# Phonon I: lattice waves Masatsugu Sei Suzuki Department of Physics, SUNY at Binghamton (February 19, 2013) 


#### Abstract

A lecture note on the lattice waves in the solid is presented. In a crystal each atom are coupled with the neighboring atoms by spring constants. The collective motion of atoms leads to a well-defined traveling wave over the whole system, leading to the collective motion, so called phonon. Here the equation of motion of atoms around thermal equilibrium position will be discussed in terms of several methods, which include numerical calculation on the eigenvalue problem (based on the Mathematica) and the translation operators in the quantum mechanics. We show that all these methods lead to the same conclusion, the existence of lattice wave, phonon in the quantum mechanics. For students who just start to study the solid state physics, it may be difficult to understand the validity of the assumption that the deviation of the displacement of the atoms from the thermal equilibrium is well described by a traveling wave. We numerically solve the eigenvalue-problem of the motion of atoms in the linear chain (typically 50-100 atoms) using Mathematica. We will give a bit of evidence for the existence of the normal modes propagating along the linear chain.


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## 1. Introduction

Phonons are a quantum mechanical version of a special type of vibrational motion, known as normal modes in classical mechanics, in which each part of a lattice oscillates with the same frequency. These normal modes are important because, according to a well-known result in classical mechanics, any arbitrary vibrational motion of a lattice can be considered as a superposition of normal modes with various frequencies; in this sense, the normal modes are the elementary vibrations of the lattice. Although normal modes are wave-like phenomena in classical mechanics, they acquire certain particle-like properties when the lattice is analyzed using quantum mechanics (see wave-particle duality.) They are then known as phonons.

There have been many excellent textbooks on the physics of lattice waves. Typical books ${ }^{1-14}$ which we read during the preparation of writing this lecture note, are presented in References. These books are very useful for our understanding physics.

## 2 Lattice waves <br> 2.1 Overview

Consider the elastic vibrations of a crystal with one atom in the primitive cell. We want to find the frequency of an elastic wave in terms of the wavevector $\boldsymbol{k}$ and the elastic constants. When a wave propagates along the $x$-direction, entire planes of atoms move in phase with displacements either parallel or perpendicular to the direction of $\boldsymbol{k}$.

We can describe with a single co-ordinate $u_{s}$ the displacement of the plane $s$ from its equilibrium position.


Fig. 1 (a)


Fig. 1 (b)
Fig.1(a) and (b)
Planes of atoms in thermal equilibrium (blue atoms ). $a$ is the nearest neighbor separation distance between planes. Planes of atoms when displaced as for a longitudinal wave (red atoms). The direction of the oscillation is parallel to that of the propagating waves. $\boldsymbol{k}$ is the wavevector. The coordinate $u_{\mathrm{s}}$ measures the displacement of the planes. Longitudinal wave. $k=n(2 \pi / N a) . N=10 . a=1 . n=4 . N$ is the total number of planes.


Fig. 1 (c)


Fig. 1 (d)

## Fig.1(c) and (d)

Plane of atoms as displaced during passage of transverse wave (red atoms). Atoms in thermal equilibrium (blue atoms). The direction of the oscillation is perpendicular to that of the propagating waves. $k=n(2 \pi / N a)$. $N=10 . a=1 . n=4$.

For each wavevector there are three modes; one of longitudinal polarization, two of transverse polarization. We assume that the elastic response of the crystal is a linear function of the forces. Or the elastic energy is a quadratic function of the relative displacement of any two points in the crystal. The forces on the plane $s$ caused by the displacement of the plane $s+p$ is proportional to the difference $u_{s+p}-u_{s}$ of their displacements.

For brevity, we consider only nearest-neighbor interactions, so that $p= \pm 1$. The total force on s comes from planes $s \pm 1$.


Fig. 2 (a) Longitidinal wave.


Fig. 2 (b) Transverse wave.

## Fig.2(a) and (b)

The displacements of atoms with mass $M$ are denoted by $u_{\mathrm{s}-1}, u_{\mathrm{s}}$, and $u_{\mathrm{s}+1}$. The repeat distance is a in the direction of the wavevector $\boldsymbol{k}$. The direction of $u_{\mathrm{s}}$ is parallel to the direction of the wavevector $\boldsymbol{k}$ for the longitudinal wave (a) and is perpendicular to the direction of the wavevector $\boldsymbol{k}$ for the transverse wave (b).

### 2.2 One dimensional case: longitudinal mode

We start our discussion with the Lagrangian for the displacement of the $s$-th plane, given by

$$
\begin{align*}
L_{s} & =T_{s}-V_{s} \\
& =\frac{1}{2} m \dot{x}_{s}^{2}-\frac{1}{2} C\left[\left(x_{s+1}-x_{s}-a\right)^{2}+\left(x_{s}-x_{s-1}-a\right)^{2}\right] .  \tag{2.1}\\
& =\frac{1}{2} m \dot{u}_{s}^{2}-\frac{1}{2} C\left[\left(u_{s+1}-u_{s}\right)^{2}+\left(u_{s}-u_{s-1}\right)^{2}\right]
\end{align*}
$$


where $x_{\mathrm{s}}$ is the position of the atom at the $s$-th site and a is assumed to be the length of the unstretched spring.

$$
x_{s}=u_{s}+s a
$$

where $a$ is the distance between the adjacent atoms in thermal equilibrium. The Lagrange's equation for this system is derived as

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\partial L}{\partial \dot{u}_{s}}\right)=\frac{\partial L}{\partial u_{s}} \tag{2.2}
\end{equation*}
$$

or

$$
\begin{align*}
m \ddot{u}_{s} & =-\frac{1}{2} C\left[2\left(u_{s+1}-u_{s}\right)(-1)+2\left(u_{s}-u_{s-1}\right)\right] \\
& =-C\left(-u_{s+1}+u_{s}+u_{s}-u_{s-1}\right)  \tag{2.3}\\
& =C\left(u_{s+1}-2 u_{s}+u_{s-1}\right)=F_{s}
\end{align*},
$$

where

$$
F_{s}=C\left(u_{s+1}-u_{s}\right)-C\left(u_{s}-u_{s-1}\right)
$$

is the effective force on the $s$-th plane (Hooke's law), $C$ is the force constant between nearest-neighbor planes, $C_{L} \neq C_{T}$ ( $C_{L}$ : force constant for longitudinal wave, $C_{T}$ : force constant for transverse wave). It is convenient hereafter to regard $C$ as defined for one atom of the plane, so that $F_{s}$ is the force on one atom in the plane $s$. The equation of motion of the plane $s$ is

$$
\begin{equation*}
M \frac{d^{2} u_{s}}{d t^{2}}=C\left(u_{s+1}-2 u_{s}+u_{s-1}\right), \tag{2.4}
\end{equation*}
$$

where $M$ is the mass of an atom in the $s$-th plane. Suppose that this equation has the traveling wave solutions of the form

$$
\begin{equation*}
u_{s}=u e^{i(k s a-\omega t)} . \tag{2.5}
\end{equation*}
$$

Note that the validity of this assumption will be verified by solving directly the eigenvalue problem (see Sec. 2.5). The boundary condition is illustrated below.


Fig. 3 One-dimensional array of equal masses and springs. This is the simplest model of a vibrational band.

Alternative representation of the Born-von Karman boundary condition. The object connecting the ion on the extreme left with the spring on the extreme right is a massless rigid rod of length $L=N a$.


Fig. 4 The Born-von Karman periodic boundary condition for the linear chain.

### 2.3 Born-von Karman boundary condition

The Born-von Karman or periodic boundary condition;

$$
\begin{align*}
& u_{s}=u_{s+N}  \tag{2.6}\\
& u_{s}=e^{i(k s a-\omega t)} . \tag{2.7}
\end{align*}
$$

The periodic boundary condition requires that

$$
\begin{equation*}
e^{i k N a}=1, \quad \text { or } \quad k=\frac{2 \pi}{a} \frac{\ell}{N}, \tag{2.8}
\end{equation*}
$$



Fig. $5 N$ modes for $-\pi / a \leq k \leq \pi / a$
where

$$
\begin{equation*}
\ell=\underbrace{-\frac{N}{2},-\frac{N}{2}+1, \cdots, \frac{N}{2}-1, \frac{N}{2}}_{N} . \tag{2.8}
\end{equation*}
$$

For each $k$, there are one longitudinal mode and two transverse modes. Then we have

$$
(1+2) N \text { modes }=3 N \text { modes }
$$

$$
\begin{equation*}
\Uparrow \tag{veryimportant}
\end{equation*}
$$

$N$ atoms: each atom has 3 freedoms $3 N$

$$
u_{s}=e^{i(k s a-\omega t)},
$$

$$
\begin{equation*}
k^{\prime}=k+\frac{2 \pi}{a} n \quad(n: \text { integer }) \tag{2.9}
\end{equation*}
$$

The displacement of the $n$-th atom for the wave $k^{\prime}$

$$
\begin{equation*}
u_{s}=e^{i\left(k^{\prime} s a-\omega t\right)}=e^{i\left[\left(k-\frac{2 \pi}{a} n\right) s a-\omega t\right]}=e^{i(k s a-\omega t)} \tag{2.10}
\end{equation*}
$$

So the displacement $u_{s}$ is the same as for the wavevector $k$, for any atom whatsoever. Thus the wave $k^{\prime}$ cannot be physically differentiated from the wave $k$.

### 2.4 First Brilouin zone

What range of $k$ is physically significant for elastic waves?
$\Rightarrow$ Only those in the first Brillouin.

$$
\begin{equation*}
\frac{u_{s+1}}{u_{s}}=e^{i k a} \tag{2.11}
\end{equation*}
$$

The range $-\pi$ to $\pi$ for the phase $k a$ covers all independent values of the exponential.

$$
\begin{equation*}
-\pi \leq k a \leq \pi \quad \text { or } \quad-\frac{\pi}{a} \leq k \leq \frac{\pi}{a} . \tag{2.12}
\end{equation*}
$$

In the continuum limit $(a=0)$,

$$
\begin{equation*}
k_{\max }=\frac{\pi}{a} \rightarrow \infty . \tag{2.13}
\end{equation*}
$$



Fig. 6 The relation between $k^{\prime}$ and $k=k^{\prime}+2 \pi / a$.
Suppose that

$$
\begin{equation*}
k^{\prime}=k+\frac{2 \pi}{a} n \quad(n: \text { integer }) \tag{2.14}
\end{equation*}
$$

Then

$$
\begin{equation*}
\frac{u_{s+1}}{u_{s}}=e^{i k a}=e^{i\left(k^{\prime}+\frac{2 \pi}{a} n\right) a}=e^{i k^{\prime} a} \tag{2.15}
\end{equation*}
$$

Thus the displacement can always be described by a wavevector within the first Brillouin zone. Note that $G=\frac{2 \pi}{a} n$ is a reciprocal lattice vector. Thus by subtraction of an appropriate reciprocal lattice vector from $\boldsymbol{k}$, we always obtain an equivalent wavevector in the 1st zone: $\boldsymbol{k}=\boldsymbol{k}^{\prime}+\boldsymbol{G}$, where $\boldsymbol{k}^{\prime}$ is the wavevector in the first Brillouin zone.

### 2.5 Normal modes

We solve the equations of motion

$$
\begin{equation*}
M \frac{d^{2} u_{s}}{d t^{2}}=C\left(u_{s+1}-2 u_{s}+u_{s-1}\right) \tag{2.16}
\end{equation*}
$$

by assuming that

$$
\begin{equation*}
u_{s}=e^{i(k s a-\omega t)} . \tag{2.17}
\end{equation*}
$$

Then we have

$$
\begin{align*}
& \frac{d^{2} u_{s}}{d t^{2}}=-\omega^{2} u_{s}  \tag{2.18}\\
& u_{s \pm 1}=e^{ \pm i k a} u_{s}  \tag{2.19}\\
& -\omega^{2} M u_{s}=C\left[e^{i k a}+e^{-i k a}-2\right] u_{s} \tag{2.20}
\end{align*}
$$

leading to the dispersion relation

$$
\begin{equation*}
\omega^{2}=\frac{C}{M}(2-2 \cos k a)=\frac{2 C}{M}(1-\cos k a)=\frac{4 C}{M} \sin ^{2}\left(\frac{k a}{2}\right), \tag{2.21}
\end{equation*}
$$

or

$$
\begin{equation*}
\omega=\sqrt{\frac{4 C}{M}}\left|\sin \left(\frac{k a}{2}\right)\right| \tag{2.22}
\end{equation*}
$$

It is usual for one to assume that $\omega$ is positive. Then the wave propagates along the positive $x$ direction for $k>0$ and along the negative $x$ direction for $k<0$.
The boundary of the first Brillouin zone lies at $k= \pm \frac{\pi}{a}$.

$$
\begin{align*}
& \frac{d \omega^{2}}{d k}=2 \omega \frac{d \omega}{d k}=\frac{2 C a}{M} \sin k a  \tag{2.23}\\
& \Rightarrow \frac{d \omega}{d k}=0 \text { at } k= \pm \frac{\pi}{a} \tag{2.24}
\end{align*}
$$



Fig. 7 The dispersion curve ( $\omega$ vs $k$ ) for a monatomic linear lattice with nearest neighbor interactions only. The first Brillouin zone is the segment between $-\pi / a$ and $\pi / a$. w is linear for small $k$ and that $\mathrm{d} \omega / \mathrm{d} k$ vanishes at the zone boundaries $(k= \pm \pi / a$.

$$
\begin{equation*}
\omega^{2}=\frac{4 C}{M} \sin ^{2}\left(\frac{k a}{2}\right) \quad \text { or } \quad \omega=\sqrt{\frac{4 C}{M}}\left|\sin \left(\frac{k a}{2}\right)\right| \quad \text { dispersion relation } \tag{2.25}
\end{equation*}
$$

### 2.6 Continuum wave equation

We consider an original wave equation given by

$$
\begin{equation*}
M \frac{d^{2} u_{s}}{d t^{2}}=C\left(u_{s+1}-2 u_{s}+u_{s-1}\right) \tag{2.26}
\end{equation*}
$$

We assume that $u_{\mathrm{s}}(x)=u(x, t)$. In the small limit of $a=\Delta x, s a=x$ and $s$ is continuous variable. Under this assumption, $u(x, t)$ can be expanded using a Taylor expansion,

$$
\begin{equation*}
u_{s+1}(t)=u(x+\Delta x, t)=u(x, t)+\frac{\Delta x}{1!} \frac{\partial u(x, t)}{\partial x}+\frac{(\Delta x)^{2}}{2!} \frac{\partial^{2} u(x, t)}{\partial x^{2}}+O(\Delta x)^{3} \tag{2.27}
\end{equation*}
$$

Then the original wave equation can be rewritten as

$$
\begin{align*}
M \frac{\partial^{2} u(x, t)}{\partial t^{2}} & =C\left[u(x, t)+\frac{\Delta x}{1!} \frac{\partial u(x, t)}{\partial x}+\frac{(\Delta x)^{2}}{2!} \frac{\partial^{2} u(x, t)}{\partial x^{2}}\right.  \tag{2.28}\\
& \left.-2 u(x, t)+u(x, t)-\frac{\Delta x}{1!} \frac{\partial u(x, t)}{\partial x}+\frac{(\Delta x)^{2}}{2!} \frac{\partial^{2} u(x, t)}{\partial x^{2}}\right]
\end{align*}
$$

or

$$
\begin{equation*}
\left.M \frac{\partial^{2} u(x, t)}{\partial t^{2}}=C(\Delta x)^{2} \frac{\partial^{2} u(x, t)}{\partial x^{2}}\right], \quad \text { (wave equation) } \tag{2.29}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{\partial^{2} u(x, t)}{\partial t^{2}}=\frac{C}{M}(\Delta x)^{2} \frac{\partial^{2} u(x, t)}{\partial x^{2}}=\frac{C}{M} a^{2} \frac{\partial^{2} u(x, t)}{\partial x^{2}}=v^{2} \frac{\partial^{2} u(x, t)}{\partial x^{2}}, \tag{2.30}
\end{equation*}
$$

which is the continuum elastic wave equation with the velocity of sound given by

$$
\begin{equation*}
v=\sqrt{\frac{C}{M}} a . \tag{2.31}
\end{equation*}
$$

((Note)) Units

$$
[\mathrm{C}]=\mathrm{N} / \mathrm{m} . \quad[\mathrm{M}]=\mathrm{N} \mathrm{~s}^{2} / \mathrm{m} . \quad[\mathrm{a}]=\mathrm{m} . \quad[\mathrm{v}]=\mathrm{m} / \mathrm{s}
$$

The solution of Eq.(2.30) is given by

$$
u(x, t)=f(x-v t),
$$

where $f$ is an arbitrary function.

