Since $d\varepsilon = \hbar c dk$, we have

$$n = \frac{N}{V} = \frac{1}{\pi^2}\frac{1}{(\hbar c)^3}\int \frac{\varepsilon^2 d\varepsilon}{\frac{1}{z}e^{\beta\varepsilon} - 1}$$

We put

$$x = \beta\varepsilon$$

Then we get

$$n = \frac{1}{\pi^2}\frac{(k_B T)^3}{(\hbar c)^3}\int_0^\infty \frac{x^2 dx}{\frac{1}{z}e^x - 1}$$

We note that

$$\int_0^\infty \frac{x^2 dx}{\frac{1}{z}e^x - 1} = 2\varsigma_3(z), \qquad \varsigma_3(z=1) = 1.20206$$

Then we have

$$n = \frac{1}{\pi^2}\frac{(k_B T)^3}{(\hbar c)^3} 2\varsigma_3(z)$$

The Bose-Einstein condensation temperature is

$$n = \frac{1}{\pi^2}\frac{(k_B T_E)^3}{(\hbar c)^3} 2\varsigma_3(z=1)$$

$$\boxed{T_E = \frac{\hbar c}{k_B}\left(\frac{n\pi^2}{2\times 1.20206}\right)^{1/3}}$$