## One dimensional Ising model (exact solution) Masatsugu Sei Suzuki Department of Physics, SUNY at Binghamton (Date: November 19, 2016)

The most popular approach to solving the 2D Ising model is via the so called transfer matrix method. We can get some idea of how this method works by using it to solve the 1D model. In particular we can use this technique to solve the 1D Ising model in the presence of an external magnetic field, something not possible with our previous direct summation approach. This solution will allow us to rigorously demonstrate that the 1D model does not exhibit a phase transition (i.e., no spontaneous magnetization).

#### 1. The transfer matrix method

We consider an *N*-site 1D Ising model with nearest neighbor ferromagnetic coupling *J* and periodic boundary conditions (i.e., i+N=i) in an external magnetic field *B*. Here we discuss the exact solutions for the thermodynamic properties of one-dimensional Ising model with N spins (spin 1/2) pointing up or down.

$$\beta H = -K \sum_{i=1}^{N} \sigma_i \sigma_{i+1} - \mu_B B \sum_{i=1}^{N} \sigma_i$$

where B is an external magnetic field,

$$K = \frac{J}{k_B T} = \beta J$$
,  $h = \frac{\mu_B B}{k_B T} = \beta(\mu_B B)$ 

We use the periodic boundary condition

$$\sigma_1 = \sigma_{N+1}$$
.

The partition function,

$$Z = \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \sum_{\sigma_3 = \pm 1} \cdots \sum_{\sigma_{N-1} = \pm 1} \sum_{\sigma_N = \pm 1} \exp(-\beta H).$$

The part related the sites *i* and i+1 in the Boltzmann factor

$$T(\sigma_i, \sigma_{i+1}) = \left\langle \sigma_i \left| \hat{T} \right| \sigma_{i+1} \right\rangle = \exp[K\sigma_i \sigma_{i+1} + \frac{h}{2}(\sigma_i + \sigma_{i+1}) + \frac{h$$

The martix T is expressed by the 2x2 matrix as

$$\hat{T} = \begin{pmatrix} \langle 1 | \hat{T} | 1 \rangle & \langle 1 | \hat{T} | -1 \rangle \\ \langle -1 | \hat{T} | 1 \rangle & \langle -1 | \hat{T} | -1 \rangle \end{pmatrix} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}$$

or

$$\hat{T} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}$$
  
=  $e^{-K} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e^{K} \sinh(h) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + e^{K} \cosh(h) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$   
=  $e^{-K} \hat{\sigma}_{x} + e^{K} \sinh(h) \hat{\sigma}_{z} + e^{K} \cosh(h) \hat{1}$ 

Using this notation, *Z* can be rewritten as

$$Z = \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \sum_{\sigma_3 = \pm 1} \cdots \sum_{\sigma_{N-1} = \pm 1} \sum_{\sigma_N = \pm 1} T(\sigma_1, \sigma_2) T(\sigma_2, \sigma_3) T(\sigma_3, \sigma_4) \cdots T(\sigma_{N-1}, \sigma_N) T(\sigma_N, \sigma_1)$$
$$= \sum_{\sigma_1 = \pm 1} \sum_{\sigma_2 = \pm 1} \sum_{\sigma_3 = \pm 1} \cdots \sum_{\sigma_{N-1} = \pm 1} \sum_{\sigma_N = \pm 1} \langle \sigma_1 | \hat{T} | \sigma_2 \rangle \langle \sigma_2 | \hat{T} | \sigma_3 \rangle \langle \sigma_3 | \hat{T} | \sigma_4 \rangle \cdots \langle \sigma_{N-1} | \hat{T} | \sigma_N \rangle \langle \sigma_N | \hat{T} | \sigma_1 \rangle$$

Here we use the closure relation (quantum mechanics)

$$\sum_{\sigma_2=\pm 1} |\sigma_2\rangle \langle \sigma_2| = \hat{1}.$$

Then we get

$$\sum_{\sigma_2=\pm 1} \langle \sigma_1 | \hat{T} | \sigma_2 \rangle \langle \sigma_2 | \hat{T} | \sigma_3 \rangle = \langle \sigma_1 | \hat{T}^2 | \sigma_3 \rangle$$