## 3. Eigenvalue-problem; solution using Mathematica

We solve the eigenvalue-problem given by

$$
\begin{equation*}
\omega^{2} u_{s}=K\left(-u_{s+1}+2 u_{s}-u_{s-1}\right), \tag{3.1}
\end{equation*}
$$

where $s=1,2, \ldots, N, K=C / M$. These equations can be expressed using a matrix $M$ (for example, $N \times N$ ) and column matrix ( $1 \mathrm{x} N$ );

$$
\begin{equation*}
M \boldsymbol{U}=\omega^{2} \boldsymbol{U} \tag{3.2}
\end{equation*}
$$

Here $\omega^{2}$ is the eigenvalue. The number of eigenvalues is $N$. For example, the matrix $M$ and the column matrix $\boldsymbol{U}$ for $N=12$ (for example), are given by

$$
M=\left(\begin{array}{cccccccccccc}
2 K & -K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{3.3}\\
-K & 2 K & -K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -K & 2 K & -K & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -K & 2 K & -K & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -K & 2 K & -K & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -K & 2 K & -K & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -K & 2 K & -K & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -K & 2 K & -K & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -K & 2 K & -K & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K & 2 K & -K & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K & 2 K & -K \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -K & 2 K
\end{array}\right),
$$

and

$$
\boldsymbol{U}=\left(\begin{array}{c}
u_{1}  \tag{3.4}\\
u_{2} \\
u_{3} \\
u_{4} \\
u_{5} \\
u_{6} \\
u_{7} \\
u_{8} \\
u_{9} \\
u_{10} \\
u_{11} \\
u_{12}
\end{array}\right), \text { for each eigenvalue. }
$$

Using the Mathematica (program called Eigenvalues), we can calculate the dispersion relation of $\omega_{\mathrm{k}}$ vs $k=(\pi / a)(n / N)$ with $N=1,2,3, \ldots, N-1, N$. For simplicity, here we choose $N=100$. For comparison we also make a plot of the solution given by $\omega=\sqrt{4 C / M}|\sin (k a / 2)|$. We find that there is a good agreement between these two curves. $K=C / M$.

## ((Mathematica-1))

Phonon dispersion relation for one atom in the unit cell.


Fig. 8 The dispersion relation with red (from Mathematica calculation) and blue $(\omega=\sqrt{4 K} \sin (k a / 2)$ with $K=C / M)$. The $x$ axis is $k=(\pi / a)(n / N)$ with $n=1$, $2, \ldots, N(=100)$. The $y$ axis is $\omega /(2 \sqrt{K})$. The agreement between these two results are excellent.

## ((Mathematica-2))

Here we use the Mathematica program (Eigensystem) to obtain the eigenvalues and the corresponding eigenfunction $\boldsymbol{U}$. For each eigenvalue, we can calculate the deviation $\left(u_{\mathrm{s}}\right)$ of the position of the atom from the equilibrium position, as a function of the equilibrium position $x_{\mathrm{s}}=s a$, where $s=1,2,3, \ldots \ldots, N$. Here we consider the eigenvalue problem $(N=50)$. We show the plot of $\boldsymbol{U}_{\mathrm{n}}=\left(u_{1}, u_{2}, u_{3}, \ldots, u_{\mathrm{N}}\right)$ as a function of the location $\left(x_{\mathrm{s}}=s a\right)$, where $n=1,2, \ldots, N .\|\mathbf{u}\|=1$. We find that the wavelength $\lambda_{\mathrm{n}}$ is given by $\lambda_{\mathrm{n}}=$ $2 N a / n$. The corresponding wavenumber $q_{\mathrm{n}}$ is equal to $k_{\mathrm{n}}=\left(2 \pi / \lambda_{\mathrm{n}}\right)=(\pi / a)(\mathrm{n} / N)$. The deviation $u_{\mathrm{s}}$ is well described by the form $\sin [\pi s(\mathrm{n} / N)]=\sin \left[k_{\mathrm{n}}(s a)\right]\left(\approx e^{i k_{n} s a}\right)$.


Fig. 9 Plot of $\boldsymbol{u}=\left(u_{1}, u_{2}, u_{3}, \ldots ., u_{50}\right)$ as a function of $x_{s}=s a$ for the eigenvalues $\omega_{1}(\mathrm{i}=$ $1,2,3, \ldots, 10) . \omega_{1}<\omega_{2}<\ldots<\omega_{10}$. Note that $\boldsymbol{u}$ is normalized $(\|\mathbf{u}\|=1) . N=50$.





\{4\}

\{8\}


| $\{10\}$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| u |  |  |  |  |  |
| 0.2 |  |  |  |  |  |
| 0.1 |  |  |  |  |  |
| 0.0 |  |  |  |  |  |
| -0.1 . | 10 | 20 | 30 |  | 0 a |
| -0.2 |  |  |  |  |  |

Fig. 10
Plot of $u=\left(u_{1}, u_{2}, u_{3}, \ldots, u_{50}\right)$ as a function of $x_{\mathrm{s}}=s a$ for the eigenvalues $\omega_{\mathrm{i}}(\mathrm{i}=1,2,3, \ldots, 10) . \omega_{1}<\omega_{2}<\ldots<\omega_{10}$. The wavenumber $k_{\mathrm{n}}=(\pi / N a) n$. The value of $n$ is denoted in each figure.


Fig. 11 Plot of $u=\left(u_{1}, u_{2}, u_{3}, \ldots, u_{50}\right)$ as a function of $x_{\mathrm{s}}=s a$ for the eigenvalues $\omega_{\mathrm{n}}(\mathrm{n}=5,10,15,20,25,30,35,40,45,50) . \omega_{5}<\omega_{10}<\ldots<\omega_{50} .\|u\|=1 . N=$ 50. The wavenumber $k_{\mathrm{n}}=(\pi / N a) n$. The value of $n$ is denoted in each figure.

## ((Note))

The eigenvalue problems for several simple systems (chains of 2 atoms, 3 atoms, 4 atoms, and 5 atoms) is discussed in detail in my web site,
http://bingweb.binghamton.edu/~suzuki/pdffiles/Oscillation_and_Waves_Note.pdf

## 4. First Brillouin zone and group velocity <br> 4.1 Definition of the group velocity

The transmission velocity of a wave pocket is the group velocity

$$
\begin{equation*}
\mathbf{v}_{g}=\frac{\partial \omega}{\partial \mathbf{k}}=\nabla_{\mathbf{k}} \omega(\mathbf{k}) \tag{4.1}
\end{equation*}
$$

This is the velocity of energy propagation in the medium.

$$
\begin{align*}
& v_{g}=\frac{\partial \omega}{\partial k}=\sqrt{\frac{C a^{2}}{M}} \cos \left(\frac{1}{2} k a\right),  \tag{4.2}\\
& v_{g}=0 \text { at } k=\frac{\pi}{a}
\end{align*}
$$

The wave is a standing wave: zero net transmission velocity for a standing wave. Note that the phase velocity is defined by $v_{p}=\omega / k$.
Long wavelength limit
When $k a \ll 1$,

$$
\begin{equation*}
1-\cos k a=1-\left[1-\frac{1}{2!}(k a)^{2}\right] \cong \frac{1}{2}(k a)^{2} . \tag{4.3}
\end{equation*}
$$

Then

$$
\begin{align*}
& \omega^{2}=\frac{2 C}{M} \frac{1}{2} k^{2} a^{2}=\frac{C}{M} k^{2} a^{2} \quad \text { or } \quad \omega=\sqrt{\frac{C a^{2}}{M}} k,  \tag{4.4}\\
& v_{g}=\frac{\partial \omega}{\partial k}=\sqrt{\frac{C}{M}} a=v_{0} \quad \text { at } \quad k a=0 . \tag{4.5}
\end{align*}
$$

The velocity of sound is independent of frequency in this limit. Thus $\omega=v k$, exactly as in the continuum theory of elastic waves --- in the continuum limit $a=0$ and thus $k a=0$.

### 4.2 The physical meaning of the first Brillouin zone

We discuss the physical meaning of the first Brillouin zone. To this end, we consider the case of $u_{s}$ with

$$
k=k_{0}=\frac{\pi}{3 a} \text { and } k=k_{0}+G=\frac{\pi}{3 a}+\frac{2 \pi}{a}=\frac{7 \pi}{3 a} .
$$

(i) $k=k_{0}=\frac{\pi}{3 a}$.

At $t=0$, we have $u_{s}=u e^{i(k s a-\omega t)}$ at $t=0$. We make a plot of

$$
\operatorname{Re}\left[u_{s}\right]=\operatorname{Re}\left[e^{i\left(\frac{\pi}{3} s-\omega t\right)}\right]=\cos \left(\frac{\pi}{3} s\right),
$$

as a function of $s$, where $s=0,1,2,3, \ldots$, where $u=1$ in Fig. 17 .
(ii) $k=k_{0}+G=\frac{\pi}{3 a}+\frac{2 \pi}{a}=\frac{7}{3 a} \pi$

We also make a plot of $\operatorname{Re}\left[u_{s}\right]=\operatorname{Re}\left[e^{i\left(\frac{7 \pi}{3} s\right)}\right]=\cos \left(\frac{7 \pi}{3} s\right)$ at $t=0$ in Fig.17.


Fig.12(a) Plot of $\operatorname{Re}\left[e^{i(k s a-\omega t)}\right]$ at $t=0$ for $\mathrm{k}=k_{0}+G=7 \pi / 3$ (the red solid line) and $k$ $=k_{0}=\pi / 3 a$ (the dotted blue line).

As shown in Fig.12, the wave represents by the solid curve ( $k=7 \pi / 3 a$ ) conveys no information not give by the dashed curve $(k=\pi / 3 a)$. Because the wave displacement is defined only at lattice points, the propagation of a large wavenumber $\left(k_{0}+G\right)$ lying outside the first Brillouin zone is identical to a short wavenumber $k_{0}$ lying inside the first Brillouin zone. In a continuum, the amplitudes would have a value everywhere, as represented by the continuous lines, so that both wavenumbers would be distinguishable.
((Exmple))


Fig.12(b) Identity of long and short waves on a lattice. $N=20 . a=1 . k=3 \pi /(10 a)$. $G=2 \pi / a . k^{\prime}=k+G=23 \pi /(10 a)$. $\operatorname{Re}\left[u_{\mathrm{s}} / u(0)\right]$ for $k=3 \pi /(10 \mathrm{a})$ (green line). $\operatorname{Re}\left[u_{\mathrm{s}} / u(0)\right]$ for $k=23 \pi /(10 \mathrm{a})$ (black line). These two lines intersect each other at the integers $s(\mathrm{~s}=1,2,3, \ldots \ldots ., 20)$.

### 4.3 Standing wave

At the zone boundary $k= \pm \pi / a$, the group velocity is equal to zero, implying that no energy is propagated. The amplitude $\operatorname{Re}\left[u_{\mathrm{s}}\right]$ at time $t$ is described

$$
\begin{equation*}
\operatorname{Re}\left[u_{s}\right]=\operatorname{Re}\left[u e^{i(k s a-\omega t)}\right]=\operatorname{Re}\left[u(-1)^{s} e^{-i \omega t}\right]=u(-1)^{s} \cos \omega t \tag{4.1}
\end{equation*}
$$

The alternate atoms oscillate in opposite phases, because $(-1)^{s}=-1$ for odd integers $s$ and $(-1)^{\mathrm{s}}=1$ for even integers $s$. Note that the wave is a standing wave. A standing wave is an example of wave motion with zero group velocity. The wave moves neither to the right nor to the left. This situation is equivalent to Bragg reflections of x-ray.


Fig. 13 The condition for the Bragg reflection in the reciprocal lattice. $k-k^{\prime}=2 \pi / a\left(\right.$ or $k=\pi / a$ and $\left.k^{\prime}=-\pi / a\right)$

The Bragg reflection arises when $k= \pm \pi / a$. Even if we excite only the state $k=\pi / a$. we should obtain $k=-\pi / a$ through the Bragg reflection. The superposition of these two wave leads to a standing wave,

$$
\begin{align*}
\frac{1}{\sqrt{2}}\left[e^{i(k s a-\omega t)} \pm e^{i\left(k^{\prime} s a-\omega t\right)}\right] & =\frac{1}{\sqrt{2}} e^{-i \omega t}\left[e^{i s \pi} \pm e^{-i s \pi}\right] \\
& =\frac{1}{\sqrt{2}} e^{-i \omega t}\left\{\begin{array}{l}
2 \cos s \pi \\
2 i \sin s \pi
\end{array}\right\}=\sqrt{2} e^{-i \omega t}\left\{\begin{array}{l}
\cos s \pi \\
i \sin s \pi
\end{array}\right. \tag{4.6}
\end{align*}
$$

This implies that the wave cannot propagate in a lattice, but through successive reflections back and forth, a standing wave is set up.

### 4.4 General property of the group velocity

There are two main features of the phonon dispersion relation;
(i) $E=\hbar \omega_{k}$ is an even function of $k ; \hbar \omega_{-k}=\hbar \omega_{k}$ or $E_{-k}=E_{k}$, where $\hbar$ is the Dirac's constant $(\hbar=h / 2 \pi)$ and $h$ is the Planck's constant. This is related to the fact that the equations of motion are invariant with respect to the time reversal.
(ii) Periodicity in $E_{k+G}=E_{k}$ with $G[=(2 \pi / a)$ times integer $]$ is the reciprocal lattice.


Fig. $14 \quad$ Values of $k$ in the first Brillouin zone related by the symmetrical relation ( $E_{k+G}=E_{k}$ and $E_{-k}=E_{k}$ ). $k= \pm \pi / a$ is the boundary of the first Brillouin zone $(|k| \leq \pi / a)$.

