(Date: August 18, 2015)

Johannes Diderik van der Waals (November 23, 1837 - March 8, 1923) was a Dutch theoretical physicist and thermodynamicist famous for his work on an equation of state for gases and liquids. His name is primarily associated with the van der Waals equation of state that describes the behavior of gases and their condensation to the liquid phase. His name is also associated with van der Waals forces (forces between stable molecules), with van der Waals molecules (small molecular clusters bound by van der Waals forces), and with van der Waals radii (sizes of molecules). He became the first physics professor of the University of Amsterdam when it opened in 1877 and won the 1910 Nobel Prize in physics.

http://en.wikipedia.org/wiki/Johannes_Diderik_van_der_Waals

Thomas Andrews (9 December 1813 - 26 November 1885) was a chemist and physicist who did important work on phase transitions between gases and liquids. He was a longtime professor of chemistry at Queen's University of Belfast.
https://en.wikipedia.org/wiki/Thomas_Andrews_\(scientist\)

https://upload.wikimedia.org/wikipedia/commons/thumb/c/cf/Andrews_Thomas.jpg/225px-
Andrews_Thomas.jpg


Fig. 1 Maxwell construction for a van der Waals system with the law of the corresponding states (we use the Mathematica for this). The critical point is at K. Phase coexistence occurs along the path $a-c-e$, when the shaded areas are equal. The line AK and AB are the spinodal lines. This figure is obtained by using the Mathematica. The van der Waals isotherms. For $t_{\mathrm{r}}<1$, there is a region $a-b-c-d-e$ in which, for a given values of reduced pressure $p_{\mathrm{r}}$ and the reduced volume $v_{r}$ is not uniquely specified by the van der Waals equation. In this region the gas transforms to liquid. The states on the path $b-d$ are unstable. The observed state follows the path ace.


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Fig. 2 ParametricPlot3D of $\left\{\mathrm{v}_{\mathrm{r}}, 10\left(1-\mathrm{t}_{\mathrm{r}}\right), \mathrm{p}_{\mathrm{r}}\right\}$ for the van der Waals system, by using the Mathematica. The co-existence boundary is shown by the blue circles. For the sake of clarity, we use $10\left(1-t_{\mathrm{r}}\right)$ instead of $\left(1-t_{\mathrm{r}}\right)$. The values of $v_{\mathrm{r}}$ and $p_{\mathrm{r}}$ for each reduced temperature are listed in the APPENDIX.

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## 1. Introduction

The van der Waals equation is a thermodynamic equation describing gases and liquids under a given set of pressure $(P)$, volume $(V)$, and temperature $(T)$ conditions (i.e., it is a thermodynamic equation of state). It was derived in 1873 by Johannes Diderik van der Waals, who received the Nobel Prize in 1910 for "his work on the equation of state for gases and liquids. The equation is a modification to and improvement of the ideal gas law, taking into account the nonzero size of atoms and molecules and the attraction between them. van der Waals equation of state, when supplemented by the Maxwell construction (equal-area rule), provides in principle a complete description of the gas and its transition to the liquid, including the shape of the coexistence boundary curve.

Here we discuss the physics of the van der Waals equation of state from numerical calculations. We use the Mathematica to determine the detail of the flat portion (the coexistence line of liquid phase and gas phase). We also discuss the critical behavior near the critical point. To this end, it is significant for us to get the appropriate Mathematica program to determine the nature of the flat portion (the coexistence of liquid and gas phase). Before we started to make our own Mathematica program for the van der Waals equation of state, we found three resources for the programs related to this equation (as far as we know). The Maxwell construction was briefly discussed using the Mathematica by Kinzel and Reents (1998). Second is form the book of Nino Boccara, Essentials of Mathematica (Springer, 2007). The third is from Paul Abbott, The Mathematica Journal vol. 8 Issue 1 (2001, Trick of the Trade, Maxwell Construction). Here we use the method with FindRoot, which is used by Abbott for the evaluation of Maxwell's construction. There is no simple analytical solution to equation for the Maxwell construction. Fairly accurate initial guesses are required. These can be obtained from the plots of the unphysical van der Waals equation. Here we show our Mathematica program to discuss the van der Waals equation of state.

Here we use the Mathematica (ContourPlot, ParametricPlot, Plot3D, ParametricPlot3D, and so on) for the calculations. Because of the nature of the nonlinearity in the van der Waals equation of state, the use of the Mathematica is essential to our understanding on the critical behavior of liquid-gas system around the critical point.

Although we spent many years in understanding the nature of the van der Waals equation of state. unfortunately our understanding was not sufficient. Thanks to the Mathematica, finally we really understand how to calculate the exact values of thermodynamic parameters at fixed temperatures such as $p_{1}, v_{1}, v_{2}, v_{3}, v_{\mathrm{m} 1}, p_{\mathrm{m} 1}, v_{\mathrm{m} 3}, p_{\mathrm{m} 3}$ (see the definitions in the text) using the Mathematica. Using these parameters we will discuss various thermodynamic properties of the van der Waals equation of state. There have been so many books and papers since the appearance of the van der Waals equation. Almost all the universal properties of van der Waals equation have been discussed thoroughly. Although there is nothing new in this article, we present our results of calculations using Mathematica.

## 2. Historical Background

The proper elucidation of the nature of gas-liquid equilibrium and the so-called critical point was gained by a series of experiments carried out by Thomas Andrews at Queen's College, Belfast, between 1861 and 1869. He chose carbon dioxide $\left(\mathrm{CO}_{2}\right)$ for his work. It is gaseous at normal temperatures and the pressure required for studying the whole range where gas and liquid are in equilibrium are relatively low. He determined, at different temperatures, the change in the volume of a given quantity of the substance
when the pressure varied. The resultant curves are called isotherms because they each refer to one and the same temperature.

The flat part of the isotherm reveals an important fact. Since the pressure remains constant, while more and more of the gas condenses into liquid, the pressure of the gas in contact with the liquid must be always the same, quite independent of whether a small or a large fraction if the volume is occupied by liquid. It also is apparent from Andrew's diagram that this equilibrium pressure rises as we go to higher isotherms, i.e., as the temperature is increased. Moreover, we also notice that the flat part becomes shorter until a singularly important isotherm is reached which has no true flat portion at all but just one point (the socalled the critical point $T_{\mathrm{c}}$ ) at which the direction of the curve changes its sign. The higher isotherms are now all ascending smoothly over the whole range of pressure and volume, and if one goes to still higher temperatures, the isotherms attain more and more the shape of true rectangular hyperbola. This is then the region in which Boyle's law is valid.



Fig. 3
Isotherms of a real gas $\left(\mathrm{CO}_{2}\right)$ as measured by Andrews. They approximate Boyle's law only at high temperatures. At low temperatures they are more complicated and below the critical point there is a region of liquefaction. The critical temperature of $\mathrm{CO}_{2}$ is $31^{\circ} \mathrm{C}$.


Fig. 4 Isotherms of a real gas $\left(\mathrm{H}_{2} \mathrm{CO}_{3}\right)$ as measured by Andrews.
Andrew's result not only yielded a wealth of new facts but they also presented a beautifully complete and satisfying picture of the relation between the gaseous and liquid states of aggregation. Andrew's careful measurements opened the way to an understanding of the strong forces of cohesion which are vested in each atom but never reach the dimension of ordinary macroscopic observation. It should also be noted that, while Andrew's observations were confined to carbon dioxide, the pattern is quite generally valid.

We have used Andrew's diagram not only for its historical interest but also because it illustrates in a clear and convincing manner the significance and the boundaries of the liquid state. Van der Waals used Andrews's terminology, and even adopted the title of Andrews's Bakerian lecture, without reference, almost verbatim as the title of his doctoral thesis of Van der Waals developed his equation of state independently, but he did compare it with Andrews's results.

Only four years elapsed before van der Waals used newly developing ideas on the kinetic theory of gases to give a plausible theoretical explanation of Andrew's experimental data. van der Waals assume that gas is made up of molecules with a hard core and a long-range mutual attraction. The range of the attractive forces was assumed to be long compared with the mean free path, and they give rise to a negative internal pressure.

$$
P_{\text {interral }}=-\frac{a}{v^{2}},
$$

where $v=V / N$ is the volume per molecule. For the hard core he made the simplest assumption that the available volume is reduced from $v$ to v-b. Hence the equation he put forward was

$$
P=P_{\text {interral }}+\frac{k_{B} T}{v-b}, \quad \text { or } \quad P+\frac{a}{v^{2}}=\frac{k_{B} T}{v-b} .
$$

The new equation, which instead of the old $P V / T=$ constant now has, when plotted, a peculiar wiggly shape. In this wiggly region van der Waal's equation has for any given pressure three solutions for the volume. A straight line, joining any of these three solutions will then result in a curve which very closely resembles the flat portion of the Andrew's isotherms. It is no doubt that in its broad concepts the van der Waals' approach was correct. The importance of this equation was quickly recognized by Maxwell who reviewed the thesis in Nature in 1874, and in a lecture to Chemical Society in 1875. It was in this lecture that Maxwell put forward his famous "equal-area construction, which completes the van der Waals treatment of liquid-gas equilibrium. The equal-area rule (Maxwell construction) can be expressed as

$$
P_{V}\left(V_{g}-V_{l}\right)=\int_{V_{V}}^{V_{G}} P d V
$$

where $P_{V}$ is the vapor pressure (flat portion of the curve), $V_{l}$ is the volume of the pure liquid phase on the diagram, and $V_{g}$ is the volume of the pure gas phase. The sum of these two volumes will equal the total volume $V$.

Thanks to such pioneering works, we now understand the essential nature of liquid phase and gas phase. A flat portion for the low temperature phase, corresponds to the region where the liquid condenses from the gas. Following any of these isotherms from large to small volume, i.e., starting on the right-hand side, we encounter the rise and then a kink where the level portion starts. Here the very first droplets of liquid appear. When now the volume is further decreased, more and more of the gas turns into liquid until, at the end of the level stretch, there is no gas left at all. From now on any further increase in pressure hardly changes the volume at all, showing that the liquid phase is highly incompressible.

Much more detail of the historical background on the van der Waals equation of state can be learned from the following books.
J.C. Maxwell, The Scientific Papers of James Clerk Maxwell vol.II, van der Waals on the Continuity of the Gaseous and Liquid States (Dover Edition). ). p.424-426.
J.S. Rowlinson and F.L. Swinton, Liquids and Liquid Mixtures, $3^{\text {rd }}$ edition (Butterworth Scientific, 1982).
C. Domb, The Critical Point: A historical introduction to the modern theory of critical phenomena (Taylor \& Francis, 1996). p.39-74.
J.L. Sengers, Howfluids in mix Discoveries by the School of Van der Waals and Kamerlingh Onnes (Royal Netherlands Academy of Arts and Sciences, 2002).
R. Flood, M. McCartney and A. Whitaker, James Clerk Maxwell: Perspectives on his Life and Work (Oxford, 2014). Chapter 8, J.S. Rowlinson, Maxwell and the theory of Liquids.
3. Origin for the van der Waals equation of state
van der Waals realized that two main factors were to be added to the ideal gas equation: the effect of molecular attraction and the effect of molecular size. The intermolecular forces would add a correction to the ideal gas pressure, whereas the molecular size would decrease the effective volume. In the case of the ideal gas there is no intermolecular attraction. The intermolecular attraction decreases the pressure from its ideal value. If $P_{\text {real }}$ is the pressure of a real gas and $P_{\text {ideal }}$ is the corresponding pressure of the ideal gas, i.e. the pressure in the absence of intermolecular forces, then

$$
P_{\text {ideal }}=P_{\text {real }}+\delta p,
$$

where $\delta p$ is the correction. Since the pressure is proportional to the number density ( $N / V$ ) (as can be seen from the ideal gas equation), $\delta p$ should be proportional to $(N / V)$ ). In addition, the total force on each molecule close to the wall of the container is also proportional to the number density ( $N / V$ ) ; hence $\delta p$ should be proportional to two factors of $(N / V))$ so that one may write:

$$
\delta p=a\left(\frac{N}{V}\right)^{2} .
$$

The correction to the volume due to the molecular size, i.e., the" excluded volume," is simply proportional to the number of molecules. Hence

$$
V_{\text {ideal }}=V-N b,
$$

in which $b$ is the correction for one mole. Substituting these values in the ideal gas equation

$$
P_{\text {ideal }} V_{\text {ideal }}=N k_{B} T .
$$

we obtain the van der Waals equation

$$
\left(P+\frac{a N^{2}}{V^{2}}\right)(V-N b)=N k_{B} T .
$$



Fig. 5 van der Waals considered molecular interaction and molecular size to improve the ideal gas equation. (a) The pressure of a real gas is less than the ideal gas pressure because intermolecular attraction decreases the speed of the molecules approaching the wall. Therefore $P_{\text {real }}=P_{\text {ideal }}-\delta p$. (b) The volume available to molecules is less than the volume of the container due to the finite size of the molecules. This "excluded" volume depends on the total number of molecules. Therefore $V_{\text {ideal }}=V-N b$. [D. Kondepudi and I. Prigogine, Modern Thermodynamics, p. 18 Figure 1.4].