We now consider the value of the group velocity at the Brillouin zone boundary. From the condition $E_{k}=E_{-k}$, we have $E(3)=E(4)$. From the condition $E_{k}=E_{k+G}$, we have $E(3)=E(5)$. Therefore, we have $E(4)=E(5)$. On taking $\delta \rightarrow 0$, the group velocity at the boundary of Brillouin zone is defined as $[E(5)-E(4)] / 2 \delta$, which reduces to zero $\left(\mathrm{d} E_{k} / \mathrm{d} k \rightarrow 0\right)$.
((Note))
It follows that from the condition $\left(E_{k}=E_{-k}\right)$, we have $E(1)=E(2)$. On taking $\delta \rightarrow 0$, the group velocity defined by $[E(2)-E(1)] / 2 \delta$ reduces to zero $\left(\mathrm{d} E_{k} / \mathrm{d} k \rightarrow 0\right)$. On applying the periodicity condition $E_{k}=E_{k+G}$ this result can immediately be extended as follows. $\mathrm{d} E_{\mathrm{k}} / \mathrm{d} k \rightarrow 0$ at $k=0, \pm 2 \pi / a, \pm 4 \pi / a, \ldots$. . This prediction (only from the symmetry consideration) that the group velocity is equal to zero at $k=0$, may be inconsistent with the result derived from the linear chain model above described. In the linear chain model, it is predicted that there is a discontinuous jump in the group velocity at $k=0$, from $-v_{0}$ $\left(v_{0}=a \sqrt{C / M}\right)$ at $k=0^{-}$to $-v_{0}$ at $k=0^{+}$. We note that this motion at $k=0$ is a translation of the crystal as a whole and it is therefore not property of a vibration.

## 5. Determination of force constants

### 5.1 The system with the nearest neighbor interaction

We can make a statement about the range of the forces from the observed dispersion relation for $\omega$. The generalization of the dispersion relation to $p$ nearest planes is found to be

$$
\begin{align*}
& \omega_{k}^{2}=\frac{2}{M} \sum_{p>0}[1-\cos (p k a)] C_{p},  \tag{5.1}\\
& M \int_{-\pi / a}^{\pi / a} d k \omega_{k}^{2} \cos (r k a)=2 \sum_{p>0} C_{p} \int_{-\pi / a}^{\pi / a} d k[1-\cos (p k a)] \cos (r k a) . \tag{5.2}
\end{align*}
$$



Fig. 15 Linear chain having the nearest-neighbor coupling $\left(C_{1}\right)$, the second nearest-neighbor coupling $\left(\mathrm{C}_{2}\right)$, and the third nearest-neighbor coupling $\left(C_{3}\right)$, and the

Here

$$
\begin{equation*}
\int_{-\pi / a}^{\pi / a} d k \cos (r k a)=\left[\frac{\sin (r k a)}{r a}\right]_{-\pi / a}^{\pi / a}=0 \tag{5.3}
\end{equation*}
$$

and

$$
\begin{align*}
\int_{-\pi / a}^{\pi / a} d k \cos (p k a) \cos (r k a) & =\int_{-\pi / a}^{\pi / a} d k \frac{1}{2}[\cos \{(p+r) k a\}+\cos \{(p-r) k a\}] \\
& =\frac{1}{2} \int_{-\pi / a}^{\pi / a} d k \cos [(p-r) k a]=\left\{\begin{array}{cl}
\frac{1}{2} \frac{2 \pi}{a} & \text { for } p=r \\
0 & \text { otherwise }
\end{array}\right. \tag{5.4}
\end{align*}
$$

Thus

$$
\begin{equation*}
M \int_{-\pi / a}^{\pi / a} d k \omega_{k}^{2} \cos (r k a)=-2 \pi \frac{C_{r}}{a}, \tag{5.5}
\end{equation*}
$$

or

$$
\begin{equation*}
C_{p}=-\frac{M a}{2 \pi} \int_{-\pi / a}^{\pi / a} d k \omega_{k}^{2} \cos (p k a) \tag{5.6}
\end{equation*}
$$

gives the force constant at range $p a$, for a structure with a monatomic basis.

### 5.2 System with long-ranged interactions



Fig. 16 Linear chain with the nearest-neighbor and the next nearest-neighbor interactions.

Here we assume that there are the second-, third,...nearest neighbor interactions in the system in addition to the nearest neighbor-interaction. Then the Lagrangian $L$ is given by

$$
\begin{align*}
L & =T-V \\
& =\sum_{s} \frac{1}{2} M \dot{u}_{s}^{2}-\sum_{s} \sum_{p=1}^{\infty} \frac{1}{2} C_{p}\left(u_{s}-u_{s+p}\right)^{2}  \tag{5.6}\\
& =\sum_{s} \frac{1}{2} M \dot{u}_{s}^{2}-\sum_{p=1}^{\infty} \frac{1}{2} C_{p}\left\{\left(u_{s}-u_{s+p}\right)^{2}+\left(u_{s}-u_{s-p}\right)^{2}\right\}-\cdots
\end{align*} .
$$

The Lagrange equation is

$$
\begin{equation*}
\frac{d}{d t} \frac{\partial L}{\partial \dot{u}_{s}}=\frac{\partial L}{\partial u_{s}} \tag{5.7}
\end{equation*}
$$

or

$$
\begin{align*}
M \ddot{u}_{s} & =-\sum_{p=1}^{\infty} C_{p}\left(u_{s}-u_{s+p}\right)-\sum_{p=1}^{\infty} C_{p}\left(u_{s}-u_{s-p}\right)  \tag{5.8}\\
& =\sum_{p=1}^{\infty} C_{p}\left(u_{s+p}-2 u_{s}+u_{s-p}\right)
\end{align*}
$$

Here we assume a traveling wave given by $u_{s}=u e^{i(k s a-\omega t)}$. Then we have

$$
\begin{equation*}
-M \omega^{2} u_{s}=\sum_{p=1}^{\infty} C_{p}\left(e^{i p k a}-2+e^{-i p k a}\right) u_{s} \tag{5.9}
\end{equation*}
$$

or

$$
\begin{equation*}
-M \omega^{2}=\sum_{p=1}^{\infty} C_{p}\left(e^{i p k a}-2+e^{-i p k a}\right) \tag{5.10}
\end{equation*}
$$

or

$$
\begin{equation*}
\omega^{2}=\frac{2}{M} \sum_{p=1}^{\infty} C_{p}(1-\cos p k a)=\frac{2 C_{1}}{M}[1-\cos (k a)]+\frac{2 C_{2}}{M}[1-\cos (2 k a)]+\ldots, \tag{5.11}
\end{equation*}
$$

which includes the higher harmonics terms through long-range spring constants. The value of $C_{p}$ can be determined experimentally from the phonon dispersion relation.

## 6A.. Vibration of square lattice ${ }^{4}$



Fig. 17 Square array of lattice constant $a$. The displacements are normal to the plane of the lattice.

We consider transverse vibrations of a planar lattice of rows and columns of identical atoms, and let $\boldsymbol{u}(l, m)$ denote the displacement normal to the plane of the lattice of the atom in the $l$-th column and $m$-th row. The mass of each atom is $M$, and $C$ is the force constant for nearest neighbor atoms.

The equation of motion is expressed by

$$
\begin{align*}
M \frac{d^{2} \boldsymbol{u}(l, m)}{d t^{2}} & =C\{[\boldsymbol{u}(l+1, m)+\boldsymbol{u}(l-1, m)-2 \boldsymbol{u}(l, m)] \\
& +[\boldsymbol{u}(l, m+1)+\boldsymbol{u}(l, m-1)-2 u(l, m)] \tag{6.1}
\end{align*}
$$

We assume that the solution of $\boldsymbol{u}(l, m)$ is given by

$$
\begin{equation*}
\boldsymbol{u}(l, m)=\boldsymbol{u}(0) \exp \left[i\left(l k_{x} a+m k_{y} a-\omega t\right),\right. \tag{6.2}
\end{equation*}
$$

where $a$ is the spacing between nearest-neighbor atoms. The equation of motion is satisfied only if

$$
\begin{equation*}
\omega=\sqrt{\frac{2 C}{M}} \sqrt{2-\cos \left(k_{x} a\right)-\cos \left(k_{y} a\right)} \tag{6.3}
\end{equation*}
$$

where the first Brillouin zone is

$$
\begin{equation*}
-\frac{\pi}{a} \leq k_{x} \leq \frac{\pi}{a} \text { and }-\frac{\pi}{a} \leq k_{x} \leq \frac{\pi}{a} \tag{6.4}
\end{equation*}
$$



Fig.18a Brillouin zone of the two-dimensional square lattice with a lattice constant a. Bragg reflections occur at the zone boundary of the first Brillouin zone.


Fig.18b Three-dimensional plot of the dispersion relation (of the 2D square lattice. The height $(\omega)$ is plotted as a function $k_{x} a$ vs $k_{y} a$ (in the 2 D wavevectorplane).


Fig.18c Contour plot of constant energy in the reciprocal lattice plane. The first Brillouin zone is shown by the red solid line. The energy of the point (the corner of the first Brillouin zone) is higher than that of the point (the middle of the zone boundary).

## 6B.Two-dimensional triangular lattice (supplement)

Problem 5-10 (H. Myer, Introductory Solid State Physics, Taylor \& Francis, 1990. London, U.K.).

Consider the following two-dimensional close-packed lattice and its unit cell:


Fig.19a Real space for the 2D triangular lattice.
Primitive cell vectors $\boldsymbol{a}_{1}$ and $\boldsymbol{a}_{2}$ are shown. Determine the first Brillouin zone appropriate to this lattice. Arrange that the reciprocal lattice vectors $\boldsymbol{b}_{1}$ and $\boldsymbol{b}_{2}$ are correctly oriented. Then, using the nearest-neighbor approximation for a vibrating net of point masses, determine the dispersion equation [let the central mass points have coordinates $\left(l \boldsymbol{a}_{1}, m \boldsymbol{a}_{2}\right)$ ]. Calculate the frequency at two-non-equivalent symmetry points on the zone boundary.
((Solution))
The primitive lattice vectors:

$$
\boldsymbol{a}_{1}=a\left(\cos \left(0^{\circ}\right), \sin \left(0^{\circ}\right)\right)=a(1,0), \quad \boldsymbol{a}_{2}=a\left(\cos \left(60^{\circ}\right), \sin \left(60^{\circ}\right)\right)=a\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)
$$

The reciprocal lattice vectors;

$$
\begin{aligned}
& \boldsymbol{b}_{1}=\frac{4 \pi}{\sqrt{3} a}\left(\left(\cos \left(-30^{\circ}\right), \sin \left(-30^{\circ}\right)\right)=\frac{4 \pi}{\sqrt{3} a}\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right),\right. \\
& \boldsymbol{b}_{2}=\frac{4 \pi}{\sqrt{3} a}\left(\left(\cos \left(90^{\circ}\right), \sin \left(90^{\circ}\right)\right)=\frac{4 \pi}{\sqrt{3} a}(0,1)\right.
\end{aligned}
$$

We set up the equation of the motion for the atom at the center. There are nearest neighbor interactions from atoms surrounding the atom at the center.

$$
\begin{align*}
M \ddot{\boldsymbol{u}}(l, m)= & C[\boldsymbol{u}(l+1, m)+\boldsymbol{u}(l, m+1)+\boldsymbol{u}(l-1, m+1) \\
& +\boldsymbol{u}(l-1, m)+\boldsymbol{u}(l, m-1)+\boldsymbol{u}(l+1, m-1)-6 \boldsymbol{u}(l, m)] \tag{1}
\end{align*}
$$

where $M$ is the mass of atom, $C$ is the spring constant, and $\boldsymbol{u}_{l+1, m}$ is the displacement vector for the atom at the position $\left(l \boldsymbol{a}_{1}, m \boldsymbol{a}_{2}\right)$. We assume the solution of the form as

$$
\boldsymbol{u}(l, m)=\boldsymbol{u}(0) \exp \left[i \boldsymbol{k} \cdot\left(l \boldsymbol{a}_{1}+m \boldsymbol{a}_{2}\right)-\omega t\right] .
$$

with

$$
\begin{aligned}
& \boldsymbol{u}(l \pm 1, m)=\exp \left( \pm i \boldsymbol{k} \cdot \boldsymbol{a}_{1}\right) \boldsymbol{u}_{l, m}, \\
& \boldsymbol{u}(l, m \pm 1)=\exp \left( \pm i \boldsymbol{k} \cdot \boldsymbol{a}_{2}\right) \boldsymbol{u}_{l, m} .
\end{aligned}
$$

Substituting these into Eq.(1), we get

$$
\begin{aligned}
-M \omega^{2} \boldsymbol{u}(l, m)= & C\left[\exp \left(i \boldsymbol{k} \cdot \boldsymbol{a}_{1}\right)+\exp \left(i \boldsymbol{k} \cdot \boldsymbol{a}_{2}\right)+\exp \left[i\left(-\boldsymbol{k} \cdot \boldsymbol{a}_{1}+\boldsymbol{k} \cdot \boldsymbol{a}_{2}\right)\right.\right. \\
& +\exp \left(-i \boldsymbol{k} \cdot \boldsymbol{a}_{1}\right)+\exp \left(-i \boldsymbol{k} \cdot \boldsymbol{a}_{2}\right)+\exp \left[i\left(\boldsymbol{k} \cdot \boldsymbol{a}_{1}-\boldsymbol{k} \cdot \boldsymbol{a}_{2}\right)-6\right] \boldsymbol{u}(l, m)
\end{aligned}
$$

Then we have the dispersion relation of $\omega$ vs $\boldsymbol{k}$ as

$$
\frac{M \omega^{2}}{2 C}=\left[3-\cos \left(\boldsymbol{k} \cdot \boldsymbol{a}_{1}\right)-\cos \left(\boldsymbol{k} \cdot \boldsymbol{a}_{2}\right)-\cos \left(\boldsymbol{k} \cdot \boldsymbol{a}_{1}-\boldsymbol{k} \cdot \boldsymbol{a}_{2}\right)\right]
$$

where

$$
\boldsymbol{k}=k_{x} \boldsymbol{e}_{x}+k_{y} \boldsymbol{e}_{y}
$$

The Brillouin zone for the triangular lattice is shown below.


Fig.19b Relation between the real space (lattice vectors $\boldsymbol{a}_{1}$ and $\boldsymbol{a}_{2}$ ) and the reciprocal lattice ( $\boldsymbol{b}_{1}$ and $\boldsymbol{b}_{2}$ ). The magnitude of $\boldsymbol{a}_{1}$ and $\boldsymbol{b}_{1}$ are chosen appropriately.
(a) At the middle of a Brillouin zone (zone boundary)

$$
\begin{aligned}
& \boldsymbol{k}=\frac{\boldsymbol{b}_{1}+\boldsymbol{b}_{2}}{2} . \\
& \boldsymbol{k} \cdot \boldsymbol{a}_{1}=\frac{1}{2} \boldsymbol{a}_{1} \cdot \boldsymbol{b}_{1}=\pi, \quad \boldsymbol{k} \cdot \boldsymbol{a}_{2}=\frac{1}{2} \boldsymbol{a}_{2} \cdot \boldsymbol{b}_{2}=\pi .
\end{aligned}
$$

Then we get

$$
\frac{M \omega^{2}}{2 C}=3-\cos (\pi)-\cos (\pi)-\cos (\pi-\pi)=4
$$

or

$$
\omega^{2}=\frac{8 C}{M}
$$

(b) At the corner of a Brillouin zone.

$$
\begin{aligned}
& \boldsymbol{k}=\frac{\boldsymbol{b}_{1}+2 \boldsymbol{b}_{2}}{3} \\
& \boldsymbol{k} \cdot \boldsymbol{a}_{1}=\frac{1}{3} \boldsymbol{a}_{1} \cdot \boldsymbol{b}_{1}=\frac{2 \pi}{3}, \quad \boldsymbol{k} \cdot \boldsymbol{a}_{2}=\frac{2}{3} \boldsymbol{a}_{2} \cdot \boldsymbol{b}_{2}=\frac{4 \pi}{3}
\end{aligned}
$$

Then we get

$$
\frac{M \omega^{2}}{2 C}=3-\cos \left(\frac{2 \pi}{3}\right)-\cos \left(\frac{4 \pi}{3}\right)-\cos \left(-\frac{2 \pi}{3}\right)=\frac{9}{2}
$$

or

$$
\omega^{2}=\frac{9 C}{M}
$$



Fig.19c Contour plot of constant energy in the reciprocal lattice plane. The first brillouin zone is shown by the blue solid line. The energy of the point K (the corner of the first Brillouin zone) is higher than that of the point $M$ ( the middle of the zone boundary).
((Note))
The Brillouin zone for the 2D triangular lattice is shown below.