## 4. Derivation of van der Waals equation: Helmholtz free energy

For ideal gas, the partition function is given by

$$
Z_{1}=n_{Q} V .
$$

so the free energy $F$ is calculated as

$$
\begin{aligned}
F & =-k_{B} T \ln \left(\frac{Z_{1}^{N}}{N!}\right) \\
& =-k_{B} T\left(N \ln Z_{1}-\ln N!\right) \\
& \approx-k_{B} T\left(N \ln Z_{1}-N \ln N+N\right) \\
& =-k_{B} T N\left[\ln \left(\frac{Z_{1}}{N}\right)+1\right] \\
& =-k_{B} T N\left[\ln \left(\frac{n_{Q}}{n}\right)+1\right]
\end{aligned}
$$

where $n=\frac{N}{V}$, and $n_{Q}$ is the quantum concentration;

$$
n_{Q}=\left(\frac{m k_{B} T}{2 \pi \hbar^{2}}\right)^{3 / 2}=\left[\frac{m k_{B} T}{2 \pi \frac{h^{2}}{4 \pi^{2}}}\right]^{3 / 2}=\left[\frac{2 \pi m k_{B} T}{h^{2}}\right]^{3 / 2}=\frac{\left(2 \pi m k_{B} T\right)^{3 / 2}}{h^{3}},
$$

where $m$ is a mass of atom.

$$
\begin{aligned}
f & =\frac{F}{N}=-k_{B} T\left\{\ln \left[n_{Q}(v-b)\right]+1\right\}-\frac{a}{v} \\
& =-k_{B} T \ln (v-b)-\frac{a}{v}-k_{B} T\left[\ln \left(n_{Q}\right)+1\right] \\
& =-k_{B} T \ln (v-b)-\frac{a}{v}+\Phi(T)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi(T) & =-k_{B} T\left[\ln \left(n_{Q}\right)+1\right] \\
& =-k_{B} T\left\{\ln \left[\frac{\left(2 \pi m k_{B} T\right)^{3 / 2}}{h^{3}}\right]+1\right\} \\
& =-k_{B} T\left\{\ln T^{3 / 2}+\ln \frac{\left(2 \pi m k_{B}\right)^{3 / 2}}{h^{3}}+1\right\} \\
& =-k_{B} T\left(\ln T^{3 / 2}+\ln \alpha+1\right) \\
& =-k_{B} T\left(\frac{3}{2} \ln T+\ln \alpha+1\right)
\end{aligned}
$$

with the constant $\alpha$

$$
\alpha=\frac{\left(2 \pi m k_{B}\right)^{3 / 2}}{h^{3}} .
$$

In summary the Helmholtz free energy is obtained as

$$
f=-\frac{a}{v}-k_{B} T\left[\ln (v-b)+\frac{3}{2} \ln T+\ln \alpha+1\right] .
$$

The pressure $P$ is obtained as

$$
P=-\left(\frac{\partial F}{\partial V}\right)_{T, N}=-\left(\frac{\partial f}{\partial v}\right)_{T, N}=\frac{k_{B} T}{v-b}-\frac{a}{v^{2}},
$$

or more simply,

$$
\left(P+\frac{a}{v^{2}}\right)(v-b)=k_{B} T,
$$

((van der Waals equation))
or

$$
P=-\frac{a}{v^{2}}+\frac{k_{B} T}{v-b} .
$$

Since $v=\frac{V}{N}$, the above van der Waals equation can be rewritten as

$$
\left(P+\frac{N^{2} a}{V^{2}}\right)(V-b N)=N k_{B} T .
$$

## 5. Law of the corresponding state

$$
\begin{equation*}
\left(\frac{\partial P}{\partial v}\right)_{T}=0, \quad\left(\frac{\partial^{2} P}{\partial v^{2}}\right)_{T}=0 \tag{1}
\end{equation*}
$$

From the condition, $\left(\frac{\partial P}{\partial v}\right)_{T}=0$, we get

$$
\begin{equation*}
\frac{2 a}{v^{3}}-\frac{k_{B} T}{(v-b)^{2}}=0 . \tag{2}
\end{equation*}
$$

From the condition, $\left(\frac{\partial^{2} P}{\partial v^{2}}\right)_{T}=0$, we get

$$
\begin{equation*}
\frac{-3 a}{v^{4}}+\frac{k_{B} T}{(v-b)^{3}}=0 . \tag{3}
\end{equation*}
$$

From Eqs.(1) - (3), we have

$$
P_{c}=\frac{a}{27 b^{2}}, \quad v_{c}=\frac{V_{c}}{N}=\frac{1}{\rho_{c}}=3 b, \quad T_{c}=\frac{8 a}{27 k_{B} b} .
$$

Note that

$$
P_{c} v_{c}=\frac{3}{8} k_{B} T_{c},
$$

or

$$
\begin{aligned}
& P_{c}\left(N_{A} v_{c}\right)=\frac{3}{8} k_{B} N_{A} T_{c}=\frac{3}{8} R T_{c}, \\
& a=\frac{27 k_{B}^{2} T_{c}^{2}}{64 P_{c}}=\frac{1}{N_{A}{ }^{2}} \frac{27 R^{2} T_{c}^{2}}{64 P_{c}} \quad b=\frac{k_{B} T_{c}}{8 P_{c}}=\frac{1}{N_{A}} \frac{R T_{c}}{8 P_{c}} \\
& \frac{a}{b}=\frac{27}{8} k_{B} T_{c} \text { (thus the unit if } a / b \text { is the energy). }
\end{aligned}
$$

Here we define the dimensionless variables by

$$
p_{r}=\frac{P}{P_{c}}, \quad v_{r}=\frac{V}{V_{c}}=\frac{v}{v_{c}}, \quad t_{r}=\frac{T}{T_{c}},
$$

and the dimensionless form of the van der Waals equation,

$$
\left(p_{r}+\frac{3}{v_{r}^{2}}\right)\left(v_{r}-\frac{1}{3}\right)=\frac{8}{3} t_{r} \quad \Rightarrow \text { Law of corresponding state. }
$$

## ((Universality)) Law of corresponding state

This equation is universal since it contains no parameters characteristic of an individual substance, and so it is equally valid for all. The variables of $p_{r}, v_{r}$, and $t_{r}$ is called the reduced variables. The thermodynamic properties of substances are the same in corresponding states, that is, states with a pair of equal reduced variables from the complete triplet of variables. In fact, the existence of such an equation implies that if two reduced variables are the same for a set of the systems, then the third reduced variable is also the same throughout the set.

## ((Visualization of the phase transition of van der Waals system by Mathematica))

The phase diagrams of $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ and $v_{\mathrm{r}}$ vs $p_{\mathrm{r}}$ are shown below. It can be obtained by using the Maxwell construction for the van der Waals system (we use the Mathematica to get this. The method will be discussed later).


Fig. 6
The phase diagrams of $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ fot $t_{r}=0.80-1.20$. The horizontal straight line for $t_{\mathrm{r}}<1$ is the coexistence line between the liquid phase and the gas phase.


Fig. $7 \quad$ The $v_{\mathrm{r}}$ vs $p_{\mathrm{r}}$ phase diagram at fixed reduced temperatures $\left(t_{\mathrm{r}}=0.80-1.20\right.$ with $\left.\Delta t_{\mathrm{r}}=0.02\right)$. The vertical straight line for $t_{\mathrm{r}}<1$ is the coexistence line between the liquid phase and gas phase.


Fig. $8 \quad \partial p_{r} / \partial v_{r}$ vs $v_{r}$ phase diagram at fixed reduced temperatures $\left(t_{\mathrm{r}}=0.80-1.20\right.$ with $\left.\Delta t_{\mathrm{r}}=0.02\right)$. $\partial p_{r} / \partial v_{r}=0$ for $v_{1} \leq v \leq v_{3}\left(t_{r}<1\right)$.



Fig. 9 Typical examples for the Maxwell construction. The phase diagrams of $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ fot $t_{\mathrm{r}}=0.99$ and 0.98. The area of closed path $a-c-b-a$ is equal to that of the closed $c-d-e-c$. The path $a-$ $c-e$ is the coexistence line.

## 6 Compressibility factor $\boldsymbol{Z}$

The compressibility factor $Z$ is defined by

$$
Z=\frac{P V}{N k_{B} T}=\frac{P v}{k_{B} T}=\frac{P_{c} p_{r} v_{c} v_{r}}{k_{B} T_{c} t_{r}}=\frac{P_{c} v_{c}}{k_{B} T_{c}} \frac{p_{r} v_{r}}{t_{r}}=\frac{3}{8} \frac{p_{r} v_{r}}{t_{r}}
$$

where

$$
\frac{P_{c} v_{c}}{k_{B} T_{c}}=\frac{3}{8}
$$

$Z$ should be equal to $3 / 8$ at the critical point $\left(p_{\mathrm{r}}=1, v_{\mathrm{r}}=1, t_{\mathrm{r}}=1\right)$ for the van der Waals systems. We make a plot of $Z$ as a function of $p_{\mathrm{r}}$ by using the ParametricPlot of Mathematica, where $t_{\mathrm{r}}$ is changed as a parameter $\left(t_{\mathrm{r}}=0.7-2\right)$. When $t_{\mathrm{r}}$ is much larger than $1, Z$ tends to 1 (it is independent of $v_{\mathrm{r}}$ ), as is expected from the Boyle's law for the ideal gas (in the non-interacting limit). The deviation from the ideal gas behavior (the Boyl's law) is clearly indicated from the compressibility factor $Z$ as a function of $v_{r}$. Note that there is a discontinuity in $Z$ at the reduced pressure corresponding to the coexistence line. It is a useful thermodynamic property for modifying the ideal gas law (Boyle's law) to account for the real gas behavior.


Fig. 10
Compressibility factor Z as a function of $p_{\mathrm{r}}$ for the van der Waals gas. $t_{\mathrm{r}}=0.70-2.0 . \mathrm{Z}=$ 1 for the ideal gas obeying the Boyle's law.

## ((Mathematica))

$$
Z=\frac{3}{8} \frac{p_{r} v_{r}}{t_{r}}=\frac{3}{8} \frac{v_{r}}{t_{r}}\left(-\frac{3}{v_{r}{ }^{2}}+\frac{8 t_{r}}{3 v_{r}-1}\right),
$$

where

$$
p_{r}=-\frac{3}{v_{r}^{2}}+\frac{8 t_{r}}{3 v_{r}-1} .
$$

We make a ParametricPlot of the co-ordinate $\left(p_{r}, Z\right)$, where $t_{\mathrm{r}}$ is fixed as a parameter and $v_{\mathrm{r}}$ is changed over the whole range of $v_{r}$.


Fig. 11 Compressibility factor $Z$ as a function of reduced pressure $p_{\mathrm{r}}$, where $t_{\mathrm{r}}$ is changed as a parameter. The fact that the data for a wide variety of fluids fall on identical curves supports the law of corresponding states. [H.E. Stanley, Introductiom to Phase Transitions and Critical Phenomena, p.73]
((Mathematica))
Clear["Global **"] ; p0[t-, $\left.v_{-}\right]:=\frac{-3}{v^{2}}+\frac{\frac{8}{3} t}{v-\frac{1}{3}} ;$
$Z\left[t_{-}, v_{-}\right]:=\frac{3}{8} \frac{p 0[t, v] v}{t} ;$
f1 = ParametricPlot[
Evaluate[Table[\{p0[t, v], Z[t, v]\},
$\{t, 1,2,0.1\}]],\{v, 0.34,15\}$,
PlotRange $\rightarrow\{\{0,8\},\{0.2,1.5\}\}$,
PlotStyle $\rightarrow$ Table[\{Hue[0.1 i], Thick\}, \{i, 0, 10\}], AspectRatio $\rightarrow$ 1];
f2 =
Graphics [
\{Text[Style["pr", Black, 12, Italic], \{6, 0.25\}], Text[Style["Z", Black, 12, Italic], $\{0.32,1.4\}]$,
Text[Style[" $\left.t_{r}=1 ", B l a c k, 12, ~ I t a l i c\right]$,
\{1.0, 0.3\}],
Text[Style["1.2", Black, 12, Italic],
\{2, 0.55\}],
Text[Style["1.4", Black, 12, Italic], $\{2.7,0.7\}]$,
Text[Style["1.6", Black, 12, Italic], \{3.3, 0.79
\}], Text[Style["1.8", Black, 12, Italic],
\{4, 0.865\}],
Text[Style["2.0", Black, 12, Italic], $\{4.5,0.925\}]\}]$;
Show[f1, f2]

## 7. Critical points and critical exponents

To examine the critical behavior, we write

$$
p_{r}=\frac{P}{P_{c}}=1+\pi
$$


$t_{r}=\frac{T}{T_{c}}=1+\tau$
where $\pi, \omega$, and $\tau$ can be regarded as small. We obtain the universal equation

$$
p_{r}=1+\pi=-\frac{3}{(1+\omega)^{2}}+\frac{8(1+\tau)}{2+3 \omega}
$$

or expanding

$$
\begin{align*}
p_{r} & =1+\pi \\
& =1+4 \tau-6 \tau \omega+9 \tau \omega^{2}-\left(\frac{3}{2}+\frac{27}{2} \tau\right) \omega^{3}+\left(\frac{21}{4}+\frac{81}{4} \tau\right) \omega^{4}-\left(\frac{99}{8}+\frac{243}{8} \tau\right) \omega^{5} \\
& +\left(\frac{393}{16}+\frac{729}{16} \tau\right) \omega^{5}+\ldots \ldots  \tag{1}\\
& =1+4 \tau-6 \tau \omega+9 \tau \omega^{2}-\frac{3}{2} \omega^{3}+O\left(\tau \omega^{2}, \omega^{4}\right)
\end{align*}
$$

The term omitted from this expression are justified post hoc in fact, we can see that $\omega \approx \sqrt{\tau}$, so Eq.(1) is indeed the lowest non-trivial order approximation to the equation of state near the critical point.

## ((Mathematica))

Critical papameter of the van der Waal's equation $\mathrm{Pc}, \mathrm{vc}$, and Tc

$$
\begin{aligned}
& \text { Clear["Global`*"]; } \\
& \mathrm{P}=\frac{\mathrm{kB} T}{\mathrm{v}-\mathrm{b}}-\frac{\mathrm{a}}{\mathrm{v}^{2}} ; \\
& \text { eq1 = } D[P, v] / / \text { Simplify } \\
& -\frac{k B T}{(b-v)^{2}}+\frac{2 a}{v^{3}} \\
& \text { eq2 }=\mathrm{D}[\mathrm{P},\{\mathrm{v}, 2\}] / / \text { FullSimplify } \\
& -\frac{6 a}{v^{4}}+\frac{2 k B T}{(-b+v)^{3}} \\
& \text { eq3 = Solve [ }\{\text { eq1 }=0 \text {, eq2 }==0\},\{v, T\}] / / S i m p l i f y \\
& \left\{\left\{\mathrm{v} \rightarrow 3 \mathrm{~b}, \mathrm{~T} \rightarrow \frac{8 \mathrm{a}}{27 \mathrm{~b} \mathrm{kB}}\right\}\right\} \\
& \{\mathrm{Pc}, \mathrm{vc}, \mathrm{Tc}\}=\{\mathrm{P}, \mathrm{v}, \mathrm{~T}\} / . \mathrm{eq} 3[[1]] \\
& \left\{\frac{a}{27 b^{2}}, 3 b, \frac{8 a}{27 \mathrm{bkB}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \text { eq5 }=P / P c / \cdot\{T \rightarrow T c t\} / / \text { FullSimplify } \\
& \mathrm{b}\left(-\frac{27 \mathrm{~b}}{\mathrm{v}^{2}}+\frac{8 t}{-\mathrm{b}+\mathrm{v}}\right) \\
& \text { eq6 }=\text { eq5 } / \cdot\{\mathrm{v} \rightarrow \mathrm{vc}(1+\omega), t \rightarrow 1+\tau\} / / \text { FullSimplify } \\
& -\frac{3}{(1+\omega)^{2}}+\frac{8(1+\tau)}{2+3 \omega}
\end{aligned}
$$

Series [eq6, $\{\omega, 0,8\},\{t, 0,8\}] / /$ Normal
$1+4 \tau-6 \tau \omega+9 \tau \omega^{2}+\left(-\frac{3}{2}-\frac{27 \tau}{2}\right) \omega^{3}+$

$$
\begin{aligned}
& \left(\frac{21}{4}+\frac{81 \tau}{4}\right) \omega^{4}+\left(-\frac{99}{8}-\frac{243 \tau}{8}\right) \omega^{5}+\left(\frac{393}{16}+\frac{729 \tau}{16}\right) \omega^{6}+ \\
& \left(-\frac{1419}{32}-\frac{2187 \tau}{32}\right) \omega^{7}+\left(\frac{4833}{64}+\frac{6561 \tau}{64}\right) \omega^{8}
\end{aligned}
$$

## (a) Critical exponent $\beta$

We start with

$$
p \approx 1+4 \tau-6 \tau \omega+9 \tau \omega^{2}-\frac{3}{2} \omega^{3}
$$



Fig. $12 \quad$ Physical meaning of $p_{r}=1+4 \tau$ at $\omega=0$.