Fig.19d Reciprocal lattice. Brillouin zone. M (the zone boundary) and K (the corner). The numbers are related to the number of Brillouin zone.

## 7. Two atoms per primitive basis

We consider a cubic crystal where atoms of mass $M_{1}$ lie on one set of planes and atoms of mass $M_{2}$ lie on planes interleaved between those of the first set.


Fig.20(a)


Fig.20(b)


Fig.20(c)

Fig. 20
(a), (b) Diatomic linear chain, There are two atoms with masses $M_{1}$ and $M_{2}$ per unit cell (lattice constant $a$ ). (b) The plane of atoms stacked alternatively (lattice constant a).

The Lagrangian of the system is given by

$$
\begin{aligned}
L= & \frac{1}{2} M_{1} \dot{X}_{s}^{2}+\frac{1}{2} M_{2} \dot{Y}_{s}^{2}+\cdots \\
& -\frac{1}{2} C\left[\left(X_{s+1}-Y_{s}-\frac{a}{2}\right)^{2}+\left(Y_{s}-X_{s}-\frac{a}{2}\right)^{2}+\left(X_{s}-Y_{s-1}-\frac{a}{2}\right)^{2}+\cdots\right]
\end{aligned}
$$

or

$$
\begin{align*}
L & =\frac{1}{2} M_{1} \dot{u}_{s}^{2}+\frac{1}{2} M_{2} \dot{v}_{s}^{2}+\cdots  \tag{7.1}\\
& -\frac{1}{2} C\left[\left(u_{s+1}-v_{s}\right)^{2}+\left(v_{s}-u_{s}\right)^{2}+\left(u_{s}-v_{s-1}\right)^{2}+\cdots\right]
\end{align*}
$$

Note that the coordinates of the positions of the atom with mass $M_{1}$ and atom with mass $M_{2}$ are described by

$$
X_{s}=s a+u_{s}, \quad Y_{s}=s a+\frac{a}{2}+v_{s} .
$$

or

$$
Y_{s}-X_{s}-\frac{a}{2}=\left(s a+\frac{a}{2}+v_{s}\right)-\left(s a+u_{s}\right)-\frac{a}{2}=v_{s}-u_{s} .
$$

The Lagrange's equations are as follows.

$$
\frac{d}{d t} \frac{\partial L}{\partial \dot{u}_{s}}=\frac{\partial L}{\partial u_{s}}
$$

or

$$
\begin{equation*}
M_{1} \ddot{u}_{s}=-\frac{1}{2} C\left[-2\left(v_{s}-u_{s}\right)+2\left(u_{s}-v_{s-1}\right)\right], \tag{7.2}
\end{equation*}
$$

or

$$
\begin{equation*}
M_{1} \ddot{u}_{s}=-C\left(-v_{s}+2 u_{s}-v_{s-1}\right)=C\left(v_{s}+v_{s-1}-2 u_{s}\right) . \tag{7.4}
\end{equation*}
$$

Similarly

$$
\begin{align*}
& \frac{d}{d t} \frac{\partial L}{\partial \dot{v}_{s}}=\frac{\partial L}{\partial v_{s}}  \tag{7.5}\\
& M_{2} \ddot{v}_{s}=-\frac{1}{2} C\left[-2\left(u_{s+1}-v_{s}\right)+2\left(v_{s}-u_{s}\right)\right]=C\left(u_{s+1}-2 v_{s}+u_{s}\right) . \tag{7.6}
\end{align*}
$$

Let $a$ denote the repeat distance of the lattice in the direction normal to the lattice plane. Equation of motion is given by

$$
\left\{\begin{array}{l}
M_{1} \frac{d^{2} u_{s}}{d t^{2}}=C\left(v_{s}+v_{s-1}-2 u_{s}\right)  \tag{7.7}\\
M_{2} \frac{d^{2} v_{s}}{d t^{2}}=C\left(u_{s+1}+u_{s}-2 v_{s}\right)
\end{array}\right.
$$

Here we assume that each plane interacts only with its nearest-neighbor and that the force constants are identical between all pairs of nearest-neighbor planes. Suppose that $u_{s}$ and $v_{s}$ have the forms of

$$
\left\{\begin{array}{l}
u_{s}=u e^{i(s k a-\omega t)}  \tag{7.8}\\
v_{s}=v e^{i(s k a-\omega t)}
\end{array}\right.
$$

Then

$$
\begin{align*}
& -\omega^{2} M_{1} u=C v\left(1+e^{-i k a}\right)-2 C u \\
& -\omega^{2} M_{2} v=C u\left(e^{i k a}+1\right)-2 C v \tag{7.9}
\end{align*}
$$

When the determinant of the coefficients of $u$ and $v$ vanishes, the homogeneous linear equations have a non-trivial solution.

$$
\left|\begin{array}{cc}
2 C-M_{1} \omega^{2} & -C\left(1+e^{-i k a}\right)  \tag{7.10}\\
-C\left(1+e^{i k a}\right) & 2 C-M_{2} \omega^{2}
\end{array}\right|=0
$$

or

$$
\begin{equation*}
M_{1} M_{2} \omega^{4}-2 C\left(M_{1}+M_{2}\right) \omega^{2}+2 C^{2}(1-\cos k a)=0 . \tag{7.11}
\end{equation*}
$$

We examine this equation in the limiting cases
(i) $k a \ll 1$
(ii) $k a= \pm \pi$ at the zone boundary
(i) $k a \ll 1$

$$
\begin{align*}
& 1-\cos k a \approx \frac{1}{2} k^{2} a^{2} \\
& M_{1} M_{2} \omega^{4}-2 C\left(M_{1}+M_{2}\right) \omega^{2}+C^{2} k^{2} a^{2}=0 \tag{7.12}
\end{align*}
$$

or

$$
\begin{align*}
\omega^{2} & =\frac{C\left(M_{1}+M_{2}\right) \pm \sqrt{C^{2}\left(M_{1}+M_{2}\right)^{2}-M_{1} M_{2} C^{2} k^{2} a^{2}}}{M_{1} M_{2}} \\
& =\frac{C\left(M_{1}+M_{2}\right)}{M_{1} M_{2}}\left[1 \pm \sqrt{1-\frac{M_{1} M_{2} C^{2} k^{2} a^{2}}{C^{2}\left(M_{1}+M_{2}\right)^{2}}}\right]  \tag{7.13}\\
& \approx \frac{C\left(M_{1}+M_{2}\right)}{M_{1} M_{2}}\left[1 \pm\left\{1-\frac{M_{1} M_{2} C^{2} k^{2} a^{2}}{2 C^{2}\left(M_{1}+M_{2}\right)^{2}}\right\}\right]
\end{align*}
$$

Then we have the two roots for $k a « 1$

$$
\begin{align*}
& \omega^{2}=2 C\left(\frac{1}{M_{1}}+\frac{1}{M_{2}}\right)(\text { optical branch })  \tag{7.14}\\
& \omega^{2}=\frac{C}{2\left(M_{1}+M_{2}\right)} k^{2} a^{2}(\text { acoustic branch }) . \tag{7.15}
\end{align*}
$$

${ }^{*}$ ) The extent of the first Brillouin zone is $-\frac{\pi}{a} \leq k \leq \frac{\pi}{a}$.
(ii) $k a= \pm \pi$

$$
\begin{align*}
& M_{1} M_{2} \omega^{4}-2 C\left(M_{1}+M_{2}\right) \omega^{2}+4 C=0 \\
& \left(M_{1} \omega^{2}-2 C\right)\left(M_{2} \omega^{2}-2 C\right)=0 \tag{7.16}
\end{align*}
$$

or

$$
\begin{equation*}
\omega^{2}=\frac{2 C}{M_{1}} ; \quad \omega^{2}=\frac{2 C}{M_{2}} . \tag{7.17}
\end{equation*}
$$



$$
\text { for } M_{1}>M_{2}
$$

Fig. 21 The dispersion curve ( $\omega$ vs $k$ ) for a diatomic linear chain with nearest neighbor atoms interacting with interaction $C$ only. The masses of the atoms are $M_{1}$ and $M_{2}$; The first Brillouin zone is the segment between $-\pi / a$ and $\pi / a$. There is an acoustic mode (lower branch) and an optical mode (upper branch).

We consider the particle displacement in TA and TO branches. For the optical branch at $k$ $=0$, we have

$$
-\omega^{2} M_{1} u=2 C v-2 C u
$$

or

$$
\begin{equation*}
\frac{u}{v}=\frac{2 C}{2 C-M_{1} \omega^{2}}=\frac{2 C}{2 C-M_{1} 2 C\left(\frac{1}{M_{1}}+\frac{1}{M_{2}}\right)}=-\frac{M_{2}}{M_{1}} \quad: \text { out-of-phase } \tag{7.18a}
\end{equation*}
$$

We also note that the velocity of the center of mass defined by

$$
\begin{equation*}
v_{C M}=\frac{M_{1} u+M_{2} v}{M_{1}+M_{2}} . \tag{7.18b}
\end{equation*}
$$

is equal to zero.
For the acoustical branch at $k=0$, we have

$$
\begin{equation*}
-\omega^{2} M_{1} u=2 C v-2 C u \tag{7.19}
\end{equation*}
$$

Since $\omega^{2}=0$, then we have

$$
u=v . \quad: \text { in-phase }
$$



Fig. 22 Nature of the vibration in the acoustical and optical branch of a vibrational spectrum.

$$
\begin{equation*}
u_{s}=u e^{i(s k a-\omega t)}, v_{s}=v e^{i(s k a-\omega t)} \tag{7.20}
\end{equation*}
$$

Acoustical waves in a diatomic linear lattice ( $k=\frac{\pi}{2 a}$ )


Fig. 23 Transverse acoustic waves in a diatomic linear lattice.

$$
\begin{equation*}
u \cos (s k a)=v \cos (s k a) \tag{7.21}
\end{equation*}
$$

Optical waves in a diatomic linear lattice ( $k=\frac{\pi}{2 a}$ )


Fig. 24 Transverse optical waves in a diatomic linear lattice. Figure represents a snapshot of the wave motion.
$v_{c} \equiv \frac{M_{1} u+M_{2} v}{M_{1}+M_{2}}=0 \Rightarrow$ The velocity of center of mass does not move.
The atoms vibrate against each other, but their center of mass is fixed. If the two atoms carry opposite charges, we may excite a motion of this type with the electric field of a light wave.


Fig. 25 Origin of the name of optical mode
So that the branch is called the optical branch. The atoms (and their center of mass) move together, as in long wavelength acoustical vibrations, whence the term acoustical branch.


Fig. 26 Origin of the name of acoustic mode
This is a frequency gap at $k_{\max }= \pm \frac{\pi}{a}$ of the first Brillouin zone.
((Note))
If we look for solutions in the gap with $\omega$ real, then $\boldsymbol{k}$ will be complex, so that the wave is damped in space.
((Mathematica)) phonon dispersion for the system with two atoms in unit cell


Fig.27(a) The dispersion curve of the diatomic linear chain with two atoms in a unit cell. The ratio $M_{2} / M_{1}$ is varied as a parameter between $2,3,4, \ldots$ and 10 .
((Note-2)) Link
Phonon dispersion relation in the Brillouin zone (Wolfram Demonstrations Project) Phonon modes in one dimensional crystal with two atoms.
http://demonstrations.wolfram.com/PhononDispersionRelationInBrillouinZone/
((Example)) Change of the unit cell size from $2 a$ to $a$ when $\mathrm{M}_{2}$ become equal to $M_{1}$.

## E. Hanamura (Problems in Solid State Physics, in Japanese) Problem 3-1.

We consider a one-dimensional chain, where two kinds of atoms with different masses $M_{1}$ and $M_{2}$ are alternatively arranged. The size of unit cell is $2 a$, where $a$ is the nearest neighbor distance between two different atoms. There are $2 N$ atoms. $N$ atoms of the mass $M_{1}$, and $N$ atoms of the mass $M_{2}$. There is only the nearest neighbor interaction between these two atoms. We consider the limit that that $M_{1}$ becomes equal to $M_{2}\left(M_{1}=M_{2}=M\right)$. In this case, there are $2 N$ atoms of the mass $M$. The nearest neighbor distance is $a$.
Discuss the dispersion relation of these systems.


Fig.
$M_{2} \neq M_{1}$. The size of unit cell is $2 a$. There are N unit cells in the system. $L$ $=2 a N$.

First we consider the case (only the longitudinal wave) where $M_{1}$ is not equal to $M_{2}$. The size of unit cell is $2 a$. The Brillouin zone is in the range $|k|<\frac{\pi}{2 a}$. The total number of atoms are $2 N$, leading to the $2 N$ modes; 1 acoustic branch ( $N$ modes) and 1 optical branch ( $N$ mode). The size of the system is $L=2 N a$.

$$
\begin{aligned}
u_{s} & =u \exp [i(s 2 a k-\omega t\}] \\
v_{s} & =v \exp [i\{s(2 a+1) k-\omega t\}]
\end{aligned}
$$

We use the periodic boundary condition;

$$
u_{N+s}=u_{s}, \quad v_{N+s}=v_{s}
$$

or

$$
\exp [i(2 N a k)]=1
$$

or

$$
2 N a k=2 \pi n, \quad \text { or } \quad k=\frac{2 \pi n}{2 N a}=\frac{\pi}{a} \frac{n}{N},
$$

There are $N$ modes for each branch (1 optical and 1 acoustic branches). We choose

$$
|k|<\frac{\pi}{2 a} .
$$

where $k=0, \pm \pi /(a N), \pm 2 \pi /(a N)$. The dispersion relation is given by

$$
M_{1} M_{2} \omega^{4}-2 C\left(M_{1}+M_{2}\right) \omega^{2}+4 C^{2} \sin ^{2}(k a)=0
$$

For $k= \pm \frac{\pi}{2 a}$,

$$
M_{1} M_{2} \omega^{4}-2 C\left(M_{1}+M_{2}\right) \omega^{2}+4 C^{2}=0 .
$$

or

$$
\left(M_{1} \omega^{2}-2 C\right)\left(\left(M_{2} \omega^{2}-2 C\right)=0\right.
$$

or

$$
\omega=\sqrt{\frac{2 C}{M_{1}}} \quad \omega=\sqrt{\frac{2 C}{M_{2}}} .
$$

In the limit of $M_{1}=M_{2}=M$, these are equal $\left(\omega=\sqrt{\frac{2 C}{M}}\right)$.

For $k=0$,

$$
M_{1} M_{2} \omega^{4}-2 C\left(M_{1}+M_{2}\right) \omega^{2}=0
$$

or

$$
\omega=0, \quad \omega=\sqrt{\frac{2 C\left(M_{1}+M_{2}\right)}{M_{1} M_{2}}}
$$

In the limit of $M_{1}=M_{2}=M$, the latter is equal to $\left(\omega=\sqrt{\frac{4 C}{M}}\right)$.


Fig. $\quad M_{1}=M_{2}$. The size of unit cell is a. There are $2 N$ cells in the system. $L=2 N a$.
Now we consider the case (only the longitudinal wave) where $M_{1}=M_{2}=M$. The size of unit cell is $a$. The Brillouin zone is in the range $|k|<\pi / a$. The total number of atoms are $2 N$, leading to the $2 N$ modes; 1 acoustic branch ( $2 N$ modes). No optical mode exists. The size of the system is $L=2 \mathrm{Na}$.

$$
u_{s}=u \exp [i(s a k-\omega t\}]
$$

We use the periodic boundary condition;

$$
u_{2 N+s}=u_{s},
$$

or

$$
\exp [i(2 N a k)]=1
$$

or

$$
2 N a k=2 \pi n, \quad \text { or } \quad k=\frac{2 \pi n}{2 N a}=\frac{2 \pi}{a} \frac{n}{2 N},
$$

where $n=-N,-N+1, \ldots \ldots . ., N-1, N$. There are $2 N$ modes for 1 acoustic branch,

$$
|k|<\frac{\pi}{a} .
$$

The dispersion relation is given by

$$
\omega=2 \sqrt{\frac{C}{M}}\left|\sin \left(\frac{a k}{2}\right)\right|
$$

In the zone boundary, $|k|=\frac{\pi}{a}$

$$
\omega=2 \sqrt{\frac{C}{M}}
$$



Fig.
The dispersion relation for $M_{1}=M_{2}$. When $M_{2}$ is slightly larger than $M_{1}$, the dispersion relation consists of the optical branch ( $N$ modes) and the acoustic branch ( $N$ modes) for the first Brillouin zone $\left(|k|<\frac{\pi}{2 a}\right)$. The
extended zone scheme for the optical branch and acoustic brance (for $M_{2} \approx$ $M_{1}$ ) is shown in this Fig.. When $M_{1}=M_{2}, \omega$ at $k= \pm \frac{\pi}{2 a}$ for the optical branch becomes equal to that for the acoustic branch. The size of the Brillouin zone becomes twice larger. Correspondingly, a part of the optical branch (denoted by green line) shifts to the acoustic branch by $-\pi / a$, while a part of the optical branch (denoted by blue line) shifts to the acoustic branch by $\pi / a$.

## F. The folding effect

We use the above examples to explain the folding effect of the first Brillouin zone.
(i) Suppose that there are one kinds of mass (the mass $M_{1}$ denoted by red circles). Here we have a chain with the total size $L=2 N a$, where $2 N$ is the number of unit cell and $a$ is the lattice constant. When we consider the lattice vibration in this system (only in the case of longitudinal waves).


Then we have one longitudinal acoustic branch with $2 N$ modes in the first Brillouin zone $(|k(a)|<\pi)$, since the lattice constant is $a$. The total number of modes is $2 N$.

(ii) Suppose that there are two kinds of masses (the mass $M_{1}(=M)$ denoted by red circles and $M_{2}\left(M_{2} \neq M_{1}\right)$ denoted by blue circles). Here we have a chain with the total size $L=2 N a$, where $2 N$ is the number of unit cell and $2 a$ is the lattice constant. When we consider the lattice vibration in this system (only in the case of longitudinal waves).


Then we have one longitudinal acoustic branch with $N$ modes, and one optical branch with $N$ modes. in the first Brillouin zone $(|k(2 a)|<\pi$, or , $|k a|<\pi / 2)$, since the lattice constant is $2 a$. The total number of modes is $2 N$. When $M_{1}$ becomes different from $M_{2}$, the lattice constant changes from a to $2 a$. Correspondingly, the size of the Brillouin zone suddenly reduces from the region of $|k a|<\pi$ to the region of $|k a|<\pi / 2$. As a result, at the zone boundary $(|k a|=\pi / 2)$, there occurs the folding effect of the acoustic branch. A part of the acoustic branch $(|k a|>\pi / 2)$ becomes the optical branches in the new first Brillouin zone ( $|k a|<\pi / 2$ ).


## 8. The number of modes; degree of freedom <br> 8.1 One-dimensional case

We consider the degree of freedom for $N$ atoms in the linear lattice chain. There are $N$ atoms (each unit cell has one atom). Each atom has three degrees of freedom. One for each of the $x, y, z$ directions. Then we have $3 N$ degree of freedoms. This indicates that there are $3 N$ modes in the system. This implies that the number of allowed $k$ values in a single branch is just $\underline{N}$ for each Brillouin zone; $2 N$ transverse acoustic (TA) modes and $N$ longitudinal acoustic (LA) mode.

2 TA mode ( $2 N$ states)
1 LA mode ( $N$ states)


Fig.27(b) Longitudinal and transverse acoustic modes. The transverse modes may be degenerate as in the cubic structures

### 8.2 Three-dimensional case

We consider the lattice waves in the 3D system. The displacement vector $\boldsymbol{u}_{\mathrm{s}}$ is given by

$$
\begin{equation*}
\mathbf{u}_{s} \sim e^{i \mathbf{k} \cdot \mathbf{R}_{n}}, \tag{7.1}
\end{equation*}
$$

where $\boldsymbol{R}_{\mathrm{n}}$ is the position vector of the atom located in the equilibrium positions of the lattice.

$$
\begin{equation*}
\mathbf{R}_{n}=n_{1} \mathbf{a}_{1}+n_{2} \mathbf{a}_{2}+n_{3} \mathbf{a}_{3}, \tag{7.2}
\end{equation*}
$$

$\boldsymbol{a}_{1}, \boldsymbol{a}_{2}$, and $\boldsymbol{a}_{3}$ are the primitive lattice vectors along the $x, y$, and $z$ directions. From the boundary condition, we have

$$
\begin{equation*}
e^{i \mathbf{k} \cdot\left(\mathbf{R}_{n}+N_{1} \mathbf{a}_{1}\right)}=e^{i \mathbf{k} \cdot \mathbf{R}_{n}}, \quad \text { or } \quad e^{i k_{x} N_{1} a_{1}}=1 \tag{7.3}
\end{equation*}
$$

leading to the selected values of wavenumber $k_{\mathrm{x}}$

$$
\begin{equation*}
k_{x}=\frac{2 \pi}{N_{1} a_{1}} l_{1}=\frac{2 \pi}{N_{1} a} l_{1} \quad \text { for } a_{1}=a_{2}=a_{3}=a . \tag{7.4}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
k_{y}=\frac{2 \pi}{N_{2} a_{2}} l_{2}=\frac{2 \pi}{N_{2} a} l_{2}, \quad \text { and } \quad k_{z}=\frac{2 \pi}{N_{3} a_{3}} l_{3}=\frac{2 \pi}{N_{3} a} l_{3} \tag{7.5}
\end{equation*}
$$

Here $l_{1}, l_{2}$, and $l_{3}$ are integers given by

$$
\left\{\begin{array}{l}
l_{1}=\underbrace{-\frac{N_{1}}{2},-\frac{N_{1}}{2}+1, \cdots, \frac{N_{1}}{2}}_{N_{1}}  \tag{7.6}\\
l_{2}=\underbrace{-\frac{N_{2}}{2},-\frac{N_{2}}{2}+1, \cdots, \frac{N_{2}}{2}}_{N_{2}} . \\
l_{3}=\underbrace{-\frac{N_{3}}{2},-\frac{N_{3}}{2}+1, \cdots, \frac{N_{3}}{2}}_{N_{3}}
\end{array} .\right.
$$

This means that there are $N\left(=N_{1} N_{2} N_{3}\right)$ modes, where $N$ is the number of unit cells in the system.

Each mode has 3 degrees of freedom (1 longitudinal mode and 2 transverse modes). Then the total number of modes is $3 \times N=3 N$.
(a)

In the case of one atom per unit cell, we have $3 N$ degree of freedom, with $N$ longitudinal acoustic mode and transverse acoustic mode 2 N (b)

We consider the number of degrees of freedom of the atoms. With $p$ atoms in the primitive cell and $N$ primitive cells, there are $p N$ atoms. Each atom has three degrees of freedom, one for each of the $x, y, z$ directions, making a total of $3 p N$ degrees of freedom for the crystal. The number of allowed $k$ values in a single branch is just $N$ for one Brillouin zone. Thus, the one LA and two TA branches have a total of $3 N$ modes. The remaining $(3 p-3) \times N$ degrees of freedom are accommodated by the optical branches.

3 acoustical branches
1 longitudinal acoustical (LA) mode
2 transverse acoustical (TA) mode
$3 p-3$ optical branches
( $p-1$ ) longitudinal optical (LO) mode
$2(p-1)$ transverse optical (TO) mode
For $p=2$, for example, we have $1 \mathrm{LA}, 1 \mathrm{LO}$ modes, and 2 TA and 2 TO modes.

## 9. Classical Model

9.1 Theory of the transverse wave in a string


Fig. 28 Oscillation of one-dimensional continuum
Suppose that a traveling wave is propagating along a string that is under a tension $T_{\mathrm{s}}$. Let us consider one small element of length $\Delta x$. The ends of the element make small angle $\theta_{\mathrm{A}}$ and $\theta_{\mathrm{B}}$ with the $x$ axis. The net force acting on the element along the $y$-axis is

$$
\begin{align*}
\sum F_{y} & =T \sin \theta_{B}-T \sin \theta_{A}  \tag{9.1}\\
& =T\left(\sin \theta_{B}-\sin \theta_{A}\right) \approx T\left(\tan \theta_{B}-\tan \theta_{A}\right)
\end{align*},
$$

or

$$
\begin{equation*}
F_{y}=T\left[\left(\frac{\partial y}{\partial x}\right)_{B}-\left(\frac{\partial y}{\partial x}\right)_{A}\right] . \tag{9.2}
\end{equation*}
$$

We now apply the Newton's second law to the element, with the mass of the element given by $m=\mu \Delta x$,

$$
\begin{equation*}
F_{y}=m a_{y}=\mu \Delta x \frac{\partial^{2} y}{\partial t^{2}} . \tag{9.3}
\end{equation*}
$$

Then we have

$$
\begin{align*}
& \mu \Delta x \frac{\partial^{2} y}{\partial t^{2}}=T_{s}\left[\left(\frac{\partial y}{\partial x}\right)_{B}-\left(\frac{\partial y}{\partial x}\right)_{A}\right] \\
& \frac{\mu}{T_{s}} \frac{\partial^{2} y}{\partial t^{2}}=\frac{\left(\frac{\partial y}{\partial x}\right)_{B}-\left(\frac{\partial y}{\partial x}\right)_{A}}{\Delta x}=\frac{\partial^{2} y}{\partial x^{2}} \tag{9.4}
\end{align*}
$$

which leads to a wave equation given by

$$
\begin{equation*}
\frac{\partial^{2} y}{\partial x^{2}}=\frac{\mu}{T_{s}} \frac{\partial^{2} y}{\partial t^{2}}=\frac{1}{v^{2}} \frac{\partial^{2} y}{\partial t^{2}}, \tag{9.5}
\end{equation*}
$$

where $v$ is the velocity of the sound,

$$
\begin{equation*}
v=\sqrt{\frac{T_{s}}{\mu}} \tag{9.6}
\end{equation*}
$$

Here we use the Taylor expansion,

$$
\begin{equation*}
\left(\frac{\partial y}{\partial x}\right)_{B}-\left(\frac{\partial y}{\partial x}\right)_{A}=\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x}-\left(\frac{\partial y}{\partial x}\right)_{x}=\Delta x \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)=\Delta x \frac{\partial^{2} y}{\partial x^{2}} . \tag{9.6}
\end{equation*}
$$

### 9.2 Energy density of the elastic wave

When the length of element changes to $\Delta x$ to $\Delta x \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^{2}}$, the work done in the system ( $=W_{\mathrm{c}}$ ) of the conserved system is given by

$$
\begin{equation*}
-T_{s} \Delta x \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^{2}}+T_{s} d \Delta=-T_{s} d \Delta\left[1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^{2}-1\right]=-\frac{T_{s}}{2} \Delta x\left(\frac{\partial y}{\partial x}\right)^{2} \tag{9.7}
\end{equation*}
$$

Since the potential energy $\Delta U$ is related to $W_{\mathrm{c}}$ by $\Delta U=-W_{c}$, the potential energy $\Delta U$ is given by

$$
\begin{equation*}
\Delta U=\frac{T_{s}}{2} \Delta x\left(\frac{\partial y}{\partial x}\right)^{2} . \tag{9.8}
\end{equation*}
$$

The kinetic energy contribution $\Delta K$ is given by

$$
\begin{equation*}
\Delta K=\mu \Delta x \frac{1}{2}\left(\frac{\partial y}{\partial t}\right)^{2} \tag{9.9}
\end{equation*}
$$

Then the energy density is given by

$$
\begin{equation*}
\Delta E=\Delta K+\Delta U=\Delta x\left[\frac{T_{s}}{2}\left(\frac{\partial y}{\partial x}\right)^{2}+\frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^{2}\right] . \tag{9.10}
\end{equation*}
$$

## 10. Quantum mechanical approach: phonon

10.1 Annihilation and creation operators ${ }^{1,3}$

We assume $\boldsymbol{U}=\left(u_{1}, u_{2}, \quad, u_{\mathrm{N}}\right)$ for the eigenvalue $\omega=\omega_{\mathrm{k}}$. The displacement $u(x)$ is expresses using a Dirac delta function,

$$
\begin{equation*}
u(x)=\sum_{s} u_{s} \delta(x-s a) \tag{10.1}
\end{equation*}
$$

with $L=N a$. The Fourier transform of $u(x)$ is given by

$$
\begin{equation*}
U_{k}=\frac{1}{\sqrt{N}} \int d x u(x) e^{-i k x}=\frac{1}{\sqrt{N}} \sum_{s} u_{s} e^{-i k s a} . \tag{10.2}
\end{equation*}
$$

The inverse Fourier transform of $U_{\mathrm{k}}$ is

$$
\begin{align*}
\frac{1}{\sqrt{N}} \sum_{k} e^{i k s a} U_{k} & =\frac{1}{\sqrt{N}} \sum_{k} \frac{1}{\sqrt{N}} \sum_{s^{\prime}} u_{s^{\prime}} e^{-i k s^{\prime} a} e^{i k s a} \\
& =\frac{1}{N} \sum_{s^{\prime}} u_{s^{\prime}} \sum_{k^{\prime}} e^{i k^{\prime} a\left(s-s^{\prime}\right)}  \tag{10.3}\\
& =\frac{N}{N} \sum_{s} u_{s} \delta_{s, s^{\prime}} \\
& =u_{s}
\end{align*}
$$

or

$$
\begin{equation*}
u_{s}=\frac{1}{\sqrt{N}} \sum_{k} e^{i k s a} U_{k} \tag{10.4}
\end{equation*}
$$

The Lagrangian of the system is given by

$$
\begin{align*}
L & =T-V \\
& =\sum_{s}\left[\frac{M}{2} \dot{u}_{s}^{2}-\frac{1}{2} C\left(u_{s+1}-u_{s}\right)^{2}\right] . \tag{10.5}
\end{align*}
$$

We use the notation

$$
\begin{equation*}
u_{s}=\frac{1}{\sqrt{N}} \sum_{k} U_{k} e^{i k s a} . \tag{10.6}
\end{equation*}
$$