When $\omega=0$,

$$
p_{r}=1+4 \tau \quad \text { with } \tau<0 .
$$

which is nearly equal to the reduced pressure $p_{1}$ for the coexistence line (the path $a-c-e$ ). Then we have

$$
-6 \tau \omega-\frac{3}{2} \omega^{3}=0
$$

or

$$
\omega=0, \quad \omega= \pm 2 \sqrt{-\tau} .
$$

where $\tau<0$. Then we have

$$
\omega_{3}=2 \sqrt{-\tau}, \quad \omega_{1}=-2 \sqrt{-\tau} .
$$

The reduced pressures $v_{1}\left(=v_{l}\right)$ at the point $a$ and $v_{3}\left(=v_{\mathrm{g}}\right)$ at the point $e$, are obtained as

$$
\begin{aligned}
& v_{1}=1+\omega_{1}=1+2 \sqrt{-\tau}, \\
& v_{3}-v_{1}=v_{g}-v_{l}=4 \sqrt{-\tau},
\end{aligned}
$$

$v_{g}-v_{l}$ depends on $(-\tau)$. It reduces to zero when $\tau \rightarrow 0_{-}$. The critical exponent $\beta$ is equal to $1 / 2$.
$\beta=\frac{1}{2} \quad$ (mean-field exponent).


Fig. 13 The points $a ;\left(v_{1}, p_{1}\right), c ;,\left(v_{2}, p_{1}\right)$, and $e ;\left(v_{3}, p_{1}\right)$, in the $p_{\mathrm{r}}-v_{\mathrm{r}}$ plane. $p_{1}=1+4 \tau$. $v_{1}=1+\omega_{1}=1+2 \sqrt{-\tau} . v_{3}=1+\omega_{3}=1-2 \sqrt{-\tau}$. $\tau$ is negative. $\varepsilon=-\tau=|\tau|$ is very small.

## (b) Critical exponent $\gamma$ and $\gamma$

The isothermal compressibility is defined by

$$
K_{T}=-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T}=-\frac{1}{P_{c} v_{r}}\left(\frac{\partial v_{r}}{\partial p_{r}}\right)_{\tau} \approx-\frac{1}{P_{c}}\left(\frac{\partial \omega}{\partial \pi}\right)_{\tau} .
$$

Since

$$
\pi=4 \tau-6 \tau \omega-\frac{3}{2} \omega^{3}, \quad\left(\frac{\partial \omega}{\partial \pi}\right)_{\tau}=-\frac{1}{6 \tau+\frac{9}{2} \omega^{2}}
$$

we get

$$
K_{T}=-\frac{1}{P_{c}}\left(\frac{\partial \omega}{\partial \pi}\right)_{\tau}=\frac{1}{P_{c}}\left(6 \tau+\frac{9}{2} \omega^{2}\right)^{-1} .
$$

For $\tau>0, \quad \omega=0$.

$$
K_{T}=\frac{1}{6 P_{c}} \tau^{-1} .
$$

For $\tau<0, \quad \omega^{2}=-4 \tau$,

$$
\kappa_{T}=\frac{1}{P_{c}}(6 \tau-18 \tau)^{-1}=\frac{1}{12 P_{c}}(-\tau)^{-1},
$$

The isothermal compressibility $\kappa_{T}$ diverges as $t_{r} \rightarrow 1 \pm 0$ with a critical exponent

$$
\gamma=\gamma^{\prime}=1 . \quad \text { (mean-field exponent). }
$$

In summary, we have
$K_{T}=\left\{\begin{array}{cc}\frac{1}{6 P_{c} \tau} & T>T_{c} \\ \frac{1}{12 P_{c}(-\tau)} & T<T_{c}\end{array}\right.$

## (c) Critical exponent for specific heat at constant volume

The specific heat predicted by the van der Waals theory is

$$
c_{V}=\left\{\begin{array}{cc}
k_{B}\left[\frac{3}{2}+\frac{9}{2}\left(1-\frac{28}{25} \varepsilon+\ldots\right)\right. & t_{r}<1 \\
k_{B} \frac{3}{2} & t_{r}>1
\end{array}\right.
$$

where $c_{V}=3 k_{B} / 2$ is the non-interacting (high-temperature) limit or ideal gas. Thus we have the critical exponent,

$$
\alpha=0 .
$$

Note that the slope of $c_{V}$ vs $t_{\mathrm{r}}$ is finite as $t_{r} \rightarrow 1$ from below, so that we have $\alpha^{\prime}=0$.
$(($ Note $))$ This discussion is repeated later for the critical behavior of the specific heat.

## (d) Critical exponent $\delta$ (critical isotherm)

$$
p_{r}=-\frac{3}{v_{r}{ }^{2}}+\frac{8 t_{r}}{3 v_{r}-1}
$$

at $t_{r}=1$. We expand $p_{\mathrm{r}}$ at $t_{\mathrm{r}}=1\left(T=T_{\mathrm{c}}\right)$ in the vicinity of $v_{r}=1$.

$$
p_{r}-1=-\frac{3}{2}\left(v_{r}-1\right)^{3}+\frac{21}{4}\left(v_{r}-1\right)^{4}-\frac{99}{8}\left(v_{r}-1\right)^{5}+\ldots
$$

This is approximated by

$$
p_{r}-1 \approx-\frac{3}{2}\left(v_{r}-1\right)^{3} \propto\left(v_{r}-1\right)^{\delta},
$$

or

$$
v_{r}-1 \approx\left(p_{r}-1\right)^{1 / \delta} .
$$

in the region very close to the critical point, leading to the critical exponent (critical isotherm)

$$
\delta=3 \quad \text { (the mean-field exponent) }
$$

## (e) Thermal expansion coefficient

The thermal expansion coefficient is given by

$$
\alpha=\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{P}=\frac{1}{T_{c} v_{r}}\left(\frac{\partial v_{r}}{\partial t_{r}}\right)_{P}=-\frac{1}{T_{c} v_{r}} \frac{\left(\frac{\partial p_{r}}{\partial t_{r}}\right)_{V}}{\left(\frac{\partial p_{r}}{\partial v_{r}}\right)_{T}}
$$

Using the expression of $K_{T}$,

$$
K_{T}=-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T}=-\frac{1}{P_{c} v_{r}}\left(\frac{\partial v_{r}}{\partial p_{r}}\right)_{T}
$$

we have

$$
\begin{aligned}
\alpha & =-\frac{1}{T_{c} v_{r}}\left(\frac{\partial v_{r}}{\partial p_{r}}\right)_{T}\left(\frac{\partial p_{r}}{\partial t_{r}}\right)_{V} \\
& =-\frac{1}{T_{c} v_{r}}\left(-P_{c} v_{r} K_{T}\right)\left(\frac{\partial p_{r}}{\partial t_{r}}\right)_{V} \\
& =\frac{P_{c}}{T_{c}} K_{T}\left(\frac{\partial p_{r}}{\partial t_{r}}\right)_{V}
\end{aligned}
$$

Noting that

$$
\left(\frac{\partial p_{r}}{\partial t_{r}}\right)_{V}=\left(\frac{\partial p_{r}}{\partial t_{r}}\right)_{v_{r}}=\frac{8}{3 v_{r}-1}
$$

we get

$$
\alpha=\frac{P_{c}}{T_{c}} K_{T} \frac{8}{3 v_{r}-1}
$$

Around the critical point, we have

$$
\alpha \rightarrow \frac{4 P_{c}}{T_{c}} K_{T}= \begin{cases}\frac{2}{3 T_{c} \tau} & T>T_{c} \\ \frac{1}{3 T_{c}(-\tau)} & T<T_{c}\end{cases}
$$

So that it is strongly divergent like $K_{T}$.

## (f) Mean-field exponent relation

From the above discussion, we find that the following relation is valid,

$$
\alpha+2 \beta+\gamma=2
$$

which is the same as that predicted from the mean-field theory. We also have the relation predicted from the mean field theory of phase transition,

$$
\beta=\frac{2-\alpha}{1+\delta}=\frac{1}{2}, \quad \gamma=\frac{(\alpha-2)(1-\delta)}{1+\delta}=1
$$

These results imply that the van der Waals theory is one of the mean field theories. Well above the critical temperature there exists the short range order due to the attractive interaction between particles. On approaching the critical temperature, short range grows gradually. At the critical temperature, a part of short range order changes into the long range order. Well below the critical temperature, the long range order extends over the entire system.

## 8. Scaled thermodynamic potential

(a) Scaled Helmholtz free energy $f$

Using the reduced variables, the Helmholtz free energy is given by

$$
\begin{aligned}
f & =-\frac{a}{v}-k_{B} T\left[\ln (v-b)+\frac{3}{2} \ln T+\ln \alpha+1\right] \\
& =\frac{a}{b}\left\{-\frac{1}{3 v_{r}}-\frac{8}{27} t_{r}\left[\ln \left(v_{r}-\frac{1}{3}\right)+\frac{3}{2} \ln t_{r}+C_{1}\right]\right\}
\end{aligned}
$$

where $C_{1}$ is constant

$$
\begin{aligned}
& C_{1}=\frac{3}{2} \ln \left(T_{c}\right)+\ln \alpha+\ln (3 b)+1=\frac{s_{0}}{k_{B}}-\frac{3}{2} . \\
& \frac{a}{b}=\frac{27}{8} k_{B} T_{c} .
\end{aligned}
$$

## (b) The law of the corresponding states

The pressure is given by

$$
\begin{aligned}
P & =-\left(\frac{\partial f}{\partial v}\right)_{T, N}=-\frac{a}{v^{2}}+\frac{k_{B} T}{v-b} \\
& =-\frac{a}{9 b^{2} v_{r}{ }^{2}}+\frac{k_{B} T_{c} t_{r}}{3 b v_{r}-b} \\
& =-\frac{a}{9 b^{2} v_{r}{ }^{2}}+\frac{8 a}{27 b^{2}} \frac{t_{r}}{3 v_{r}-1} \\
& =\frac{a}{b^{2}}\left(-\frac{1}{9 v_{r}{ }^{2}}+\frac{8}{27} \frac{t_{r}}{3 v_{r}-1}\right)
\end{aligned}
$$

The reduced pressure $p_{\mathrm{r}}$ is given by

$$
p_{r}=\frac{P}{P_{c}}=\frac{1}{\frac{a}{27 b^{2}}} \frac{a}{b^{2}}\left(-\frac{1}{9 v_{r}{ }^{2}}+\frac{8}{27} \frac{t_{r}}{3 v_{r}-1}\right)=-\frac{3}{v_{r}{ }^{2}}+\frac{8 t_{r}}{3 v_{r}-1}
$$

or

$$
p_{r}=-\frac{3}{v_{r}^{2}}+\frac{8 t_{r}}{3 v_{r}-1}
$$

(c) Scaled internal energy $\boldsymbol{u}$

The internal energy is determined by standard thermodynamics,

$$
\begin{aligned}
u & =\frac{U}{N} \\
& =-T^{2} \frac{\partial}{\partial T}\left(\frac{f}{T}\right)_{v} \\
& =-\frac{a}{v}+\frac{3}{2} k_{B} T \\
& =-\frac{a}{3 b v_{r}}+\frac{3}{2} k_{B} T_{c} t_{r} \\
& =\frac{a}{b}\left(-\frac{1}{3 v_{r}}+\frac{4 t_{r}}{9}\right)
\end{aligned}
$$

or

$$
u=\frac{a}{b}\left(-\frac{1}{3 v_{r}}+\frac{4 t_{r}}{9}\right)
$$

with

$$
\frac{a}{b}=\frac{27}{8} k_{B} T_{c} .
$$

(d) Scaled entropy $s$

The entropy $S$ can be similarly determined by standard thermodynamics

$$
s=\frac{S}{N}
$$

where

$$
\begin{aligned}
s & =-\left(\frac{\partial f}{\partial T}\right)_{v}=k_{B} \ln (v-b)+k_{B}\left[\ln \frac{\left(2 \pi m k_{B} T\right)^{3 / 2}}{h^{3}}+\frac{5}{2}\right] \\
& =k_{B} \ln (v-b)+\frac{3}{2} k_{B} \ln T+k_{B}\left(\ln \alpha+\frac{5}{2}\right) \\
& =k_{B} \ln \left(3 b v_{r}-b\right)+\frac{3}{2} k_{B} \ln \left(t_{r} T_{c}\right)+k_{B}\left(\ln \alpha+\frac{5}{2}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
s & =k_{B}\left\{\ln \left(v_{r}-\frac{1}{3}\right)+\frac{3}{2} \ln t_{r}+\frac{3}{2} \ln T_{c}+\ln \alpha+\ln (3 b)+\frac{5}{2}\right\} \\
& =k_{B}\left[\ln \left(v_{r}-\frac{1}{3}\right)+\frac{3}{2} \ln t_{r}\right]+s_{0}
\end{aligned}
$$

where

$$
s_{0}=k_{B}\left[\ln \alpha+\ln (3 b)+\frac{3}{2} \ln T_{c}+\frac{5}{2}\right]
$$

In the adiabatic process ( $s=$ constant, isentropic process), we have

$$
\ln \left[\left(v_{r}-\frac{1}{3}\right) t_{r}^{3 / 2}\right]=\text { const. }
$$

or

$$
\left(v_{r}-\frac{1}{3}\right) t_{r}^{3 / 2}=\mathrm{constant}
$$

Note that for the ideal monatomic gas,

$$
v_{r} t_{r}^{3 / 2}=\text { constant. }
$$

## (e) Scaled Gibbs free energy $g$

The Gibbs free energy $G$ is given by

$$
\begin{aligned}
g(T, P) & =\frac{G(T, P)}{N} \\
& =f+P v \\
& =-\frac{2 a}{v}+\frac{k_{B} T v}{v-b}-k_{B} T\left[\ln (v-b)+\frac{3}{2} \ln T+\ln \alpha+1\right]
\end{aligned}
$$