Since $u_{\mathrm{s}}$ is real, it is required that

$$
\begin{equation*}
U_{-k}=U_{k}^{*} . \tag{10.7}
\end{equation*}
$$

Using these relations, we have

$$
\begin{equation*}
\sum_{s} \dot{u}_{s}^{2}=\frac{1}{N} \sum_{k} \sum_{k^{\prime}} \sum_{s} \dot{U}_{k} \dot{U}_{k^{\prime}} e^{i\left(k+k^{\prime}\right) s a}=\sum_{k} \sum_{s} \dot{U}_{k} \dot{U}_{-k} \tag{10.8}
\end{equation*}
$$

Here we use

$$
\begin{equation*}
\sum_{s=1}^{N} e^{i\left(k+k^{\prime}\right) s a}=N \delta_{k^{\prime},-k} \tag{10.9}
\end{equation*}
$$

Similarly

$$
\begin{equation*}
\sum_{s}\left(u_{s+1}-u_{s}\right)^{2}=2 \sum_{k} U_{k} U_{-k}(1-\cos k a)=\sum_{k} \frac{1}{k} M \omega_{k}^{2} U_{k} U_{-k} \tag{10.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega_{k}^{2}=\frac{2 C}{M}(1-\cos k a) \tag{10.11}
\end{equation*}
$$

Then $L$ can be rewritten as

$$
\begin{equation*}
L=\sum_{k}\left(\frac{M}{2} \dot{U}_{k} \dot{U}_{-k}-\frac{M \omega_{k}^{2}}{2} U_{k} U_{-k}\right) . \tag{10.12}
\end{equation*}
$$

The linear momentum $P_{k}$ conjugate to $U_{k}$ is defined by

$$
\begin{equation*}
P_{k}=\frac{\partial L}{\partial \dot{U}_{k}}=M \dot{U}_{-k} . \tag{10.13}
\end{equation*}
$$

Then Hamiltonian $H$ is obtained as

$$
\begin{equation*}
H=\sum_{k} P_{k} \dot{U}_{k}-L=\frac{1}{2} \sum_{k}\left(\frac{1}{M} P_{k} P_{-k}+\frac{1}{2} M \omega_{k}^{2} U_{k} U_{-k}\right) . \tag{10.14}
\end{equation*}
$$

Note that we define the Fourier transform of the linear momentum by

$$
\begin{align*}
& p_{s}=\frac{1}{\sqrt{N}} \sum_{k} e^{-i k s a} P_{k}  \tag{10.15}\\
& P_{k}=\frac{1}{\sqrt{N}} \sum_{s} e^{i k s a} p_{s}
\end{align*}
$$

with

$$
\begin{equation*}
P_{-k}=P_{k}^{+} . \tag{10.16}
\end{equation*}
$$

Let us work it out by the operator technique, starting from the commutation relations,

$$
\begin{equation*}
\left[u_{s}, p_{s^{\prime}}\right]=i \hbar \delta_{s, s^{\prime}} . \tag{10.17}
\end{equation*}
$$

Then the commutation relation is preserved in $\left[U_{k}, P_{k^{\prime}}\right]=i \hbar \delta_{k, k^{\prime}}$, since

$$
\begin{align*}
{\left[U_{k}, P_{k^{\prime}}\right] } & =\left[\frac{1}{\sqrt{N}} \sum_{s} e^{-i k s a} u_{s}, \frac{1}{\sqrt{N}} \sum_{s} p_{s}^{-i k^{\prime} s a}\right] \\
& =\frac{1}{N} \sum_{s} \sum_{s^{\prime}} e^{-i k s a}\left[u_{s}, p_{s^{\prime}}\right] e^{i k^{\prime} s^{\prime} a}  \tag{10.18}\\
& =\frac{i \hbar}{N} \sum_{s} \sum_{s^{\prime}} e^{-i k s a} e^{i k^{\prime} s^{\prime} a} \delta_{s, s^{\prime}} \\
& =\frac{i \hbar}{N} \sum_{s} e^{-i\left(k-k^{\prime}\right) s a}=i \hbar \delta_{k, k^{\prime}}
\end{align*}
$$

Thus our new displacements and momenta are canonically conjugate, and noncommuting, if they are of the same wavenumber; otherwise they are dynamically independent operators. The Hamiltonian is rewritten as

$$
\begin{equation*}
H=\frac{1}{2} \sum_{k}\left(\frac{1}{M} P_{k}^{+} P_{k}+\frac{1}{2} M \omega_{k}^{2} U_{k}^{+} U_{k}\right) . \tag{10.19}
\end{equation*}
$$

The final step is the introduction of annihilation and creation operators defined by

$$
\begin{align*}
& a_{k}^{+}=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{M \omega_{k}}{\hbar}} U_{k}^{+}-\frac{1}{\sqrt{M \hbar \omega_{k}}} i P_{k}\right) \\
& a_{k}=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{M \omega_{k}}{\hbar}} U_{k}+\frac{1}{\sqrt{M \hbar \omega_{k}}} i P_{k}^{+}\right) \tag{10.20}
\end{align*}
$$

where $a_{\mathrm{k}}$ and ${a_{\mathrm{k}}}^{+}$act to destroy, and create a phonon of waveuumber, k and energy $\hbar \omega_{k}$, respectively. One can get the expressions for $P_{\mathrm{k}}$ and $U_{\mathrm{k}}$ from the above equations for $a_{k}$ and $a_{k}{ }^{+}$

$$
\begin{equation*}
P_{k}=i \sqrt{\frac{M \hbar \omega_{k}}{2}}\left(a_{k}^{+}-a_{-k}\right), \tag{10.21}
\end{equation*}
$$

$$
\begin{equation*}
U_{k}=\sqrt{\frac{\hbar}{2 M \omega_{k}}}\left(a_{k}+{a_{-k}}^{+}\right), \tag{10.22}
\end{equation*}
$$

The annihilation and creation operators satisfy the commutation

$$
\begin{align*}
& {\left[a_{k}, a_{k^{\prime}}^{+}\right]=\delta_{k, k^{\prime}},}  \tag{10.23}\\
& {\left[a_{k}, a_{k^{\prime}}\right]=\left[a_{k}^{+}, a_{k^{\prime}}^{+}\right]=0,} \tag{10.24}
\end{align*}
$$

The transformed Hamiltonian is

$$
\begin{equation*}
H=\sum_{k} \hbar \omega_{k}\left(a_{k}^{+} a_{k}+\frac{1}{2}\right)=\sum_{k} \hbar \omega_{k}\left(N_{k}+\frac{1}{2}\right), \tag{10.25}
\end{equation*}
$$

where $N_{k}=a_{k}^{+} a_{k}$. So that each phonon may be regarded as possessing an energy $\hbar \omega_{k}$. The total Hamiltonian is the sum of the Hamiltonian of independent linear oscillators of angular frequency $\omega_{\mathrm{k}}$. The various properties of the operators and eigenstates of the Hamiltonian are seen in the Appendix A.

### 10.2 Symmetry of lattice and translation operator ${ }^{13,15}$

We now consider the displacement defined by

$$
\begin{equation*}
u(x)=\langle x \mid u\rangle=\sum_{s} u_{s}\langle x \mid s\rangle . \tag{10.26}
\end{equation*}
$$

When we use the Dirac notation $\langle x \mid s\rangle=\delta(x-s a)$, the ket vector is described by

$$
\begin{equation*}
|u\rangle=\sum_{s} u_{s}|s\rangle . \tag{10.27}
\end{equation*}
$$

We introduce the translation operator given by $T(a)$. The Hamiltonian $H$ is invariant under the translation of the system by the lattice constant $a$. In other words, $T(a)$ commutes with the Hamiltonian $H$. The eigenstate of $H$ should be simultaneously the eigenstate of $T(a) . T(a)$ is a unitary operator, but not a Hermite operator. Then the eigenvalue of $T(a)$ is a complex number (see the Appendix B for more detail).

Since $T(a)|s\rangle=|s+1\rangle$, the ket $|s\rangle$ is not an eigenstate of $T(a)$. Suppose that $u_{\mathrm{s}}$ has the form of $\exp ($ isak $)$, where $k$ is a real parameter. $|u\rangle$ is a linear combinationof $|s\rangle$ with $s=0$, $1, \ldots, N-1$.

$$
\begin{equation*}
|u\rangle=\sum_{s} e^{i s a k}|s\rangle . \tag{10.28}
\end{equation*}
$$

When $T(a)$ is applied to the eigen ket $|u\rangle$, we get

$$
\begin{align*}
T(a)|u\rangle & =\sum_{s} e^{i s a k} T(a)|s\rangle=\sum_{s} e^{i s a k}|s+1\rangle \\
& =e^{-i k a} \sum_{s} e^{i(s+1) a k}|s+1\rangle=e^{-i k a}|u\rangle \tag{10.29}
\end{align*}
$$

which means that $|u\rangle$ is the eigenket of $T(a)$. Similarly, we repeat this process $N$ times,

$$
\begin{equation*}
[T(a)]^{N}|u\rangle=e^{-i k N a}|u\rangle=|u\rangle, \tag{10.30}
\end{equation*}
$$

since $[T(a)]^{N}=1$ (we use the periodic boundary condition). Then we have $e^{-i N k a}=1$, or $k=\frac{2 \pi}{N a} n$ where $n=0,1,2, \ldots, N-1\left(0 \leq k<\frac{2 \pi}{a}\right)$. Alternatively we chose the value of $k$ as the $N$ states for $-\frac{\pi}{a} \leq k<\frac{\pi}{a}$ (the first Brillouin zone).

It follows from this discussion that when $|u\rangle=\left|u_{k}\right\rangle$ are introduced, the $N$ harmonic oscillators ( $N$ being the number of atoms in the crystal) becomes uncoupled and that to each $\left|u_{k}\right\rangle$ corresponds to one separate oscillator with the angular frequency $\omega_{k}$. This motion is called a normal mode of vibration or a normal mode. This motion does not describe the motion of a single atom in the crystal but, rather, of all atoms in it.

A few examples will show how this works. We take a normal mode for which $k$ equals the reciprocal lattice $G(=2 \pi / a)$. In this case, the coefficient $u_{\mathrm{s}}$ equals unity. This motion is a translation of the crystal as a whole and it is therefore not property a vibration. The case just considered corresponds of course to the center $(k=0)$ of the Brillouin zone.

In order to have another example, we consider the case where $k$ equal to a point at the zone boundary of the Brillouin zone. The displacement vector is out of phase in going from one cell to the next, whether the successive lines show the evolution of the motion in time, the harmonic function depicted corresponding to the particular frequency of this normal mode.

## 11. Crystal momentum

A phonon of k will interact with particles such as photons, neutrons, and electrons as if it had a momentum $\hbar \mathbf{k}$. However, a phonon does not carry physical momentum. The physical momentum of a crystal is given by

$$
\begin{align*}
P_{k} & =\frac{\partial L}{\partial \dot{U}_{k}}=M \dot{U}_{-k} \\
& =M \frac{1}{\sqrt{N}} \sum_{s} \dot{u}_{s} e^{i k s a} \tag{11.1}
\end{align*}
$$

where

$$
\begin{equation*}
U_{k}=\frac{1}{\sqrt{N}} \int d x u(x) e^{-i k x}=\frac{1}{\sqrt{N}} \sum_{s} u_{s} e^{-i k s a} \tag{11.2}
\end{equation*}
$$

We assume that $\dot{u}_{s}$ is independent of $s ; \dot{u}_{s}=\dot{u}$. Then the momentum $P_{\mathrm{k}}$ is evaluated as

$$
\begin{align*}
P_{k} & =\frac{M}{\sqrt{N}} \dot{u} \sum_{s} e^{i k s a} \\
& =\frac{M}{\sqrt{N}} \dot{u}\left[1+e^{i k a}+e^{2 i k a}+\ldots .+e^{i k a(N-1)}\right],  \tag{11.3}\\
& =M \sqrt{N} \dot{u} \delta_{k, 0}
\end{align*}
$$

where $k=\frac{2 \pi r}{N a}(r=0, \pm 1, \pm 2, \ldots, \pm N / 2)$. The mode $k=0$ represents a uniform translation of the crystal as a whole. Such a translation does carry momentum. For most practical purpose, a phonon acts as if its momentum were $\hbar \boldsymbol{k}$, sometimes called the crystal momentum.

## 12 Semiclassical approach

### 12.1 Simple case

The energy of a lattice vibration is quantized. The quantum of energy is called a phonon. Elastic waves in crystals are made up of phonons. Thermal vibrations in crystals are thermally excited phonons. The energy of an elastic mode of angular frequency $\omega_{k}$ is

$$
\begin{equation*}
\varepsilon_{k}=\left(\left\langle n_{k}\right\rangle+\frac{1}{2}\right) \hbar \omega_{k}, \tag{12.1}
\end{equation*}
$$

when the mode is excited to quantum number $\left\langle n_{k}\right\rangle$, where the mode is occupied by $n$ photons. The term $\frac{1}{2} \hbar \omega$ is the zero pint energy of the mode. We consider the wave of the mode $k$ with the amplitude.

$$
\begin{equation*}
u=u_{k} \cos \left(\omega_{k} t-k x\right) \tag{12.2}
\end{equation*}
$$

where $u$ is the displacement of a volume element from its equilibrium positions at $x$ in the crystal.

$$
\begin{equation*}
u_{k}=u_{0} \cos (\omega t-k x)=u_{0}(\cos \omega t \cos k x+\sin \omega t \sin k x) . \tag{12.3}
\end{equation*}
$$

The energy in the mode is

$$
\begin{equation*}
\varepsilon_{k}=[\langle K\rangle+\langle P\rangle]=2\langle K\rangle \quad(\because\langle K\rangle=\langle P\rangle) \tag{12.4}
\end{equation*}
$$

when it is averaged over time.

$$
\begin{align*}
& K=\frac{1}{2} \rho\left(\frac{\partial u}{\partial t}\right)^{2}  \tag{12.5}\\
& \frac{\partial u}{\partial t}=u_{0} \omega_{k}\left[-\sin \omega_{k} t \cos k x+\cos \omega_{k} t \sin k x\right] \tag{12.6}
\end{align*}
$$

where $\rho$ is the mass density.

$$
\begin{align*}
\int K d V= & \frac{1}{2} \rho u_{0}^{2} \omega_{k}^{2} \int_{V} d V\left[-\sin ^{2} \omega_{k} t \cos ^{2} k x+\cos ^{2} \omega_{k} t \sin ^{2} k x\right.  \tag{12.7}\\
& \left.-2 \sin \omega_{k} t \cos \omega_{k} t \sin k x \cos k x\right]
\end{align*}
$$

Note that

$$
d V=A d x
$$

where $A$ is the total area in the $y-z$ plane (the x axis is normal to this plane), and the total volume $V$ is given by $V=A L=A N a$. Here we have

$$
\left\{\begin{array}{l}
x=0 \sim N a(=L)  \tag{12.8}\\
k=\frac{\pi}{a\left(\frac{N}{2}\right)} \ell=\frac{2 \pi}{a N} \ell \quad\left(\ell=0, \pm 1, \cdots, \pm \frac{N}{2}\right),
\end{array}\right.
$$

from the boundary condition: $\cos (k N A)=1 \Rightarrow k N a=2 \pi \ell$ ). Then we obtain

$$
\begin{align*}
& \int_{0}^{N a} d x \cos ^{2} k x=\int_{0}^{N a} d x \frac{1}{2}(1+\cos 2 k x)=\frac{1}{2} N a+[k \sin 2 k x]_{0}^{N a}=\frac{1}{2} N a .  \tag{12.9}\\
& \left(\because 2 k N a=2\left(\frac{2 \pi}{a N} \ell\right) N a=4 \pi \ell\right) .
\end{align*}
$$

Similarly, we have

$$
\begin{equation*}
\int_{0}^{N a} d x \sin ^{2} k x=\frac{1}{2} N a, \quad \int_{0}^{N a} d x \sin k x \cos k x=0 \tag{12.10}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\int K d V=\int K A d x=\frac{V}{4} \rho u_{0}^{2} \omega^{2}\left(\sin ^{2} \omega t+\cos ^{2} \omega t\right)=\frac{V}{4} \rho u_{0}^{2} \omega^{2} . \tag{12.11}
\end{equation*}
$$

The time average kinetic energy is

$$
\begin{equation*}
\langle K\rangle=\frac{V}{4} \rho u_{0}^{2} \omega^{2} \frac{1}{T} \int_{0}^{T} d t=\frac{V}{4} \rho u_{0}^{2} \omega^{2} . \tag{12.12}
\end{equation*}
$$

Since $\langle K\rangle=\frac{1}{2} \varepsilon$, we have

$$
\begin{equation*}
\frac{V}{4} \rho u_{0}^{2} \omega_{k}^{2}=\frac{1}{2} \varepsilon=\frac{1}{2}\left(\left\langle n_{k}\right\rangle+\frac{1}{2}\right) \hbar \omega_{k}, \tag{12.13}
\end{equation*}
$$

or

$$
\begin{equation*}
u_{0}^{2}=\frac{2 \hbar\left(\left\langle n_{k}\right\rangle+\frac{1}{2}\right)}{\rho V \omega_{k}} \Rightarrow u_{e f f}=\frac{1}{\sqrt{2}} u_{0} . \tag{12.15}
\end{equation*}
$$

Since $\rho V=N M$,

$$
\begin{equation*}
u_{0}^{2}=\frac{2 \hbar\left(\left\langle n_{k}\right\rangle+\frac{1}{2}\right)}{N M \omega_{k}} . \tag{12.16}
\end{equation*}
$$

This relates the displacement in a given mode to the phonon occupancy $n$ of the mode.
An optical mode with $\omega$ close to zero is called a soft mode.

### 12.2 General case

The Lagrangian $L$ is given by

$$
\begin{align*}
L & =T-V=\sum_{s}\left[\frac{M}{2} \dot{u}_{s}^{2}-\frac{1}{2} C\left(u_{s+1}-u_{s}\right)^{2}\right] .  \tag{12.17}\\
& =L\left(\dot{u}_{1}, \dot{u}_{2}, . ., \dot{u}_{N-1}, u_{1}, u_{2}, . ., u_{N-1}\right)
\end{align*}
$$

The linear momentum conjugate to us, is given by

$$
\begin{equation*}
p_{s}=\frac{\partial L}{\partial \dot{u}_{s}}=M \dot{u}_{s} . \tag{12.18}
\end{equation*}
$$

Then the Hamiltonian $H$ can be derived as

$$
\begin{align*}
H & =\sum_{s} p_{s} \dot{u}_{s}-L \\
& =\sum_{s} M \dot{u}_{s}^{2}-\sum_{s} \frac{M}{2} \dot{u}_{s}^{2}+\sum_{s} \frac{1}{2} C\left(u_{s+1}-u_{s}\right)^{2} .  \tag{12.19}\\
& =\sum_{s} \frac{M}{2} \dot{u}_{s}^{2}+\sum_{s} \frac{1}{2} C\left(u_{s+1}-u_{s}\right)^{2}
\end{align*}
$$

Now we calculate the total energy for the case of $u_{s}=u \cos (s k a-\omega t)$.

$$
\begin{equation*}
\dot{u}_{s}=u(-1)(-\omega) \sin (s k a-\omega t), \tag{12.20}
\end{equation*}
$$

or

$$
\begin{align*}
E= & \frac{1}{4} M u^{2} \omega^{2} \sum_{s}[1-\cos 2(\omega t-s k a)] \\
& +\frac{1}{2} C u^{2}(1-\cos k a) \sum_{s}\left[1-\cos 2\left(\omega t-s k a-\frac{1}{2} k a\right)\right] \tag{12.21}
\end{align*}
$$

with $\rho=\frac{N M}{V}$ (mass density), $V=N a$. The dispersion relation is given by

$$
\begin{equation*}
\omega^{2}=\frac{2 C}{M}(1-\cos k a) \tag{12.22}
\end{equation*}
$$

Then the total energy is

$$
\begin{equation*}
E=\frac{1}{4} M u^{2} \omega^{2}\left\{\sum_{s}[1-\cos 2(\omega t-s k a)]+\sum_{s}\left[1-\cos 2\left(\omega t-s k a-\frac{1}{2} k a\right)\right]\right\} . \tag{12.23}
\end{equation*}
$$

The time-average is

$$
\begin{align*}
\langle E\rangle & =\frac{1}{T} \int_{0}^{T} E d t=\frac{1}{2} M u^{2} \omega^{2} \sum_{s} 1=\frac{1}{2} M u^{2} \omega^{2} N  \tag{12.24}\\
& =\frac{1}{2} \rho V u^{2} \omega^{2}
\end{align*}
$$

where

$$
T=\frac{2 \pi}{\omega}, \rho=\frac{N M}{V}=\frac{N M}{N a}, N M=\rho V .
$$

In general, we use

$$
\begin{equation*}
u_{s}=\sum_{k} u_{k} \cos (s k a-\omega t) . \tag{12.25}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\langle E\rangle=\frac{1}{2} \rho V \sum_{k} u_{k}^{2} \omega_{k}^{2} . \tag{12.26}
\end{equation*}
$$

This energy is compared with the result derived from the quantum mechanics.

$$
\frac{1}{2} \rho V \sum_{k} u_{k}^{2} \omega_{k}^{2}=\sum_{k} \hbar \omega_{k}\left(\left\langle\hat{n}_{k}\right\rangle+\frac{1}{2}\right) .
$$

or

$$
\begin{equation*}
\frac{1}{2} \rho V u_{k}^{2} \omega_{k}^{2}=\hbar \omega_{k}\left(\left\langle\hat{n}_{k}\right\rangle+\frac{1}{2}\right) \tag{12.27}
\end{equation*}
$$

or

$$
\begin{equation*}
u_{k}^{2}=\frac{2 \hbar\left(\left\langle\hat{n}_{k}\right\rangle+\frac{1}{2}\right)}{\rho V \omega_{k}}=\frac{2 \hbar\left(\left\langle\hat{n}_{k}\right\rangle+\frac{1}{2}\right)}{N M \omega_{k}} \tag{12.29}
\end{equation*}
$$

where $\rho V=N M$. Here we define the effective amplitude as

$$
\begin{equation*}
u_{k}^{e f f}=\frac{1}{\sqrt{2}} u_{k}, \tag{12.30}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(u_{k}^{e f f}\right)^{2}=\frac{\hbar\left(\left\langle\hat{n}_{k}\right\rangle+\frac{1}{2}\right)}{N M \omega_{k}} \tag{12.31}
\end{equation*}
$$

## Conclusion

We have shown that the well-defined lattice waves propagate over the crystal, forming a so-called phonon as a quantization of the lattice waves. Phonon has the dual characters of wave and particle, which is essential to the quantum mechanics. Phonon is one of bosons, obeying the Bose-Einstein statistics. Phonons will be seen to play an important role in any phenomena for which the energy of importance is comparable to $\hbar \omega$, the energy of the phonon in question. In the BCS (Bardeen-Cooper-Schrieffer) model for the superconductivity, a specific interaction between electrons can lead to an
energy gap separated from excited states by a energy gap. The formation of Cooper pairs is due to the electron-phonon interaction. The first electron interacts with the lattice and deforms it; the second electron sees the deformed lattice. Thus the second electron interacts with the first electron through the lattice deformation.

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## APPENDIX

A Simple harmonics (1D) in quantum mechanics ${ }^{15}$
(a) Annihilation and creation operator

The commutation relation

$$
\begin{equation*}
[\hat{x}, \hat{p}]=i \hbar . \tag{A.1}
\end{equation*}
$$

The Hamiltonian of the simple harmonics is

$$
\begin{equation*}
\hat{H}=\frac{1}{2 m} \hat{p}^{2}+\frac{m \omega_{0}^{2}}{2} \hat{x}^{2} \tag{A.2}
\end{equation*}
$$

The eigenvalue-problem of the simple harmonics

$$
\begin{equation*}
\hat{H}|n\rangle=\varepsilon_{n}|n\rangle, \tag{A.3}
\end{equation*}
$$

with

$$
\begin{equation*}
\varepsilon_{n}=\left(n+\frac{1}{2}\right) \hbar \omega_{0} \tag{A.4}
\end{equation*}
$$

where $n=0,1,2,3, \ldots$.
In the $\{|x\rangle\}$ representation, the wave function of the simple harmonics can be described as

$$
\begin{equation*}
\left(-\frac{\hbar^{2}}{2 m} \frac{d^{2}}{d x^{2}}+\frac{m \omega_{0}^{2}}{2} x^{2}\right)\langle x \mid n\rangle=\varepsilon_{n}\langle x \mid n\rangle . \tag{A.5}
\end{equation*}
$$

Here we introduce the creation operator and annihilation operators given by

$$
\begin{align*}
& \hat{a}=\frac{\beta}{\sqrt{2}}\left(\hat{x}+\frac{i \hat{p}}{m \omega_{0}}\right)=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{m \omega_{0}}{\hbar}} \hat{x}+\frac{i \hat{p}}{\sqrt{m \hbar \omega_{0}}}\right),  \tag{A.6}\\
& \hat{a}^{+}=\frac{\beta}{\sqrt{2}}\left(\hat{x}-\frac{i \hat{p}}{m \omega_{0}}\right)=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{m \omega_{0}}{\hbar}} \hat{x}-\frac{i \hat{p}}{\sqrt{m \hbar \omega_{0}}}\right), \tag{A.7}
\end{align*}
$$

with

$$
\begin{align*}
& \beta=\sqrt{\frac{m \omega_{0}}{\hbar}}  \tag{A.8}\\
& \hat{x}=\frac{1}{\sqrt{2} \beta}\left(\hat{a}+\hat{a}^{+}\right)=\sqrt{\frac{\hbar}{2 m \omega_{0}}}\left(\hat{a}+\hat{a}^{+}\right),  \tag{A.9}\\
& \hat{p}=\frac{1}{\sqrt{2} \beta} \frac{m \omega_{0}}{i}\left(\hat{a}-\hat{a}^{+}\right)=\frac{1}{i} \sqrt{\frac{m \hbar \omega_{0}}{2}}\left(\hat{a}-\hat{a}^{+}\right),  \tag{A.10}\\
& {[\hat{x}, \hat{p}]=\frac{1}{(\sqrt{2} \beta)^{2}} \frac{m \omega_{0}}{i}\left[\hat{a}+\hat{a}^{+}, \hat{a}-\hat{a}^{+}\right]=-\frac{\hbar}{i}\left[\hat{a}, \hat{a}^{+}\right],} \tag{A.11}
\end{align*}
$$

or

$$
\begin{align*}
& {\left[\hat{a}, \hat{a}^{+}\right]=\hat{1}}  \tag{A.12}\\
& \hat{a}^{+} \hat{a}=\frac{\beta^{2}}{2}\left(\hat{x}-\frac{i \hat{p}}{m \omega_{0}}\right)\left(\hat{x}+\frac{i \hat{p}}{m \omega_{0}}\right)=\frac{\beta^{2}}{2}\left(\hat{x}^{2}+\frac{\hat{p}^{2}}{m^{2} \omega_{0}^{2}}-i \frac{1}{m \omega_{0}}[\hat{p}, \hat{x}]\right) \tag{A.13}
\end{align*}
$$

or

$$
\begin{equation*}
\hat{a}^{+} \hat{a}=\frac{1}{\hbar \omega_{0}}\left(\hat{H}-\frac{1}{2} \hbar \omega_{0}\right) \tag{A.14}
\end{equation*}
$$

or

$$
\begin{equation*}
\hat{H}=\hbar \omega_{0}\left(\hat{N}+\frac{1}{2}\right) \tag{A.15}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{N}=\hat{a}^{+} \hat{a} \tag{A.16}
\end{equation*}
$$

The operator $\hat{N}$ is Hermitian since

$$
\begin{equation*}
\hat{N}^{+}=\left(\hat{a}^{+} \hat{a}\right)^{+}=\hat{a}^{+} \hat{a}=\hat{N} . \tag{A.17}
\end{equation*}
$$

The eigenvectors of $\hat{H}$ are those of $\hat{N}$, and vice versa since $[\hat{H}, \hat{N}]=0$,

$$
\begin{align*}
& {[\hat{N}, \hat{a}]=\left[\hat{a}^{+} \hat{a}, \hat{a}\right]=\hat{a}^{+} \hat{a} \hat{a}-\hat{a} \hat{a}^{+} \hat{a}=\left[\hat{a}^{+}, \hat{a}\right] \hat{a}=-\hat{a},}  \tag{A.18}\\
& {\left[\hat{N}, \hat{a}^{+}\right]=\left[\hat{a}^{+} \hat{a}, \hat{a}^{+}\right]=\hat{a}^{+} \hat{a} \hat{a}^{+}-\hat{a}^{+} \hat{a}^{+} \hat{a}=\hat{a}^{+}\left[\hat{a}, \hat{a}^{+}\right]=\hat{a}^{+} .} \tag{A.19}
\end{align*}
$$

Thus we have the relations

$$
\begin{equation*}
[\hat{N}, \hat{a}]=-\hat{a}, \tag{A20}
\end{equation*}
$$

and

$$
\begin{align*}
& {\left[\hat{N}, \hat{a}^{+}\right]=\hat{a}^{+},}  \tag{A.21}\\
& \hat{N}|n\rangle=\hbar \omega_{0}\left(n+\frac{1}{2}\right)|n\rangle . \tag{A.22}
\end{align*}
$$

From the relation $\quad[\hat{N}, \hat{a}]|n\rangle=-\hat{a}|n\rangle$,

$$
\begin{equation*}
(\hat{N} \hat{a}-\hat{a} \hat{N})|n\rangle=-\hat{a}|n\rangle, \tag{A.23}
\end{equation*}
$$

or

$$
\begin{equation*}
\hat{N}(\hat{a}|n\rangle)=(n-1) \hat{a}|n\rangle . \tag{A.24}
\end{equation*}
$$

$\hat{a}|n\rangle$ is the eigenket of $\hat{N}$ with the eigenvalue ( $n-1$ ).

$$
\begin{equation*}
\hat{a}|n\rangle \approx|n-1\rangle . \tag{A.25}
\end{equation*}
$$

From the relation

$$
\begin{align*}
& {\left[\hat{N}, \hat{a}^{+}\right]|n\rangle=\hat{a}^{+}|n\rangle,}  \tag{A.26}\\
& (\hat{N} \hat{a}-\hat{a} \hat{N})|n\rangle=-\hat{a}|n\rangle, \tag{A.27}
\end{align*}
$$

or

$$
\begin{equation*}
\hat{N}\left(\hat{a}^{+}|n\rangle\right)=(n+1) \hat{a}^{+}|n\rangle . \tag{A.28}
\end{equation*}
$$