$g(T, P)$ can be rewritten as

$$
\begin{aligned}
g(T, P) & =-\frac{2 a}{3 b\left(v_{r}\right)}+\frac{k_{B} t_{r} T_{c} 3 b\left(v_{r}\right)}{3 b\left(v_{r}\right)-b}-k_{B} t_{r} T_{c}\left\{\ln \left[3 b\left(v_{r}\right)-b\right]\right. \\
& \left.\left.+\frac{3}{2} \ln \left(t_{r} T_{c}\right)+\ln \alpha+1\right]\right\} \\
& =-\frac{2 a}{3 b\left(v_{r}\right)}+\frac{8 a}{27 b} \frac{t_{r} v_{r}}{v_{r}-\frac{1}{3}}-\frac{8 a}{27 b} t_{r}\left[\ln \left(v_{r}-\frac{1}{3}\right)\right. \\
& \left.+\frac{3}{2} \ln t_{r}+\frac{3}{2} \ln \left(T_{c}\right)+\ln \alpha+\ln (3 b)+1\right] \\
& =\frac{a}{b}\left\{-\frac{2}{3 v_{r}}+\frac{8}{27}\left(\frac{t_{r} v_{r}}{v_{r}-\frac{1}{3}}\right)-\frac{8}{27} t_{r}\left[\ln \left(v_{r}-\frac{1}{3}\right)\right.\right. \\
& \left.\left.+\frac{3}{2} \ln t_{r}+\frac{3}{2} \ln \left(T_{c}\right)+\ln \alpha+\ln (3 b)+1\right]\right\} \\
& =\frac{a}{b}\left\{-\frac{2}{3 v_{r}}+\frac{8}{27}\left(\frac{t_{r} v_{r}}{v_{r}-\frac{1}{3}}\right)-\frac{8}{27} t_{r}\left[\ln \left(v_{r}-\frac{1}{3}\right)+\frac{3}{2} \ln t_{r}+C_{1}\right]\right\}
\end{aligned}
$$

Here we have

$$
C_{1}=\frac{3}{2} \ln \left(T_{c}\right)+\ln \alpha+\ln (3 b)+1=\frac{s_{0}}{k_{B}}-\frac{3}{2} .
$$

where $s_{0}$ is the constant entropy. Note that the above equation gives $g$ as a function of $v$ and $T$. The natural variables for $g$ are $P$ and $T$,

$$
\left(p_{r}+\frac{3}{v_{r}^{2}}\right)\left(v_{r}-\frac{1}{3}\right)=\frac{8}{3} t_{r} .
$$

We note that the Gibbs free energy can also be obtained by the following approach.

$$
\begin{aligned}
g & =\frac{G}{N} \\
& =u-T s+P v \\
& =\frac{a}{b}\left(-\frac{1}{3 v_{r}}+\frac{4 t_{r}}{9}\right)-T_{c} t_{r} k_{B}\left[\ln \left(v_{r}-\frac{1}{3}\right)+\frac{3}{2} \ln t_{r}+\frac{s_{0}}{k_{B}}\right]-\frac{a}{3 b v_{r}}+\frac{k_{B} T_{c} t_{r} 3 b v_{r}}{3 b v_{r}-b} \\
& =\frac{a}{b}\left(-\frac{1}{3 v_{r}}+\frac{4 t_{r}}{9}\right)-\frac{8 a}{27 b} t_{r}\left[\ln \left(v_{r}-\frac{1}{3}\right)+\frac{3}{2} \ln t_{r}+\frac{s_{0}}{k_{B}}\right]-\frac{a}{3 b v_{r}}+\frac{8 a}{27 b} \frac{t_{r} v_{r}}{v_{r}-\frac{1}{3}} \\
& =\frac{a}{b}\left\{-\frac{2}{3 v_{r}}+\frac{8}{27} \frac{t_{r} v_{r}}{v_{r}-\frac{1}{3}}-\frac{8}{27} t_{r}\left[\ln \left(v_{r}-\frac{1}{3}\right)+\frac{3}{2} \ln t_{r}-\frac{3}{2}+\frac{s_{0}}{k_{B}}\right]\right\} \\
& =\frac{a}{b}\left\{-\frac{2}{3 v_{r}}+\frac{8}{27} \frac{t_{r} v_{r}}{v_{r}-\frac{1}{3}}-\frac{8}{27} t_{r}\left[\ln \left(v_{r}-\frac{1}{3}\right)+\frac{3}{2} \ln t_{r}+C_{1}\right]\right\}
\end{aligned}
$$

## (f) Thermodynamics surface

From the expression of $u$, the temperature $T$ is calculated as

$$
k_{B} T=\frac{2}{3}\left(\frac{a}{v}+u\right) .
$$

The thermodynamics surface $u(s, v)$ is obtained by eliminating $T$, as

$$
s-k_{B} \ln (v-b)-s_{1}=\frac{3}{2} k_{B} \ln T=\frac{3}{2} k_{B} \ln \frac{2\left(\frac{a}{v}+u\right)}{3 k_{B}}
$$

or

$$
s=k_{B}\left[\ln (v-b)+\frac{3}{2} \ln T\right]+s_{1}
$$

with

$$
s_{1}=k_{B}\left(\ln \alpha+\frac{5}{2}\right)
$$

Then we get

$$
\begin{aligned}
\frac{2\left(\frac{a}{v}+u\right)}{3 k_{B}} & =\exp \left\{\frac{2}{3 k_{B}}\left[s-s_{1}-k_{B} \ln (v-b)\right]\right\} \\
& =\exp \left[\frac{2}{3 k_{B}}\left(s-s_{1}\right)\right] \exp \left[-\frac{2}{3} \ln (v-b)\right] \\
& =(v-b)^{-2 / 3} \exp \left[\frac{2}{3 k_{B}}\left(s-s_{1}\right)\right]
\end{aligned}
$$

Thus u depends on $v$ and $s$,

$$
\begin{aligned}
u & =-\frac{a}{v}+\frac{3 k_{B}}{2}(v-b)^{-2 / 3} \exp \left[\frac{2\left(s-s_{1}\right)}{3 k_{B}}\right] \\
& =-\frac{a}{v}+c_{0}(v-b)^{-2 / 3}
\end{aligned}
$$

with

$$
c_{0}=\frac{3 k_{B}}{2} \exp \left[\frac{2\left(s-s_{1}\right)}{3 k_{B}}\right]
$$

## 9. Mathematica program

Method how to determine the values of characteristic reduced pressure and volume at a fixed reduced temperature

The van der Waals equation is given by

$$
\begin{equation*}
p_{r}=-\frac{3}{v_{r}{ }^{2}}+\frac{8 t_{r}}{3 v_{r}-1}, \tag{1}
\end{equation*}
$$

(i) The local maximum point (the point $d$ ) and local minimum point (the point $b$ ) satisfies:

$$
\begin{equation*}
\frac{\partial p_{r}}{\partial v_{r}}=\frac{6}{v_{r}^{3}}-\frac{24 t_{r}}{\left(3 v_{r}-1\right)^{2}}=0 . \tag{2}
\end{equation*}
$$

(ii) Maxwell's construction:

$$
\begin{equation*}
p_{r}\left(t_{1}, v_{1}\right)=p_{r}\left(t_{1}, v_{3}\right)=p_{1}, \tag{3}
\end{equation*}
$$

$$
\begin{equation*}
-\frac{3}{v_{1}}+\frac{3}{v_{3}}+\frac{8}{3} t_{1}\left[-\ln \left(3 v_{1}-1\right)+\ln \left(3 v_{3}-1\right)\right]=p_{1}\left(v_{3}-v_{1}\right) . \tag{4}
\end{equation*}
$$

(iii) $\quad v_{1}(a), v_{2}(c)$, and $v_{3}(e)$ are the roots of

$$
p_{r}\left(t_{r}, v_{r}\right)=p_{1}=-\frac{3}{v_{r}^{2}}+\frac{8 t_{r}}{3 v_{r}-1}
$$

Clear["Global`*"] ; p0[t_, $\left.v_{-}\right]:=\frac{-3}{v^{2}}+\frac{\frac{8}{3} t}{v-\frac{1}{3}}$;
Eq1[t_, v1_, v3_] := p0[t, v1] =: $\mathrm{p} 0[t, v 3]$;
$\operatorname{Eq} 2\left[t_{-}, v 1_{-}, v 3_{-}\right]=-\frac{3}{v 1}+\frac{3}{v 3}+\frac{8}{3} t(-\log [-1+3 v 1]+\log [-1+3 v 3])=$ $\mathrm{p} 0[\mathrm{t}, \mathrm{v} 1](\mathrm{v} 3-\mathrm{v} 1)$;


Fig. $14 \quad$ The phase diagrams of $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ fot $t_{r}=0.93$. The area of closed path $a-c-b-a$ is equal to that of the closed $c-d-e-c$. The path $a-c-e$ is the coexistence line.

For a fixed $t_{\mathrm{r}}\left(=t_{1}<1\right)$, the locations of the points $a$ and $e$ can be determined from
$a: \quad\left(v_{1}, p_{1}\right)$,
$c: \quad\left(v_{2}, p_{1}\right)$,
$e: \quad\left(v_{3}, p_{1}\right)$
$b: \quad\left(v_{m 1}, p_{m 1}\right), \quad d: \quad\left(v_{m 3}, p_{m 3}\right)$.

## (a) Appropriate method to find boundary conditions for $\boldsymbol{\nu}_{1}$ and $\boldsymbol{\nu}_{\mathbf{3}}$

For a fixed reduced temperature $t_{\mathrm{r}}$, we determine the values of $v_{\mathrm{m} 1}$ and $v_{\mathrm{m} 3}$ from Eq.(2) using DSolve. The initial values $v_{1}$ ' for $v_{1}$ and $v_{3}$ ' for $v_{3}$ are obtained as follows. First calculate the value of $\bar{v}$ as average of $v_{\mathrm{m} 1}$ and $v_{\mathrm{m} 2}$ as

$$
\bar{v}=\frac{1}{2}\left(v_{m 1}+v_{m 3}\right) .
$$

The corresponding pressure is obtained as

$$
\bar{p}=p_{r}\left(t_{r}, \bar{v}\right) .
$$

Next we solve the equation

$$
\bar{p}=p_{r}\left(t_{r}, v_{r}\right)
$$

This equation has three roots, $v=v_{1}{ }^{\prime}, v_{3}{ }^{\prime}$, and $\bar{v}$. Figure shows the $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ curve at $t_{\mathrm{r}}=0.86$.

$$
v_{m 1}=0.680031, \quad v_{m 3}=1.68212
$$

Then we have

$$
\begin{aligned}
& \bar{v}=\frac{v_{m 1}+v_{m 3}}{2}=1.18107 \\
& \bar{p}=p_{r}\left(t_{r}, \bar{v}\right)=0.554593 .
\end{aligned}
$$

Using Eq.(1), we solve

$$
\bar{p}=p_{r}\left(t_{r}, v\right),
$$

and get the three solutions,

$$
v_{1}^{\prime}=0.559688, \quad \text { and } \quad v_{3}^{\prime}=2.72774
$$

as well as the root $(\bar{v}=1.18107)$. The two outer ones $\left(v_{1}{ }^{\prime}\right.$, and $\left.v_{3}{ }^{\prime}\right)$ are suitable boundary values. Note that these values are already close to the values which we are looking for; $v_{1}=0.561955$ and $v_{3}=2.9545$.


Fig. 15 How to get the boundary values. $v_{1}^{\prime}$ amd $v_{3}{ }^{\prime}$ for finding the values of $v_{1}$ and $v_{3}$.
(b) Subroutine program to determine the boundary values of $v_{1}$ and $v_{3}$.

```
initial[t_ /; t< 1] :=
    Module[{v11, v31, v1, v2, v3, v, vi, eq11, eq12, eq21, eq22, t1},
    t1 = t;
    eq11 = D[p0[t1, v], v] // Simplify;
    eq12 = NSolve[eq11 == 0, v];
    v11 = v /. eq12[[1]]; v31 = v /. eq12[[2]];
    eq21 = NSolve[p0[t1,v]== p0[t1, v11 + v31
    eq22 = Sort[{v / . eq21[[1]], v / . eq21[[2]], v /. eq21[[3]]},
        #1 < #2 &];
    vi = {eq22[[1]], eq22[[3]]}]
```

((Subroutine program to determine the values of $v_{1 \mathrm{~m}}$ and $v_{3 \mathrm{~m}}$ (local maximum and local minimum))

```
Deriv1[t_ /; t< 1] := Module[{v, eq11, eq12, eq2, t1, N1, h1, k1},
    t1 = t;
    eq11 = D[p0[t1, v], v] // Simplify;
    eq12 = NSolve[eq11 == 0, v, Reals];
    N1 = Length[eq12];
    h1 = Table[v /. eq12[[i]], {i, 1, N1}];
    eq2 = Sort[h1, #1 < #2 &] ;
    k1 = Which[N1 == 1, {eq2[[1]], eq2[[1]], eq2[[1]]}, N1 == 2,
        {eq2[[1]], eq2[[1]], eq2[[2]]}, N1 == 3,
        {eq2[[1]], eq2[[2]], eq2[[3]] }]; {k1[[2]], k1[[3]]}];
```


## (c) Maxwell's construction

Using the initial values for $v_{1}$ and $v_{3}$, we determine the final values of $v 1$ and $v 3$ by using FindRoot for two equations,

$$
\begin{equation*}
p_{r}\left(t, v_{1}\right)=p_{r}\left(t, v_{3}\right) \tag{3}
\end{equation*}
$$

$$
\begin{equation*}
-\frac{3}{v_{1}}+\frac{3}{v_{3}}+\frac{8}{3} t\left[-\ln \left(3 v_{1}-1\right)+\ln \left(3 v_{3}-1\right)\right]=\left(-\frac{3}{v_{1}{ }^{2}}+\frac{8 t}{3 v_{1}-1}\right)\left(v_{3}-v_{1}\right) . \tag{4}
\end{equation*}
$$

with

$$
p_{1}=p_{r}\left(t, v_{1}\right)=-\frac{3}{v_{1}{ }^{2}}+\frac{8 t}{3 v_{1}-1} .
$$

## (d) Determination of $\boldsymbol{v}_{\mathbf{2}}$

For each $t=t_{1}(<1)$, we get the required values,

$$
\begin{array}{ll}
a: & \left(v_{1}, p_{1}\right), \\
b: & \left(v_{m 1}, p_{m 1}\right), \\
d: & \left(v_{m 3}, p_{m 3}\right) . \\
e: & \left(v_{3}, p_{1}\right)
\end{array}
$$

We also need the value of $v_{2}$ for $p_{\mathrm{r}}=p_{1}$ at the point $c$. Using the equation

$$
p_{1}=-\frac{3}{v^{2}}+\frac{8 t}{3 v-1}
$$

Using Solve, we get the solution of this equation as $v=v_{2}$, as well as $v=v_{1}$ and $v_{3}$.

$$
c: \quad\left(v_{2}, p_{1}\right)
$$

(e) Subroutine program to determine the value of $\boldsymbol{v}_{\mathbf{2}}$

```
FV2[t_, p_] := Module[\{v, g1, g2, t1, p1, N1, h1, k1, eq1, eq2\},
    t1 = t;
    p1 = p;
    \(\mathrm{g} 1=\mathrm{p} 0[\mathrm{t} 1, \mathrm{v}]\);
    g2 = g1 == p1;
    eq1 = NSolve[g2, v, Reals];
    N1 = Length [eq1];
    h1 = Table[v /.eq1[[i]], \{i, 1, N1\}];
    eq2 = Sort[h1, \#1 < \#2 \&] ;
    \(\mathrm{k} 1=\) Which [N1 = 1 , \(\{\) eq2[[1]], eq2[[1]], eq2[[1]]\}, N1 =: 2,
        \(\{e q 2[[1]], e q 2[[1]]\), eq2[[2]]\}, N1 == 3,
        \{eq2[[1]], eq2[[2]], eq2[[3]]\}]];
```

(f) Subroutine program to determine the values of $v_{1}$ and $\nu_{3}$ at $p_{1}$ based on the Maxwell construction

```
LG1[t_ /; t< 1] :=
    Module[{t1, eq1, v1, v2, v3, v11, v13, v21, v22, v23,
        vm21, pm21, vm23, pm23, p1, p21}, t1 = t;
    {v11, v13} = initial[t1];
    eq1 = FindRoot[Evaluate[{Eq1[t, v1, v3], Eq2[t, v1, v3]}],
        {v1, v11}, {v3, v13}];
    v21 = v1 /. eq1[[1]];
    v23 = v3 / . eq1[[2]];
    p21 = p0[t1, v21];
    vm21 = Deriv1[t1][[1]];
    vm23 = Deriv1[t1][[2]];
    v22 = FV2[t1, p21][[2]];
    pm21 = p0[t1, vm21];
    pm23 = p0[t1, vm23];
    {t1, p21, v21, pm21, vm21, v22, pm23, vm23, v23}];
```

( $\boldsymbol{p}_{\mathrm{r}}$-vs $\boldsymbol{v}_{\mathrm{r}}$ curve for $\left.\boldsymbol{t}<\mathbf{1}\right)$ )
$p_{\mathrm{r}}$ can be expressed by

$$
p_{r}=\left\{\begin{array}{cc}
-\frac{3}{v_{r}^{2}}+\frac{8 t}{3 v_{r}-1} & v_{r}<v_{1} \\
p_{1} & v_{1}<v_{r}<v_{3} \\
-\frac{3}{v_{r}^{2}}+\frac{8 t}{3 v_{r}-1} & v_{r}>v_{3}
\end{array}\right.
$$

with the co-existence line $\left(p_{1}\right)$ between $v_{1}$ and $v_{3}$.
(g) Subroutine program for the $p_{r}-v_{r}$ phase with the coexistence line

```
P3D[t_ , v_] := Module[{t1, p1, v1, vm1, pm1, v2, vm3, pm3, v3, h1},
    t1 = t;
    a = 10;
    {t1, p1, v1, pm1, vm1, v2, pm3, vm3, v3} = LG1[t1];
    h1[x_] := Which[0.5<x<v1, p0[t1, x], v1<x<v3, p1,
        x > v3, p0[t1, x]];
    {v, h1[v], a (1-t1)}]; P3U[t_, v_] := Module[{t1, h1}, t1 = t;
    h1[x_] := p0[t1, x];
    {v, h1[v], a (1-t1)}];
```



Fig. 16 Maxwell's construction. The phase diagram of $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ at $t_{\mathrm{r}}=0.88$.

```
h11 = ParametricPlot3D[
    Evaluate[Table[P3D[t, v], {t, 0.85, 0.999, 0.0025}],
        {v, 0.35, 3}],
    PlotStyle }->\mathrm{ Table[{Hue[0.01 i], Thick}, {i, 0, 70}],
    AspectRatio -> Full] ; a = 10;
h12 =
    Graphics3D[{Thick, Red, Line[{{1/3,0.5,1.5},{3, 0.5, 1.5}}],
        Blue, Line[{{1/3, 0.5, 1.5}, {1/3, 1.2, 1.5}}], Black,
        Line[{{1/3, 0.5, 0}, {1/3, 0.5, 1.5}}],
        Text[Style["K", Black, 15], {1, 1, 0}],
        Text[Style["pr", Black, 15], {0.4, 1.1, 1.5}],
        Text[Style["v
        Text[Style["a(1-tre)", Black, 15], {0.4, 1.2, 0.9}]}];
h21 = ParametricPlot3D[
    Evaluate[Table[P3U[t, v], {t, 1, 2, 0.0025}], {v, 0.35, 3}],
    PlotStyle }->\mathrm{ Table[{Hue[0.1 i], Thick}, {i, 0, 70}],
    AspectRatio }->\mathrm{ Full];
Show[h11, h12, h21, PlotRange }->{{1/3,3},{0.5,1.2}, {-0.2, 1.5}}
```



Clear["Global *"] ; p0[t_, $\left.v_{-}\right]:=\frac{-3}{v^{2}}+\frac{\frac{8}{3} t}{v-\frac{1}{3}}$;
Eq1[ $\left.t_{-}, v 1_{-}, v 3 \_\right]:=p 0[t, v 1]==p 0[t, v 3] ;$
$\mathrm{Eq} 2\left[t_{-}, v 1_{-}, v 3_{-}\right]=-\frac{3}{v 1}+\frac{3}{v 3}+\frac{8}{3} t(-\log [-1+3 v 1]+\log [-1+3 v 3])=$ $\mathrm{p} 0[\mathrm{t}, \mathrm{v} 1](\mathrm{v} 3-\mathrm{v} 1)$;
initial[t_ / $t<1]:=$
Module[\{v1m, v2m, v3m, v1, v2, v3, v, vi, eq11, eq12, eq21, eq22, t1\},
t1 = $t$;
eq11 = D[p0[t1, v], v] // Simplify;
eq12 = NSolve[eq11 =: 0, v];
$\mathrm{v} 1 \mathrm{~m}=\mathrm{v} / . \mathrm{eq} 12[[1]] ; \mathrm{v} 3 \mathrm{~m}=\mathrm{v} / . \mathrm{eq} 12[[2]]$;
eq21 $=$ NSolve $\left[p 0[t 1, v]==p 0\left[t 1, \frac{v 1 m+v 3 m}{2}\right], v\right]$;
eq22 $=\operatorname{Sort}[\{v / . \operatorname{eq21[[1]],v/.eq21[[2]],v/.eq21[[3]]\} ,\# 1<\# 2\& ];~}$
vi $=\{$ eq22[[1] ], eq22[[3]]\}]
FV2[t_ , $\left.p_{-}\right]:=$Module[\{ g1, g2, t1, p1, N1, h1, k1, k2, eq1, eq2\}, t1 = $t$;
p1 = p;
$\mathrm{g} 1=\mathrm{p} 0[\mathrm{t} 1, \mathrm{v}]$;
g2 = g 1 = p 1 ;
eq1 = NSolve[g2, v, Reals];
N1 = Length [eq1] ;
h1 = Table[v /.eq1[[i]], \{i, 1, N1\}];
eq2 = Sort[h1, \#1 < \#2 \& ] ;
$\mathrm{k} 2=$ Which $[\mathrm{N} 1=1$, $\{\mathrm{eq} 2[[1]], \mathrm{eq} 2[[1]]$, eq2[[1]]\}, $\mathrm{N} 1=2$, \{eq2[[1]], eq2[[1]], eq2[[2]]\}, N1 == 3, \{eq2[[1]], eq2[[2]], eq2[[3]]\}]];

```
Deriv1[t_] := Module[{v, eq11, eq12, eq2, t1, N1, h1, k1}, t1 = t;
    eq11 = D[p0[t1, v], v] // Simplify;
    eq12 = NSolve[eq11 == 0, v, Reals];
    N1 = Length[eq12];
    h1 = Table[v /. eq12[[i]], {i, 1, N1}];
    eq2 = Sort[h1, #1 < #2 &] ;
    k1 = Which[N1 == 1, {eq2[[1]], eq2[[1]], eq2[[1]]}, N1 == 2,
        {eq2[[1]], eq2[[1]], eq2[[2]]}, N1 == 3, {eq2[[1]], eq2[[2]], eq2[[3]] }];
    {k1[[2]], k1[[3]]}];
```

```
LG1[t_ / ; t< 1] :=
    Module[{t1, eq1, v1, v2, v3, v11, v13, v21, v22, v23, vm21, pm21,
        vm23, pm23, p1, p21}, t1 = t;
        {v11, v13} = initial[t1];
        eq1 = FindRoot[Evaluate[{Eq1[t, v1, v3], Eq2[t, v1, v3]}],
            {v1, v11}, {v3, v13}];
    v21 = v1 / . eq1 [[1]];
    v23 = v3 / . eq1[[2]];
    p21 = p0[t1, v21];
    vm21 = Deriv1[t1][[1]];
    vm23 = Deriv1[t1][[2]];
    v22 = FV2[t1, p21][[2]] ;
    pm21 = p0[t1, vm21];
    pm23 = p0[t1, vm23];
    {t1, p21, v21, pm21, vm21, v22, pm23, vm23, v23}];
MAX1[t_] :=
    Module[{\delta, \sigma, t1, p1, v1, vm1, pm1, v2, vm3, pm3, v3, f11, f12,
        g11, h11, J1}, t1 = t;
        {t1, p1, v1, pm1, vm1, v2, pm3, vm3, v3} = LG1[t1];
        \delta={-0.01, 0.017};
        \sigma={0.25, 0};
        f11 = Graphics[{Red, Thick, Line[{{v1, p1}, {v3, p1}}]}];
        f12 = Plot[Evaluate[p0[t1, v]], {v, 0.35, 4}, PlotStyle }->\mathrm{ {Blue, Thick}];
        g11 = Graphics[{Text[Style["a", Italic, Black, 12], {v1, p1} + \delta],
        Text[Style["b", Italic, Black, 12], {vm1, pm1} + \delta],
        Text[Style["c", Italic, Black, 12], {v2, p1} + \delta],
        Text[Style["d", Italic, Black, 12], {vm3, pm3} + \delta],
        Text[Style["e", Italic, Black, 12], {v3, p1} + \delta],
        Text[Style["tr=" <> ToString[t1], Italic, Black, 12], {v3, p1} + \sigma]}];
    h11 =
        RegionPlot[Evaluate[{v1 < x < v2 && p0[t1, x] < y < p1,
            v2 <x < v3&& p1<y<p0[t1, x]}], {x, 0.35, 4}, {y, 0, 1},
        PlotPoints }->\mathrm{ 100,
        PlotStyle }->\mathrm{ {{Opacity[0.2], Green}, {Opacity[0.2], Green}},
        PlotRange }->\mathrm{ All];
J1 = Show[f12, f11, g11, h11]];
```

$\mathrm{S} 1=\operatorname{Show}[\operatorname{MAX} 1[0.98], \operatorname{MAX1}[0.96], \operatorname{MAX1}[0.93], \operatorname{MAX1}[0.90], \operatorname{MAX1}[0.87]$, PlotRange $\rightarrow$ All];
f1 = Table[LG1[t], \{t, 0.5, 0.999, 0.001\}];
£1 / /
TableForm[\#,
TableHeadings $\rightarrow$
\{None, \{"tr", "p1", "v1", "pm1", "vm1", "pm1", "v2", "pm3", "vm3", "v3"\}\}] \&;
N1 = Length[f1];
g1 = Table[\{f1[[i, 3]], f1[[i, 2]]\}, \{i, 1, N1\}];
$\mathrm{g} 2=\mathrm{Table}[\{\mathrm{f} 1[[\mathrm{i}, 5]]$, f1[[i, 4]]\}, \{i, 1, N1\}];
g3 = Table[\{f1[[i, 8]], f1[[i, 7]]\}, \{i, 1, N1\}];
g4 = Table[\{f1[[i, 9]], f1[[i, 2]]\}, \{i, 1, N1\}];
J1 $=$ ListPlot $[\{g 1, ~ g 2, ~ g 3, ~ g 4\}, ~ J o i n e d ~ \rightarrow T r u e, ~, ~$ PlotStyle $\rightarrow$ Table[\{Hue[0.15i], Thick\}, \{i, 0, 5\}], PlotRange $\rightarrow\{\{0,4\},\{0.4,1\}\}]$;

J2 = Graphics[\{Text[Style["vr", Italic, Black, 12], \{3, 0.40\}],
Text[Style["pr", Italic, Black, 12], \{0.3, 1.2\}], Black, Thick, Arrowheads [0.02], Arrow [\{\{0, 0.5\}, \{4, 0.5\}\}], Arrow [\{\{0.5, 0.1\}, $\{0.5,1.3\}\}]\}$;

Show[S1, J1, J2, PlotRange $\rightarrow\{\{0,4\},\{0.1,1.3\}\}]$


```
J2 = Graphics[{Text[Style["K", Black, 12], {1, 1.03}],
    Text[Style["A", Black, 12], {0.75, 0.55}],
    Text[Style["B", Black, 12], {1.9, 0.55}],
    Text[Style["Liquid", Black, 12], {0.65, 0.85}],
    Text[Style["Gas", Black, 12], {2.5, 0.68}],
    Text[Style["\mp@subsup{v}{r}{}", Italic, Black, 12], {3.2, 0.55}],
    Text[Style["pr", Italic, Black, 12], {0.50, 1.2}], Black, Thick,
    Arrowheads[0.02], Arrow[{{0, 0.5}, {4, 0.5}}],
    Arrow[{{0.5, 0.1}, {0.5, 1.3}}], PointSize[0.010], Point[{1, 1}]}];
```

Show [ MAX1[0.90], J1, J2]

10. Maxwell construction using the Gibbs free energy
(a) Maxwell construction for the $\boldsymbol{\nu}_{\mathbf{r}}-\boldsymbol{p}_{\mathbf{r}}$ phase diagram

Unfortunately we cannot conveniently put $G$ into an analytic form as a function of $P$ instead of $V$. We need

$$
G(T, P, N)=N \mu(T, P) .
$$

It is $\mu$ that determines the phase co-existence relation; $\mu_{l}=\mu_{g}$. At any $T$, the lowest branch represents the stable phase. The point $a\left(v_{\mathrm{r}}=v_{1}=v_{l}\right)$ and the point $e\left(v_{\mathrm{r}}=v_{3}=v_{\mathrm{g}}\right)$ are on the coexistence line denoted by the path $a-c-e$.


Fig. 17
The $v_{\mathrm{r}}$ vs $p_{\mathrm{r}}$ phase diagram with a fixed reduced temperature $t_{\mathrm{r}}$ (in this case $t_{\mathrm{r}}=0.96$ ). The area $a-b-c-a$ is equal to the area $e-d-c-e$. (Maxwell construction)

The reduced volumes $v_{1}$ and $v_{3}$ are determined by the condition that

$$
\mu_{l}\left(t_{r}, p_{r}\right)=\mu_{g}\left(t_{r}, p_{r}\right)
$$

along the horizontal line between $v_{1}$ and $v_{3}$. This will occur if the shaded area below the line is equal to the shaded area above the line.

$$
d G=-S d T+V d P+\mu d N
$$

For $N=$ const. and $T=$ constant,

$$
d g=v_{r} d p_{r}
$$

for the scaled Gibbs free energy, and

$$
g_{g}-g_{l}=\int v_{r} d p_{r}
$$

The integral is just the sum of the shaded area (Maxwell construction).
(b) Maxwell construction for the $p_{r}$ vs $\nu_{r}$ phase diagram


Fig. $18 \quad$ The phase diagram of $v_{\mathrm{r}}$ vs $p_{\mathrm{r}}$ at $t_{\mathrm{r}}=0.96$.