$\hat{a}^{+}|n\rangle$ is the eigenket of $\hat{N}$ with the eigenvalue ( $n+1$ ).

$$
\begin{equation*}
\hat{a}^{+}|n\rangle \approx|n+1\rangle \tag{A.29}
\end{equation*}
$$

Now we need to show that $n$ should be either zero or positive integers: $n=0,1,2,3, \ldots$. We note that

$$
\begin{align*}
& \langle n| \hat{a}^{+} \hat{a}|n\rangle=n\langle n \mid n\rangle \geq 0,  \tag{A.30}\\
& \langle n| \hat{a} \hat{a}^{+}|n\rangle=\langle n| \hat{a}^{+} \hat{a}+1|n\rangle=(n+1)\langle n \mid n\rangle \geq 0 . \tag{A.31}
\end{align*}
$$

The norm of a ket vector is non-negative and the vanishing of the norm is a necessary and sufficient condition for the vanishing of the ket vector. In other words, $n \geq 0$. If $n=0$, $\hat{a}|n\rangle=0$. If $n \neq 0, \hat{a}|n\rangle$ is a nonzero ket vector of norm $n\langle n \mid n\rangle$.
If $n>0$, one successively forms the set of eigenkets,
$\hat{a}|n\rangle, \hat{a}^{2}|n\rangle, \hat{a}^{3}|n\rangle, \ldots . \hat{a}^{p}|n\rangle$, , belonging to the eigenvalues, $n-1, n-2, n-3, \ldots \ldots, n-p$,
This set is certainly limited since the eigenvalues of $\hat{N}$ have a lower limit of zero. In other words, the eigenket $\hat{a}^{p}|n\rangle \approx|n-p\rangle$, or $n-p=0$. Thus n should be a positive integer. Similarly, one successively forms the set of eigenkets,
$\hat{a}^{+}|n\rangle, \hat{a}^{+^{2}}|n\rangle, \hat{a}^{+3}|n\rangle, \ldots . \hat{a}^{+p}|n\rangle$, belonging to the eigenvalues, $n+1, n+2, n+3, \ldots \ldots, n+p$, Thus the eigenvalues are either zero or positive integers: $n=0,1,2,3,4$,

The properties of $\hat{a}^{+}$and $\hat{a}$

1. $\hat{a}|0\rangle=0$,
since $\langle 0| \hat{a}^{+} \hat{a}|0\rangle=0$.
2. $\quad \hat{a}^{+}|n\rangle=\sqrt{n+1}|n+1\rangle$,
$\left[\hat{N}, \hat{a}^{+}\right]|n\rangle=\hat{a}^{+}|n\rangle$,
$\hat{N} \hat{a}^{+}|n\rangle=\hat{a}^{+} \hat{N}|n\rangle+\hat{a}^{+}|n\rangle=(n+1) \hat{a}^{+}|n\rangle$.
$\hat{a}^{+}|n\rangle$ is an eigenket of $\hat{N}$ with the eigenvalue $(n+1)$.
Then

$$
\begin{equation*}
\hat{a}^{+}|n\rangle=c|n+1\rangle . \tag{A.36}
\end{equation*}
$$

Since

$$
\begin{equation*}
\langle n| \hat{a} \hat{a}^{+}|n\rangle=|c|^{2}\langle n+1 \mid n+1\rangle=|c|^{2}, \tag{A.37}
\end{equation*}
$$

or

$$
\begin{equation*}
\langle n| \hat{a}^{+} \hat{a}+1|n\rangle=n+1=|c|^{2}, \tag{A.39}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
|c|=\sqrt{n+1} \tag{A.40}
\end{equation*}
$$

3. 

$$
\begin{align*}
& \hat{a}|n\rangle=\sqrt{n}|n-1\rangle,  \tag{A.41}\\
& {[\hat{N}, \hat{a}]|n\rangle=-\hat{a}|n\rangle,}  \tag{A.42}\\
& \hat{N} \hat{a}|n\rangle=\hat{a} \hat{N}|n\rangle-\hat{a}|n\rangle=(n-1) \hat{a}|n\rangle, \tag{A.43}
\end{align*}
$$

$\hat{a}|n\rangle$ is an eigenket of $\hat{N}$ with the eigenvalue (n-1)
Then

$$
\begin{equation*}
\hat{a}|n\rangle=c|n-1\rangle . \tag{A.44}
\end{equation*}
$$

Since

$$
\begin{align*}
& \langle n| \hat{a}^{+} \hat{a}|n\rangle=|c|^{2}\langle n-1 \mid n-1\rangle=|c|^{2}=n,  \tag{A.45}\\
& |c|=\sqrt{n} \tag{A.46}
\end{align*}
$$

(b) Basis vectors in terms of $|0\rangle$

We use the relation

$$
\begin{aligned}
& |1\rangle=\hat{a}^{+}|0\rangle, \\
& |2\rangle=\frac{1}{\sqrt{2}} \hat{a}^{+}|1\rangle=\frac{1}{\sqrt{2!}}\left(\hat{a}^{+}\right)^{2}|0\rangle, \\
& |3\rangle=\frac{1}{\sqrt{3}} \hat{a}^{+}|2\rangle=\frac{1}{\sqrt{3!}}\left(\hat{a}^{+}\right)^{3}|0\rangle,
\end{aligned}
$$

$$
\begin{equation*}
|n\rangle=\frac{1}{\sqrt{n}} \hat{a}^{+}|n-1\rangle=\frac{1}{\sqrt{n!}}\left(\hat{a}^{+}\right)^{n}|0\rangle, \tag{A.48}
\end{equation*}
$$

The expression for $\hat{x}|n\rangle$ and $\hat{p}|n\rangle$

$$
\begin{align*}
& \hat{x}|n\rangle=\sqrt{\frac{\hbar}{2 m \omega_{0}}}\left(\hat{a}+\hat{a}^{+}\right)|n\rangle=\sqrt{\frac{\hbar}{2 m \omega_{0}}}(\sqrt{n+1}|n+1\rangle+\sqrt{n}|n-1\rangle),  \tag{A.49}\\
& \hat{p}|n\rangle=\sqrt{\frac{m \hbar \omega_{0}}{2}} i\left(\hat{a}^{+}-\hat{a}\right)|n\rangle=\sqrt{\frac{m \hbar \omega_{0}}{2}} i(\sqrt{n+1}|n+1\rangle-\sqrt{n}|n-1\rangle) . \tag{A.50}
\end{align*}
$$

Therefore the matrix elements of $\hat{a}, \hat{a}^{+}, \hat{x}$, and $\hat{p}$ operators in the $\{|n\rangle\}$ representation are as follows.

$$
\begin{align*}
& \left\langle n^{\prime}\right| \hat{a}|n\rangle=\sqrt{n} \delta_{n^{\prime}, n-1},  \tag{A.51}\\
& \left\langle n^{\prime}\right| \hat{a}^{+}|n\rangle=\sqrt{n+1} \delta_{n^{\prime}, n+1},  \tag{A.52}\\
& \left\langle n^{\prime}\right| \hat{x}|n\rangle=\sqrt{\frac{\hbar}{2 m \omega_{0}}}\left(\sqrt{n+1} \delta_{n^{\prime}, n+1}+\sqrt{n} \delta_{n^{\prime}, n-1}\right),  \tag{A.53}\\
& \left\langle n^{\prime}\right| \hat{p}|n\rangle=i \sqrt{\frac{m \hbar \omega_{0}}{2}}\left(\sqrt{n+1} \delta_{n^{\prime}, n+1}-\sqrt{n} \delta_{n^{\prime}, n-1}\right) . \tag{A.54}
\end{align*}
$$

Mean values and root-mean-square deviations of $\hat{x}$ and $\hat{p}$ in the state $|n\rangle$.

$$
\begin{align*}
& \langle n| \hat{x}|n\rangle=0,  \tag{A.55}\\
& \langle n| \hat{p}|n\rangle=0,  \tag{A.56}\\
& (\Delta x)^{2}=\langle n| \hat{x}^{2}|n\rangle=\left(n+\frac{1}{2}\right) \frac{\hbar}{m \omega_{0}},  \tag{А.57}\\
& (\Delta p)^{2}=\langle n| \hat{p}^{2}|n\rangle=\left(n+\frac{1}{2}\right) m \hbar \omega_{0} . \tag{A.58}
\end{align*}
$$

The product $\Delta x \Delta p$ is

$$
\begin{equation*}
\Delta x \Delta p=\left(n+\frac{1}{2}\right) \hbar \geq \frac{1}{2} \hbar \quad \text { (Heisenberg's principle of uncertainty), } \tag{A.59}
\end{equation*}
$$

Note that

$$
\begin{align*}
& \hat{x}^{2}=\frac{\hbar}{2 m \omega_{0}}\left(\hat{a}^{+}+\hat{a}\right)\left(\hat{a}^{+}+\hat{a}\right)=\frac{\hbar}{2 m \omega_{0}}\left(\hat{a}^{+} \hat{a}^{+}+\hat{a} \hat{a}+\hat{a}^{+} \hat{a}+\hat{a} \hat{a}^{+}\right),  \tag{A.60}\\
& \hat{p}^{2}=\frac{m \hbar \omega_{0}}{2}\left(\hat{a}^{+}-\hat{a}\right)\left(\hat{a}^{+}-\hat{a}\right)=\frac{m \hbar \omega_{0}}{2}\left(\hat{a}^{+} \hat{a}^{+}+\hat{a} \hat{a}-\hat{a}^{+} \hat{a}-\hat{a} \hat{a}^{+}\right), \tag{A.61}
\end{align*}
$$

and

$$
\begin{align*}
& \langle n|\left(\hat{a}^{+}\right)^{2}|n\rangle=0,  \tag{A.62}\\
& \langle n| \hat{a}^{2}|n\rangle=0,  \tag{A.63}\\
& \langle n| \hat{a}^{+} \hat{a}+\hat{a} \hat{a}^{+}|n\rangle=\langle n| 2 \hat{a}^{+} \hat{a}+1|n\rangle=2 n+1, \tag{A.64}
\end{align*}
$$

Mean potential energy

$$
\begin{equation*}
\langle V\rangle=\frac{1}{2} m \omega^{2}\langle n| \hat{x}^{2}|n\rangle=\frac{1}{2} m \omega^{2}(\Delta x)^{2}=\frac{1}{2} \varepsilon_{n} . \tag{A.65}
\end{equation*}
$$

Mean kinetic energy

$$
\begin{equation*}
\langle K\rangle=\frac{1}{2 m}\langle n| \hat{p}^{2}|n\rangle=\frac{1}{2 m}(\Delta p)^{2}=\frac{1}{2} \varepsilon_{n} . \tag{A.66}
\end{equation*}
$$

Thus we have

$$
\begin{equation*}
\langle V\rangle=\langle K\rangle . \tag{A.67}
\end{equation*}
$$

(Virial theorem)

## B. Translation operator in quantum mechanics ${ }^{\mathbf{1 5}}$

Here we discuss the translation operator $\hat{T}(a)$ in quantum mechanics,

$$
\begin{equation*}
\left|\psi^{\prime}\right\rangle=\hat{T}(a)|\psi\rangle \tag{B.1}
\end{equation*}
$$

or

$$
\begin{equation*}
\left\langle\psi^{\prime}\right|=\left\langle\psi^{\prime}\right| \hat{T}^{+}(a) \tag{B.2}
\end{equation*}
$$

In an analogy from the classical mechanics, it is predicted that the average value of $\hat{x}$ in the new state $\left|\psi^{\prime}\right\rangle$ is equal to that of $\hat{x}$ in the old state $|\psi\rangle$ plus the $x$-displacement $a$ under the translation of the system

$$
\left\langle\psi^{\prime}\right| \hat{x}\left|\psi^{\prime}\right\rangle=\langle\psi| \hat{x}+a|\psi\rangle
$$

or

$$
\langle\psi| \hat{T}^{+}(a) \hat{x} \hat{T}(a)|\psi\rangle=\langle\psi| \hat{x}+a|\psi\rangle
$$

or

$$
\begin{equation*}
\hat{T}^{+}(a) \hat{x} \hat{T}(a)=\hat{x}+a \hat{1} . \tag{B.3}
\end{equation*}
$$

Normalization condition:

$$
\left\langle\psi^{\prime} \mid \psi^{\prime}\right\rangle=\langle\psi| \hat{T}^{+}(a) \hat{T}(a)|\psi\rangle=\langle\psi \mid \psi\rangle
$$

or

$$
\begin{equation*}
\hat{T}^{+}(a) \hat{T}(a)=\hat{1} \tag{B.4}
\end{equation*}
$$

[ $\hat{T}(a)$ is an unitary operator].
From Eqs.(B3) and (B4), we have

$$
\hat{x} \hat{T}(a)=\hat{T}(a)(\hat{x}+a)=\hat{T}(a) \hat{x}+a \hat{T}(a)
$$

or the commutation relation:

$$
\begin{equation*}
[\hat{x}, \hat{T}(a)]=a \hat{T}(a) . \tag{B.5}
\end{equation*}
$$

From this, we have

$$
\hat{x} \hat{T}(a)|x\rangle=\hat{T}(a) \hat{x}|x\rangle+a \hat{T}(a)|x\rangle=(x+a) \hat{T}(a)|x\rangle
$$

Thus, $\hat{T}(a)|x\rangle$ is the eigenket of $\hat{x}$ with the eigenvalue $(x+a)$.
or

$$
\begin{equation*}
\hat{T}(a)|x\rangle=|x+a\rangle \tag{B.6}
\end{equation*}
$$

or

$$
\begin{equation*}
\hat{T}^{+}(a) \hat{T}(a)|x\rangle=\hat{T}^{+}(a)|x+a\rangle=|x\rangle \tag{B.7}
\end{equation*}
$$

When $x$ is replaced by $x-a$ in Eq.(B7), we get

$$
\begin{equation*}
|x-a\rangle=\hat{T}^{+}(a)|x\rangle, \tag{B.8}
\end{equation*}
$$

or

$$
\begin{equation*}
\langle x-a|=\langle x| \hat{T}(a) . \tag{B.9}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\left\langle x \mid \psi^{\prime}\right\rangle=\langle x| \hat{T}(a)|\psi\rangle=\langle x-a \mid \psi\rangle=\psi(x-a) . \tag{B.10}
\end{equation*}
$$

The average value of $\hat{p}$ in the new state $\left|\psi^{\prime}\right\rangle$ is equal to the average value of $\hat{p}$ in the old state $|\psi\rangle$ under the translation of the system

$$
\begin{equation*}
\left\langle\psi^{\prime}\right| \hat{p}\left|\psi^{\prime}\right\rangle=\langle\psi| \hat{p}|\psi\rangle, \tag{B.11}
\end{equation*}
$$

or

$$
\langle\psi| \hat{T}^{+}(a) \hat{p} \hat{T}(a)|\psi\rangle=\langle\psi| \hat{p}|\psi\rangle,
$$

or

$$
\begin{equation*}
\hat{T}^{+}(a) \hat{p} \hat{T}(a)=\hat{p} . \tag{B.12}
\end{equation*}
$$

So we have the commutation relation

$$
[\hat{T}(a), \hat{p}]=\hat{0} .
$$

From this commutation relation, we have

$$
\hat{p} \hat{T}(a)|p\rangle=\hat{T}(a) \hat{p}|p\rangle=p \hat{T}(a)|p\rangle .
$$

Thus, $\hat{T}(a)|p\rangle$ is the eigenket of $\hat{p}$ associated with the eigenvalue $p$.