At $t_{r}=t_{1}$,

$$
g\left(t_{r}=t_{1}, p_{r}\right)=g\left(t_{1}, p_{a}\right)+\int_{p_{a}}^{p} v_{r}\left(p_{r}, t_{1}\right) d p_{r},
$$

We assume that $p_{0}=p_{e}$ (the pressure at the point $e$ ). Then we have

$$
g\left(t_{r}=t_{1}, p_{e}\right)=g\left(t_{1}, p_{a}\right)+\int_{a b c} v_{r}\left(p_{r}, t_{1}\right) d p_{r}+\int_{c d e} v_{r}\left(p_{r}, t_{1}\right) d p_{r},
$$

Since $g\left(t_{1}, p_{a}\right)=g\left(t_{1}, p_{e}\right)$, we have

$$
\int_{a b c} v_{r}\left(p_{r}, t_{1}\right) d p_{r}+\int_{c d e} v_{r}\left(p_{r}, t_{1}\right) d p_{r}=0
$$

or

$$
\int_{e d} v_{r}\left(p_{r}, t_{1}\right) d p_{r}+\int_{d c} v_{r}\left(p_{r}, t_{1}\right) d p_{r}=\int_{a b} v_{r}\left(p_{r}, t_{1}\right) d p_{r}+\int_{b c} v_{r}\left(p_{r}, t_{1}\right) d p_{r} .
$$

We note that

$$
\int_{e d} v_{r}\left(p_{r}, t_{r}\right) d p_{1}>0, \quad \int_{d c} v_{r}\left(p_{r}, t_{1}\right) d p_{r}=-\int_{c d} v_{r}\left(p_{r}, t_{1}\right) d p_{r}
$$

and

$$
\int_{b c} v_{r}\left(p_{r}, t_{1}\right) d p_{r}>0, \quad \int_{a b} v_{r}\left(p_{r}, t_{1}\right) d p_{r}=-\int_{b a} v_{r}\left(p_{r}, t_{1}\right) d p_{r}
$$

Then we have

$$
\int_{e d} v_{r}\left(p_{r}, t_{1}\right) d p_{r}-\int_{c d} v_{r}\left(p_{r}, t_{1}\right) d p_{r}=\int_{b c} v_{r}\left(p_{r}, t_{1}\right) d p_{r}-\int_{b a} v_{r}\left(p_{r}, t_{1}\right) d p_{r},
$$

which means that the area of the region $e-d-c$ is the same as that of the region $a-b-c$. Note that

$$
p_{a}=p_{c}=p_{e}=p_{1}
$$

It is only after the nominal (non-monotonic) isotherm has been truncated by this equal area construction that it represents a true physical isotherm.

In summary, In the $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ phase diagram,
(i) The a-c-e- is the coexistence line ( $p_{r}=p_{1}$ and $t_{r}=t_{1}$ ) of the liquid phase and the gas phase.
(ii) The area $(a-b-c-a)$ is the same as the area $(c-d-e-c)$ [Maxwell construction].
(iii) K is the critical point $\left(p_{\mathrm{r}}=v_{\mathrm{r}}=t_{\mathrm{r}}=1\right)$.
(iv) The line KA and the line AB are the spinodal lines.


Fig. 19 The phase diagram of $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ at $t_{\mathrm{r}}=0.96$.
(c) Example: the area for the $\boldsymbol{v}_{\mathrm{r}}$ vs $\boldsymbol{p}_{\mathrm{r}}$ and the area for the $\boldsymbol{p}_{\mathrm{r}}$ vs $\boldsymbol{v}_{\mathrm{r}}$ for $\boldsymbol{t}_{\mathrm{r}}=\mathbf{0 . 9 5}$



Fig. $20 \quad p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ curve at $t_{\mathrm{r}}=0.95 . a:\left(v_{1}, p_{1}\right) ; b\left(v_{1 \mathrm{~m}}, p_{1 \mathrm{~m}}\right), c\left(v_{2}, p_{1}\right), d\left(v_{3 \mathrm{~m}}, p_{3 \mathrm{~m}}\right), e\left(v_{3}, p_{1}\right)$. Maxwell (equal-area) construction. The pressure $p$ where two phase coexistence begins for $t_{\mathrm{r}}=0.95$ is determined so that the areas above $(c-e-d)$ and below the horizontal line $(a-b-c)$ are equal. In this case, $p=0.811879$ (the pressures at $a$ and $e$ ).

$$
\begin{array}{lll}
t_{\mathrm{r}}=0.95 & p_{1}=0.811879, & \\
v_{1}=0.684122, & v_{2}=1.04247 & v_{3}=1.72707 \\
& v_{3 \mathrm{~m}}=1.33004, & p_{3 \mathrm{~m}}=0.845837 \\
v_{1 \mathrm{~m}}=0.786967, & p_{1 \mathrm{~m}}=0.74049 & \text { (local maximum point) } \\
& & \\
& & \\
g\left(t_{r}, v_{1}\right)=0.313223 . & g\left(t_{r}, v_{3 m}\right)=0.319189 \\
g\left(t_{r}, v_{1 m}\right)=0.307563 & &
\end{array}
$$

## (d) The Gibbs energy at the critical point (K)

Let us plot the $p_{r}-v_{r}$ plane an isotherm of the liquid and gas. According to the thermodynamic inequality we have

which implies that $p_{r}$ is a decreasing function of $v_{r}$. The segments $a-b$ and $d$ - e of the isotherms correspond to metastable super-heated liquid state and super-cooled vapor state, in which the thermodynamic inequality is still satisfied.

A complete-equilibrium isothermal change of state between the points $a$ and $e$ corresponds to the horizontal segment $a-c-e$, on which separation into two phases occur. If we use the fact that the points $a$
and $e$ have the same ordinate $p_{r}=p_{1}$, it is clear that the two parts of the isotherm cannot pass continuously into each other: there must be a discontinuity between them.

The isotherms terminates at $b$ and $e$, where

$$
\left(\frac{\partial p_{r}}{\partial v_{r}}\right)_{t_{r}}=0
$$

Curve A-K-B on which the thermodynamic inequality is violated for a homogeneous body; boundary of a region in which the body can never exist in a homogeneous state.

Near the critical point, the specific volumes of the liquid and gas are almost the same, denoting them by $v_{r}$ and $v_{r}+\delta v_{r}$, we can write the condition for equal pressure of the two phases

$$
p_{r}\left(v_{r}, t_{r}\right)=p\left(v_{r}+\delta v_{r}, t_{r}\right),
$$

or

$$
\left(\frac{\partial p_{r}}{\partial v_{r}}\right)_{t_{r}}+\frac{1}{2} \delta v_{r}\left(\frac{\partial^{2} p_{r}}{\partial v_{r}^{2}}\right)_{t_{r}}+\ldots=0 .
$$

Hence we see that, when $\delta v_{r} \rightarrow 0$ (at the critical point),


## (e) Properties of the Gibbs free energy in the metastable state and unstable state

To see the qualitative behavior of the Gibbs function $g\left(t_{r}, p_{r}\right)$ as a function of $p_{r}$, we use the relation

$$
v_{r}=\left(\frac{\partial g}{\partial p_{r}}\right)_{t_{r}}
$$

or

$$
g\left(t_{r}, p_{r}\right)=g\left(t_{r}, p_{0}\right)+\int_{p_{0}}^{p} v_{r} d p_{r}
$$

On the $g\left(t_{r}, p_{r}\right)$ curve as a function of $p_{r}, v_{r}$ represents the slope, $\left(\frac{\partial g}{\partial p_{r}}\right)_{t_{r}}$ :

$$
v_{r}=\left(\frac{\partial g}{\partial p_{r}}\right)_{t_{r}} .
$$

We take the van der Waals isotherm $a-b-c-d-e$ in the $p_{\mathrm{r}}-v_{\mathrm{r}}$ diagram. We make a plot of the corresponding $g\left(t_{r}, p_{r}\right)$ curve as function of $p_{r}$ at $t_{\mathrm{r}}=t_{1}$ (in this case, $t_{\mathrm{r}}=0.95$ ).


Fig. $21 \quad$ Gibbs free energy as a function of $p_{\mathrm{r}}$ at $t_{\mathrm{r}}=0.95$.
(i) Around the point $d$ (on the path $d-e-\mathrm{g}$ ) where $\left(\frac{\partial p_{r}}{\partial v_{r}}\right)_{t_{1}}=0$, and $\xi_{m 3}=-\left(\frac{\partial^{2} p_{r}}{\partial v_{r}{ }^{2}}\right)_{t_{1}}>0, p_{r}$ can be expressed by using the Taylor expansion, as

$$
p_{r}-p_{m 3} \approx\left(\frac{\partial p_{r}}{\partial v_{r}}\right)_{t_{1}}\left(v_{r}-v_{m 3}\right)+\frac{1}{2}\left(\frac{\partial^{2} p_{r}}{\partial v_{r}^{2}}\right)_{t_{1}}\left(v_{r}-v_{m 3}\right)^{2}
$$

or

$$
p_{r}-p_{m 3} \approx-\frac{\xi_{m 3}}{2}\left(v_{r}-v_{m 3}\right)^{2},
$$

or

$$
v_{r}-v_{m 3}=\sqrt{2} \xi_{m 3}{ }^{1 / 2}\left(p_{m 3}-p_{r}\right)^{1 / 2} .
$$

Then we have

$$
\left(\frac{\partial v_{r}}{\partial p_{r}}\right)_{t_{1}}=-\frac{\xi_{m 3}^{1 / 2}}{\sqrt{2}}\left(p_{m 3}-p_{r}\right)^{-1 / 2} .
$$

So that $\left(\frac{\partial v_{r}}{\partial p_{r}}\right)_{t_{1}}$ become infinite at the point $d$. Then $g\left(t_{1}, p_{r}\right)$ curve has a cusp.
(ii) From $b$ to $d$ (on the path $b-c-d) . v_{r}=\left(\frac{\partial g}{\partial p_{r}}\right)_{t_{1}}$ increases as $p_{\mathrm{r}}$ increases. $\left(\frac{\partial v_{r}}{\partial p_{r}}\right)_{t_{1}}$ is positive (this portion is unstable).
(iii) Around the point $b$ (on the path l-a-b) where $\left(\frac{\partial p_{r}}{\partial v_{r}}\right)_{t_{1}}=0$ and, $\xi_{m 1}=\left(\frac{\partial^{2} p_{r}}{\partial v_{r}{ }^{2}}\right)_{t_{1}}>0, p_{r}$ can be expressed by using the Taylor expansion, as

$$
p_{r}-p_{m 1} \approx\left(\frac{\partial p_{r}}{\partial v_{r}}\right)_{t_{1}}\left(v_{r}-v_{m 1}\right)+\frac{1}{2}\left(\frac{\partial^{2} p_{r}}{\partial v_{r}^{2}}\right)_{t_{1}}\left(v_{r}-v_{m 1}\right)^{2},
$$

or

$$
p_{r}-p_{m 1} \approx \frac{1}{2} \xi_{m 1}\left(v_{r}-v_{m 1}\right)^{2}
$$

or

$$
v_{r}-v_{m 1}=\sqrt{2} \xi_{m 1}^{1 / 2}\left(p_{r}-p_{m 1}\right)^{1 / 2} .
$$

Then we have

$$
\left(\frac{\partial v_{r}}{\partial p_{r}}\right)_{t_{1}}=\frac{\xi_{m 1}^{1 / 2}}{\sqrt{2}}\left(p_{r}-p_{m 1}\right)^{-1 / 2}
$$

So that $\left(\frac{\partial v_{r}}{\partial p_{r}}\right)_{t_{1}}$ become infinite at the point $b$ and $g\left(t_{1}, p_{r}\right)$ curve has another cusp.
(iv) From $l$ to $b$ (on the path $l-a-b) . v_{r}=\left(\frac{\partial g}{\partial p_{r}}\right)_{t_{r}}$ decreases as $p_{r}$ increases. $\left(\frac{\partial v_{r}}{\partial p_{r}}\right)_{t_{r}}$ is negative, but becomes small as the point $l$ is approached.

In summary, the path $d-c-b$ corresponds to unstable region and the paths e- $d$ and $b-a$ are metastable.

## (f) Numerical calculation

We can make a plot of $g$ vs $p_{r}$ where $t_{\mathrm{r}}$ is fixed, using the ParametricPlot of the Mathematica. The scaled Gibbs free energy $g$ and the reduced pressure $p_{r}$ are given by

$$
g=-\frac{2}{3 v_{r}}+\frac{8}{27}\left(\frac{t_{r} v_{r}}{v_{r}-\frac{1}{3}}\right)-\frac{8}{27} t_{r}\left[\ln \left(v_{r}-\frac{1}{3}\right)+\frac{3}{2} \ln t_{r}-\frac{3}{2}\right],
$$

and

$$
p_{r}=-\frac{3}{v_{r}{ }^{2}}+\frac{8 t_{r}}{3 v_{r}-1}
$$

So we make a ParametricPlot of the co-ordinate $\left(p_{r}, g\right)$ when $t_{\mathrm{r}}$ is given as a fixed parameter and $v_{\mathrm{r}}$ is continuously changed as a variable. The Mathematica which we use is as follows.
((Mathematica))

$$
\begin{aligned}
& \text { Clear["Global`*"]; } \\
& \text { g1[tr_, vr_]:= } \\
& \frac{-2}{3 v r}+\frac{8}{27} \frac{t r v r}{v r-\frac{1}{3}}-\frac{8}{27} \operatorname{tr}\left(\log \left[v r-\frac{1}{3}\right]+\frac{3}{2} \log [t r]-\frac{3}{2}\right) ; \\
& \operatorname{pr}\left[t r_{-}, v r_{-}\right]:=\frac{-3}{v r^{2}}+\frac{\frac{8}{3}}{v r-\frac{1}{3}} \operatorname{tr} ; \\
& \operatorname{tr}=0.95 ; \\
& \text { s1 }=\text { ParametricPlot[Evaluate[\{pr[tr, vr], g1[tr, vr]\}], } \\
& \quad \text { \{vr, } 0.5,10\}, \text { PlotStyle } \rightarrow\{\operatorname{Red}, \mathrm{Thick}\}, \\
& \quad \text { PlotRange } \rightarrow\{\{0.70,0.9\},\{0.29,0.32\}\}, \\
& \text { AspectRatio } \rightarrow \text { Full]; }
\end{aligned}
$$

## (i) $t_{\mathrm{r}}=0.99$

The $p_{\mathrm{r}}$ dependence of the scaled Gibbs energy is shown below. We note that the scaled Gibbs energy is the same at the points $a$ and $e$. The Gibbs energy along the path $a-b$ (the metastable state), along the path $b-c-d$ (the unstable state), and along the path d-e is higher than that along the path $l$ (liquid) $-a$ and along the path $e-g$ (gas). This means that the coexistence line $(a-c-e)$ is the equilibrium state. It is seen that the Gibbs free energy vs $p_{\mathrm{r}}$ shows a thermodynamically invisible bow tie.


Fig. 22
$v_{\mathrm{r}}$ vs $p_{\mathrm{r}}$ for $t_{\mathrm{r}}=0.99$. The line $a-c-e$ is the co-existence line between the gas and liquid phases. K: critical point. The lines A-K and B-K are spinodal lines. The path $a-c-e$ is the co-existence line of the liquid and gas phases.


Fig. $23 \quad$ Scaled Gibbs free energy $g$ vs $p_{\mathrm{r}}$ for $t_{\mathrm{r}}=0.99 . g$ is in the units of $(a / b)$. The path $b-c-d$ is unstable. $p_{1}=0.960479 . p_{\mathrm{m} 1}=0.955095($ point $b) . p_{\mathrm{m} 3}=0.964369($ point $d)$


Fig. 24
The phase diagram of $p_{r}$ vs $v_{\mathrm{r}}$ for $t_{\mathrm{r}}=0.99 . v_{1}=0.830914$ (point $a$ ). $v_{3}=1.24295$ (point $e$ ). $p_{1}=0.960479$.
(ii) $t_{\mathrm{r}}=0.98$


Fig. 25
$v_{\mathrm{r}}$ vs $p_{\mathrm{r}}$ for $t_{\mathrm{r}}=0.98$. The line $a-c-e$ is the coexistence line. The path $b-c-d$ is unstable. The path $a-b$ and the path $d$-e is unstable. The area enclosed by $a-b-c$ is the same as that by $c$ -$d-e$ (Maxwell construction). $p_{1}=0.921912 . v_{1}=0.775539 . v_{3}=1.3761$.


Fig. 26
Scaled Gibbs energy $g$ (in the units of $a / b$ ) vs $p_{\mathrm{r}}$ for $t_{\mathrm{r}}=0.98$. The path $b-c-d$ is unstable. The shape of the $b-c-d$ is similar to spine (the spinodal decomposition). The path $a-b$ and the path $d$-e are unstable. $p_{1}=0.921912 . p_{\mathrm{m} 1}=0.905756$ (point $b$ ). $p_{\mathrm{m} 3}=0.932089$ (point d).


Fig. $27 \quad p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ for $t_{\mathrm{r}}=0.98$. The line $a-c-e$ is the coexistence line. The path $b-c-d$ is unstable. The path $a-b$ and the path $d$-e are metastable. The area enclosed by $a-b-c-a$ is the same as that by $c-d-e-c$ (Maxwell's construction). $p_{1}=0.921912 . v_{1}=0.775539 . v_{3}=1.3761$.
(iii) $\quad t_{\mathrm{r}}=0.97$


Fig. $28 \quad$ Phase diagram of $v_{\mathrm{r}}$ vs $p_{\mathrm{r}}$ at $t_{\mathrm{r}}=0.97$.


Fig. 29 Gibbs free energy as a function of $p_{\mathrm{r}}$ at $t_{\mathrm{r}}=0.97 . p_{1}=0.884294 . p_{\mathrm{m} 1}=0.853279 . p_{\mathrm{m} 3}=$ 0.901849 .


Fig. $30 \quad$ Phase diagram of $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ at $t_{\mathrm{r}}=0.97$.
(iv) $\quad t_{r}=0.96$


Fig. $31 \quad$ Phase diagram of $v_{\mathrm{r}}$ vs $p_{\mathrm{r}}$ at $t_{\mathrm{r}}=0.96$.


Fig. $32 \quad$ Gibbs free energy as a function of $p_{\mathrm{r}}$ at $t_{\mathrm{r}}=0.96 . p_{1}=0.847619 . p_{\mathrm{m} 1}=0.798108 . p_{\mathrm{m} 3}=$ 0.873186 .


Fig. $33 \quad$ Phase diagram of $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$ at $t_{\mathrm{r}}=0.96 . p_{1}=0.847619 . v_{1}=0.708189 . v_{3}=1.61181$.
(g) Plot3D of the Gibbs free energy $g\left(t_{r}, p_{r}\right)$

We make a Plot3D of $g\left(t_{\mathrm{r}}, p_{\mathrm{r}}\right)$ in the $\left(t_{\mathrm{r}}, p_{\mathrm{r}}\right)$ plane by using the Mathematica.


Fig. 34 Gibbs surface for the van der Waals gas in the vicinity of the critical point. We use the Mathematica (ParametricPlot3D).

As $t_{\mathrm{r}}$ is raised and $v_{3}-v_{1}$ diminishes, the two branches $\mathrm{L}-\mathrm{a}-\mathrm{b}$ and d-e-G intersect more and more nearly tangentially. The cusped region becomes steadily smaller until at the critical temperature the curve degenerates into a single continuous curve. Just at the critical temperature the gradient of the curve, which is equal to $v_{\mathrm{r}}$, is everywhere continuous, but the curvature $\left(\frac{\partial v_{r}}{\partial p_{r}}\right)_{t_{r}}$ becomes momentarily infinite at the critical pressure, since at this point the van der Waals gas is infinitely compressible. Above the critical temperature, the curves for $g$ are everywhere continuous in all their derivatives. Here we show the ParametricPlot3D for $\left\{t_{r}, p\left(t_{r}, v_{r}\right), g\left(t_{r}, v_{r}\right)\right\}$ in the vicinity of the critical point.
((Mathematica))

```
Clear ["Global` *"] ; p0 [t_, v_] := \(\frac{-3}{v^{2}}+\frac{\frac{8}{3} t}{v-\frac{1}{3}}\);
\(g 1\left[t r_{-}, v r_{-}\right]:=\frac{-2}{3 v r}+\frac{8}{27} \frac{\operatorname{tr} v r}{v r-\frac{1}{3}}-\frac{8}{27} \operatorname{tr}\left(\log \left[v r-\frac{1}{3}\right]+\frac{3}{2} \log [t r]-\frac{3}{2}\right) ;\)
\(\mathrm{f} 1=\) ParametricPlot3D [Evaluate [\{tr, p0[tr, vr], g1[tr, vr]\}],
    \{tr, \(0.85,0.999\}\), \{vr, \(0.4,10\}\),
    PlotStyle \(\rightarrow\) \{Green, Red, Opacity [0.12]\},
    PlotRange \(\rightarrow\{\{0.85,0.999\},\{0,1\},\{0.2,0.4\}\}\), Mesh \(\rightarrow 10\}\);
f2 = Graphics3D [\{Black, Thick, Arrowheads [0.015],
    Arrow [\{\{0.85, 0, 0.20\(\},\{0.99,0,0.20\}\}]\),
    Arrow [\{\{0.85, 0, 0.20\(\},\{0.85,1,0.20\}\}]\),
    Arrow [\{\{0.85, 0, 0.20\(\},\{0.85,0,0.40\}\}]\),
    Text [Style ["t \(t_{r}\) ", Italic , Black, 20], \{0.97, 0, 0.21\(\left.\}\right]\),
    Text [Style [" \(\mathrm{p}_{\mathrm{r}}\) ", Italic, Black, 20], \{0.85, 0.95, 0.21\}],
    Text [Style ["g(tr, \(\left.\mathrm{p}_{\mathrm{r}}\right) \mathrm{"} ,\mathrm{Italic} \mathrm{} ,\mathrm{Black} \mathrm{}, \mathrm{20]}, \mathrm{\{0.85}, \mathrm{0}, \mathrm{0.41} \mathrm{\}]} \mathrm{\}];}\)
Show [f1, f2]
```


## (g) Proof of Maxwell construction by Enrico Fermi

We find this proof in the book by E. Fermi (Thermodynamics). It seems that this proof is much simpler than the Maxwell construction based on the Gibbs free energy and Helmholtz free energy. The area of closed path $a-b-c-a$ is equal to that of the closed path $c-d-e-c$. We show that the work done on the system $W$ during a reversible isothermal cycle is equal to zero. We now consider the reversibly isothermal cycle $a-b-c-d-e-c-a$. According to the first law of thermodynamics

$$
\begin{aligned}
& \Delta U=\oint d U=\oint d Q-\oint P d V \\
& \oint d U=0 .
\end{aligned}
$$

For a reversible cycle, we have

$$
\oint d S=\oint \frac{d Q}{T}=0 .
$$

In this case, the cycle is isothermal. So we can remove $1 / T$ from under the integral sign,

$$
\oint d Q=0 .
$$

Then we have

$$
\oint P d V=0 .
$$

This integral consists of two parts,

$$
\int_{a b c a} P d V+\int_{c d e c} P d V=0
$$

or

$$
\int_{a b c a} P d V=\int_{c e d c} P d V, \quad \text { (Maxwell construction), }
$$

since

$$
-\int_{c d e c} P d V=\int_{c e d c} P d V,
$$

which is positive.

## 11. Double-tangent construction based on Helmholtz free energy

Maxwell construction based on the Gibbs free energy is equivalent to the double-tangent construction based on the Helmholtz free energy. Here we discuss the double-tangent construction using the concept of the Helmholtz free energy.
(a) Double-tangent line (coexistence line)

Using the relation

$$
p_{r}=-\frac{\partial f\left(v_{r}, t_{r}\right)}{\partial v_{r}},
$$

the Helmholtz free energy can be obtained as

$$
f\left(t_{r}, v_{r},\right)=f\left(t_{r 0}, v_{r 0},\right)-\int_{v_{0}}^{v_{r}} p_{r}\left(v_{r}, t_{r}\right) d v_{r} .
$$

The Helmholtz free energy is related to the Gibbs free energy as

$$
g\left(t_{r}, p_{r}\right)=f\left(t_{r}, v_{r}\right)+p_{r} v_{r} .
$$

According to the Maxwell construction from the Gibbs free energy, we have

$$
g_{1}=g_{3},
$$

at the point $a\left(p_{r}=p_{1}, v_{r}=v_{1}\right)$ and point $e\left(p_{r}=p_{1}, v_{r}=v_{3}\right)$, where

$$
\begin{array}{ll}
g_{1}=f_{1}+p_{1} v_{1}, & \text { at the point } a, \\
g_{3}=f_{3}+p_{1} v_{3}, & \text { at the point } e .
\end{array}
$$

In the diagram of $f$ vs $v_{r}$ the point $a$ is located at the co-ordinate $\left(v_{1}, f_{1}\right)$, while the point $e$ is located at the co-ordinate $\left(v_{3}, f_{3}\right)$. Note that the point $a$ and point $b$ are on the coexistence line in the $p_{\mathrm{r}} \mathrm{vs} v_{\mathrm{r}}$ diagram for $t_{\mathrm{r}}<1$. The straight line (double-tangent line) connecting the points $a$ and $e$ can be given by

$$
f_{D T}=f_{1}+\left(\frac{f_{3}-f_{1}}{v_{3}-v_{1}}\right)\left(v-v_{1}\right) .
$$

We note that

$$
\frac{\partial f_{D T}}{\partial v}=\left(\frac{f_{3}-f_{1}}{v_{3}-v_{1}}\right)=-p_{1}
$$

The reduced pressure $p_{1}$ corresponds to a negative of the slope of the straight line (the double-tangent line) connecting the point $a$ and the point $e$. We note that

$$
\frac{\partial^{2} f_{D T}}{\partial v^{2}}=0
$$

We make a plot of the reduced Helmholtz free energy as a function of the reduced volume $v_{\mathrm{r}}$ at fixed temperature (in this case $t_{\mathrm{r}}=0.85$ ). The double-tangent line is denoted by the straight line connecting the points $a$ and $e$. The tangential line at the point $a$ coincides with that at the point $e$. Note that the Helmholtz free energy at fixed $t_{\mathrm{r}}$ is higher than the corresponding double-tangent line between $v_{1}$ and $\nu_{3}$. This means that this double tangent line is the coexistence line between the points $a$ and $e$.


Fig. 35
The isothermal Helmholtz energy $f$ as a function of the reduced volume. $t_{\mathrm{r}}=0.85$. The Helmholtz free energy has two inflection points at the point $b$ and the point $d$ below the critical point. The double-tangent line (the straight line $a-e$ ) represents coexisting vapor and liquid phases. The Helmholtz free energy with double-tangent line (the path $a-e$ ) is lower than the metastable (the path $a-b$ and the path $d-e$ ) and unstable part (the path $b-c-d$ ) between $v_{1}$ (the point $a$ ) and $v_{3}$ (the point $e$ ); double-tangent construction. $v_{1}=0.55336$. $v_{3}$ $=3.12764 . p_{1}=0.504492$.

(a)

(b)

Fig. 36 (a) and (b)
Tangential line (green line) of the Helmholtz free energy vs $v_{\mathrm{r}}$, corresponding to $-p_{\mathrm{r}}$ at fixed reduced temperature $t_{\mathrm{r}}$ ( $=0.8$, in this case). The double-tangent line (black line) is the co-existence curve with $p_{r}=p_{1}$ between $v_{\mathrm{r}}=v_{1}$ (the state $a$ ) and $v_{3}$ (the state $e$ ). The tangential line at the point $a$, coincides with that at the point $e . v_{1}=0.517409 . v_{2}=1.20827 . v_{3}=4.17246$.


Fig. 37
$f\left(t_{r}, v_{r}\right)$ vs $v_{\mathrm{r}}$ at each fixed reduced temperature $t_{\mathrm{r}}\left(t_{\mathrm{r}}=0.75,0.80,0.85,0.90\right.$, and 0.95$)$.
The double-tangent lines are also denoted by the black straight lines connecting between the point $a$ and the point $e$.

## (b) Pressure as a function of the reduced volume

The reduced pressure is given by

$$
p_{r}=-\left(\frac{\partial f}{\partial v_{r}}\right)_{t_{r}}
$$

using the Helmholtz free energy. The reduced pressure $p_{\mathrm{r}}$ is plotted as a function of the reduced volume $\left(t_{\mathrm{r}}=0.85\right)$. Above the critical point, as $v_{\mathrm{r}}$ increases, the Helmholtz free energy decreases, corresponding to the monotonic decrease in $p_{\mathrm{r}}$ vs $v_{\mathrm{r}}$. This is a typical of any temperature above the critical point. Below the critical point, we see that the path $l$ (liquid)- $a$ and the path $e-g$ (gas) in which the reduced pressure decreases monotonically as $v_{\mathrm{r}}$ increases. These are joined by a straight line. The path $a$-e touching the path $l-a$ at the point $a$ and touching the path $e-g$ at the point $e$. The three portions correspond to the liquid phase, to the gas phase, and to a two-phase liquid-gas system. This typically happens when $t_{\mathrm{r}}<1$. Note that the path $a-b$ represents superheated liquid. The path $d-e$ represent super-cooled vapor. We see that all states represented by these portions of curves are metastable.


Fig. $38 \quad p_{r}=-\left(\partial f / \partial v_{r}\right)_{t_{r}}$ as a function of $v_{\mathrm{r}} \cdot t_{\mathrm{r}}=0.85$. The path $a-c-e$ is the coexistence boundary.
(c) Metastable state and unstable state


Fig. 39 $\left(\partial^{2} f / \partial v_{r}^{2}\right)_{t_{r}}$ as a function of $v_{\mathrm{r}} . t_{\mathrm{r}}=0.85 .\left(\partial^{2} f / \partial v_{r}{ }^{2}\right)_{t_{r}}<0$ for the path $b-c-d$, indicating that the curve $f$ vs $v_{\mathrm{r}}$ is concave upwards (unstable). $\left(\partial^{2} f / \partial v_{r}{ }^{2}\right)_{t_{r}}>0$ for the path $a$ - $b$ and path $d-e$, indicating that the curve is concave downwards (metastable).

The Helmholtz free energy of the superheated liquid (the path $a-b$ ) or the supersaturated vapor (the path $d-e$ ) is greater than that for the double-tangent line. The portions curve $a-b$ and $d-e$ are in the metastable state. They have curvature concave upwards so that

$$
\frac{\partial p_{r}}{\partial v_{r}}=-\left(\frac{\partial^{2} f}{\partial v_{r}^{2}}\right)<0
$$

because of the definition of mathematics,


Fig. $40 \quad\left(\frac{\partial^{2} f}{\partial v_{r}{ }^{2}}\right)>0, \quad$ (concave downwards, or convex upwards),


Fig. $41 \quad\left(\frac{\partial^{2} f}{\partial v_{r}{ }^{2}}\right)<0, \quad$ (concave upwards, or convex downwards).

We note that the path $b-c$-d has a curvature concave downwards. This would correspond to a positive value of

$$
\frac{\partial p_{r}}{\partial v_{r}}=-\left(\frac{\partial^{2} f}{\partial v_{r}^{2}}\right)>0
$$

leading to unstable states. Such states are not realized.
Since the tangent line $f\left(v_{r}\right)$ maintains the same slope between $v_{1}$ and $v_{3}$, the pressure remains constant between $v_{1}$ and $v_{3}$ :

$$
\left(\frac{\partial f\left(t_{r}, v_{r}\right)}{\partial v_{r}}\right)_{\tau}=-p_{1}
$$

In other words, the line connecting points on the $p_{r}-v_{r}$ plot is horizontal and the two coexistence phases are in thermal equilibrium. For each temperature below $t_{\mathrm{r}}=1$, the phase transformation occurs at a welldefined pressure $p_{1}$, the so-called vapor pressure. Two stable branches $g$ (gas)-e-d and $b-a-l$ (liquid) correspond to different phases: the branch $g-e-d$ (gas phase) and the branch $b-a-l$ (liquid phase). The branch $e-c-a$ is the co-existence line between the gas phase and liquid phase. The branch $e-d$ is a metastable gas phase, while the branch $b-a$ is a metastable liquid phase.

## (d) Difference $\Delta f=f-f_{D T}$

Here we define as

$$
\Delta f=\Delta f\left(t_{r}, v_{r}\right)=f\left(t_{r}, v_{r}\right)-f_{D T}\left(t_{r}, v_{r}\right)
$$

where $f_{D T}$ is given by

$$
f_{D T}\left(t_{r}, v_{r}\right)=f_{1}+\frac{f_{3}-f_{1}}{v_{3}-v_{1}}\left(v_{r}-v_{1}\right),
$$

for the double-tangent line. Since $\partial^{2} f_{D T}\left(t_{r}, v_{r}\right) / \partial v_{r}{ }^{2}=0$, it follows that

$$
\partial^{2} \Delta f\left(t_{r}, v_{r}\right) / \partial v_{r}^{2}=\partial^{2} f\left(t_{r}, v_{r}\right) / \partial v_{r}^{2} .
$$

The plots of $\Delta f$ vs $v_{\mathrm{r}}$ is shown for the range $\left(v_{1}<v<v_{3}\right)$, where $t_{\mathrm{r}}$ is changed as a parameter. We note that the difference $\Delta f$ is equal to zero at $v_{\mathrm{r}}=v_{1}$ and $v_{3}$. It shows a peak at $v_{\mathrm{r}}=v_{2}$. We show the deviation $\Delta f$ vs $v_{\mathrm{r}}$ between $v_{1}$ and $v_{3}$ at $t_{\mathrm{r}}=0.95$. The points $a\left(v_{\mathrm{r}}=v_{1}\right), b, c\left(v_{\mathrm{r}}=v_{2}\right), d$, and e $\left(v_{\mathrm{r}}=v_{3}\right)$, are shown in this figure. $\partial^{2} \Delta f / \partial v_{r}{ }^{2}>0$ for the path $a-b$ (the superheated state) and the path $d$-e (the super-cooled state). $\partial^{2} \Delta f / \partial v_{r}^{2}<0$ for the path $b-c-d$ (unstable state).




Fig. 42
(a), (b), and (c) The deviation $\Delta f$ vs $v_{\mathrm{r}}$ between $v_{1}$ and $v_{3}$ at $t_{\mathrm{r}}=0.95$. The points $a\left(v_{\mathrm{r}}=v_{1}\right)$, $b, c\left(v_{\mathrm{r}}=v_{2}\right), d$, and $\mathrm{e}\left(v_{\mathrm{r}}=v_{3}\right)$, are shown in this figure. $\partial^{2} \Delta f / \partial v_{r}^{2}>0$ for the path $a-b$ (the superheated state) and the path $d$-e (the super-cooled state). $\partial^{2} \Delta f / \partial v_{r}{ }^{2}<0$ for the path $b$ -$c-d$ (unstable state).


Fig. 43 $f\left(t_{r}, v_{1}\right), f\left(t_{r}, v_{m 1}\right), f\left(t_{r}, v_{2}\right), f\left(t_{r}, v_{m 3}\right)$, and $f\left(t_{r}, v_{3}\right)$ as a function of $t_{\mathrm{r}}$. Note that $v_{1}, v_{\mathrm{m} 1}$, $v_{2}, v_{\mathrm{m} 3}$, and $v_{3}$ are dependent on $t_{\mathrm{r}}$ according to Maxwell construction or double-tangent construction.

## (e) The lever rule for the Helmholtz free energy

The straight line connecting the point $a$ and the point $e$ is given by

$$
f=f_{1}+\left(\frac{f_{3}-f_{1}}{v_{3}-v_{1}}\right)\left(v-v_{1}\right),
$$

where $v_{1} \leq v \leq v_{3}$. This can be written as

$$
\begin{aligned}
f & =f_{1}-\left(\frac{f_{3}-f_{1}}{v_{3}-v_{1}}\right) v_{1}+\left(\frac{f_{3}-f_{1}}{v_{3}-v_{1}}\right) v \\
& =f_{1}\left(\frac{v_{3}-v}{v_{3}-v_{1}}\right)+f_{3}\left(\frac{v-v_{1}}{v_{3}-v_{1}}\right)
\end{aligned}
$$

or

$$
f=\theta_{1} f_{1}+\theta_{3} f_{3}
$$

corresponding to the lever rule for the reduce volume, where

$$
\theta_{1}=\frac{v_{3}-v}{v_{3}-v_{1}}, \quad \theta_{3}=\frac{v-v_{1}}{v_{3}-v_{1}}
$$

$f$ can be written as

$$
f=\left(\frac{v_{3}-v}{v_{3}-v_{1}}\right) f_{1}+\left(\frac{v-v_{1}}{v_{3}-v_{1}}\right) f_{3}=\frac{1}{v_{3}-v_{1}}\left[f_{1}\left(v_{3}-v\right)+\left(v-v_{1}\right) f_{3}\right]
$$

which is the straight line passing through the two points $\left(v_{1}, f_{1}\right)$ and $\left(v_{3}, f_{3}\right)$. We recognize this as the common tangent line.

$$
\left(\frac{\partial f}{\partial v}\right)_{\tau}=\frac{f_{3}-f_{1}}{v_{3}-v_{1}}=-p_{1}
$$

and

$$
\left(\frac{\partial^{n} f}{\partial v^{n}}\right)_{\tau}=0
$$

## (f) Summary

According to the Maxwell relation

$$
p_{r}=-\left(\frac{\partial f\left(t_{r}, v_{r}\right)}{\partial v_{r}}\right)_{t_{r}}
$$

the Helmholtz free energy can be obtained as the area under the isotherm:

$$
f\left(t_{r}, v_{r}\right)=-\int_{\text {isotherm }} p_{r} d v_{r} .
$$

Note that $v_{1}$ and $v_{3}$ are defined by the double-tangent construction. At any point along the tangent, the Helmholtz free energy is a linear combination of those at $a$ and $e$, and thus represent a mixture of the liquid and gas phases. Note that the value of $f\left(t_{r}, v_{r}\right)$ for $v_{1}<v_{r}<v_{3}$ is larger than that on the double tangent line, as is obvious from the graphical construction. Thus the phase-separated state is the equilibrium state. The states $a$ and $e$ are defined by the condition

$$
\begin{array}{ll}
p_{1}=-\frac{\partial f}{\partial v_{1}}=-\frac{\partial f}{\partial v_{3}}, & \text { (equal pressure) } \\
\frac{\partial f}{\partial v_{1}}=\frac{\partial f}{\partial v_{3}}=\frac{f_{3}-f_{1}}{v 3-v_{1}}, & \text { (common tangent) } \\
\int_{v_{1}}^{v_{3}} p_{r} d v_{r}=p_{1}\left(v_{3}-v_{1}\right), & \text { (Maxwell construction) }
\end{array}
$$

or

$$
\int_{v_{1}}^{v_{3}}\left[p_{r}-p_{1}\right] d v_{r}=0
$$

For local stability at any point the curve $f(T, v)$ must always lie above its tangent, and for global stability this tangent must not cut the primitive $f(T, v)$ curve at any other point. If it does, the substance will split into a mixture of two phases with values of volume $v_{1}, v_{2}$ corresponding to the two points of contact of the tangent. The double-contact tangent corresponds to the co-existence of two phases in equilibrium.

## 12. Critical behavior of $\boldsymbol{v}_{1}$ and $v_{3}$ around the critical point

We make a plot of the values of characteristic reduced volumes ( $v_{1}, v_{\mathrm{m} 1}, v_{\mathrm{m} 3}$, and $v_{3}$ ) as a function of $t_{\mathrm{r}}$ for $t_{\mathrm{r}} \leq 1$.


Fig. $44 \quad t_{\mathrm{r}}$ vs $v_{1}, t_{\mathrm{r}}$ vs $v_{\mathrm{m} 1}, t_{\mathrm{r}}$ vs $v_{\mathrm{m} 3}$, and $\mathrm{t}_{\mathrm{r}}$ vs $v_{3}$ with $v_{1}$ and $v_{3}$ lines (bimodal lines) and $v_{\mathrm{m} 1}$ and $v_{\mathrm{m} 3}$ lines (spinodal lines).
(a) The least squares fitting of $\Delta v=\frac{1}{2}\left(v_{1}+v_{3}\right)-1$ vs $\varepsilon=|\tau|$
(i) The result of $\Delta v=\frac{1}{2}\left(v_{1}+v_{3}\right)-1$ vs $\varepsilon=|\tau|$ (for $0<\varepsilon \leq 0.010$ ) is best fitted by a polynomial given by
$\Delta v=\frac{1}{2}\left(v_{1}+v_{3}\right)-1=3.6 \varepsilon+9.13377 \varepsilon^{2}+19.8942 \varepsilon^{3}+41.5431 \varepsilon^{4}$.


Fig. 45

$$
\Delta v=\frac{1}{2}\left(v_{1}+v_{3}\right)-1 \text { vs } \varepsilon=|\tau|(\text { for } 0<\varepsilon \leq 0.010) .
$$

(ii) The result of $\Delta v=\frac{1}{2}\left(v_{1}+v_{3}\right)-1$ vs $\varepsilon=|\tau|$ (for $0<\varepsilon \leq 0.001$ ) is best fitted by a straight line given by

$$
\Delta v=\frac{1}{2}\left(v_{1}+v_{3}\right)-1=3.64535 \varepsilon,
$$



Fig. $46 \Delta v=\frac{1}{2}\left(v_{1}+v_{3}\right)-1$ vs $\varepsilon=|\tau|($ for $0<\varepsilon \leq 0.001)$.
(b) The least squares fitting of $\Delta v=\frac{1}{2}\left(v_{3}-v_{1}\right)$ vs $\varepsilon=|\tau|$
(i) The result of $\Delta v=\frac{1}{2}\left(v_{3}-v_{1}\right)$ vs $\varepsilon=|\tau|$ (for $0<\varepsilon \leq 0.010$ ) is best fitted by a polynomial given by
$\Delta v=\frac{1}{2}\left(v_{3}-v_{1}\right)=2.000 \varepsilon^{1 / 2}+5.87746 \varepsilon^{3 / 2}+14.174 \varepsilon^{5 / 2}$


Fig. 47

$$
\Delta v=\frac{1}{2}\left(v_{3}-v_{1}\right) \text { vs } \varepsilon=|\tau|(\text { for } 0<\varepsilon \leq 0.010)
$$

(i) The result of $\Delta v=\frac{1}{2}\left(v_{3}-v_{1}\right)$ vs $\varepsilon=|\tau|$ (for $0<\varepsilon \leq 0.001$ ) is best fitted by a polynomial given by
$\Delta v=\frac{1}{2}\left(v_{3}-v_{1}\right)=2.02437 \varepsilon^{0.501368}$


Fig. 48

$$
\Delta v=\frac{1}{2}\left(v_{3}-v_{1}\right) \text { vs } \varepsilon=|\tau|(\text { for } 0<\varepsilon \leq 0.001)
$$

## 13. Lever rule for the reduced volume in the coexistence line

At the point $e$ the substance is entirely in the gas phase with volume $v_{1}$; at the point $a$, it is entirely in the liquid phase with volume $v_{1}$. At any point on the line $a-c-e$ (the coexistence line), the reduced volume can be described by

$$
v=\theta_{1} v_{1}+\theta_{3} v_{3}
$$

using the lever rule, where $\theta_{1}$ is the fraction of the liquid phase and $\theta_{3}$ is the fraction of the gas phase,

$$
\theta_{1}+\theta_{3}=1 .
$$

Then we have

$$
\theta_{1}=\frac{v_{3}-v}{v_{3}-v_{1}},
$$

$$
\theta_{3}=\frac{v-v_{1}}{v_{3}-v_{1}}
$$

For $v=v_{1}, \quad \theta_{1}=1$ and $\theta_{3}=0 \quad$ (the pure liquid phase).
For $v=v_{3}, \quad \theta_{1}=0$ and $\theta_{3}=1 . \quad$ (the pure gas phase).