The partition function can be expressed by a simple form

$$Z = \sum_{\sigma_1 = \pm 1} \langle \sigma_1 | \hat{T}^N | \sigma_1 \rangle = Tr[\hat{T}^N]$$

We now calculate the partition function. The matrix T can be expressed using the Pauli matrix

$$\hat{T} = e^{K} \sinh(h) \hat{\sigma}_{z} + e^{-K} \hat{\sigma}_{x} + e^{K} \cosh(h) \hat{1}$$
  
=  $\sqrt{e^{2K} \sinh^{2}(h) + e^{-2k}} \left[ \frac{e^{K} \sinh(h)}{\sqrt{e^{2K} \sinh^{2}(h) + e^{-2k}}} \hat{\sigma}_{z} + \frac{e^{-K}}{\sqrt{e^{2K} \sinh^{2}(h) + e^{-2k}}} \hat{\sigma}_{x} \right]$   
+  $e^{K} \cosh(h) \hat{1}$ 

where the Pauli matrices are given by

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note that  $\left|\sigma_{i}\right\rangle$  is the eigenket of  $\hat{\sigma}_{z}$ ,

$$\sigma_{z}|\sigma_{i}\rangle = \sigma_{i}|\sigma_{i}\rangle$$

with  $\sigma_i = \pm 1$ . We introduce the angle  $\theta$  from the *z* axis in the *z*-*x* plane. Then we have

$$\hat{T} = \sqrt{e^{2K} \sinh^2(h) + e^{-2k}} (\cos\theta\hat{\sigma}_z + \sin\theta\hat{\sigma}_z) + e^K \cosh(h)\hat{l}$$
$$= \sqrt{e^{2K} \sinh^2(h) + e^{-2k}} (\hat{\sigma} \cdot \boldsymbol{n}) + e^K \cosh(h)\hat{l}$$

where **n** is the unit vector;  $\mathbf{n} = (\sin \theta, 0, \cos \theta)$ ,

$$\cos\theta = \frac{e^{K}\sinh(h)}{\sqrt{e^{2K}\sinh^{2}(h) + e^{-2k}}}, \qquad \sin\theta = \frac{e^{-K}}{\sqrt{e^{2K}\sinh^{2}(h) + e^{-2k}}}$$

or

$$\tan\theta = \frac{e^{-2K}}{\sinh(h)}$$



The eigenvalue and eigenstate for the operator  $\hat{T}$  are as follows.

$$\begin{aligned} (\hat{\sigma} \cdot \boldsymbol{n}) &|+\boldsymbol{n}\rangle = |+\boldsymbol{n}\rangle, \qquad (\hat{\sigma} \cdot \boldsymbol{n}) |-\boldsymbol{n}\rangle = -|-\boldsymbol{n}\rangle \\ \hat{T}|+\boldsymbol{n}\rangle &= \sqrt{e^{2K} \sinh^2(h) + e^{-2k}} (\hat{\sigma} \cdot \boldsymbol{n}) |+\boldsymbol{n}\rangle + e^K \cosh(h) \hat{I}|+\boldsymbol{n}\rangle \\ &= [\sqrt{e^{2K} \sinh^2(h) + e^{-2k}} + e^K \cosh(h)] |+\boldsymbol{n}\rangle \\ &= \lambda_+ |+\boldsymbol{n}\rangle \\ \hat{T}|-\boldsymbol{n}\rangle &= \sqrt{e^{2K} \sinh^2(h) + e^{-2k}} (\hat{\sigma} \cdot \boldsymbol{n}) |-\boldsymbol{n}\rangle + e^K \cosh(h) \hat{I}|-\boldsymbol{n}\rangle \\ &= [-\sqrt{e^{2K} \sinh^2(h) + e^{-2k}} + e^K \cosh(h)] |-\boldsymbol{n}\rangle \\ &= \lambda_- |-\boldsymbol{n}\rangle \end{aligned}$$

Note that

(i) One of the eigenkets of  $\hat{T}$  is

$$|+\boldsymbol{n}\rangle = \hat{U}|+z\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\\ \sin\frac{\theta}{2} \end{pmatrix},$$

with the eigenvalue

$$\lambda_{+} = \sqrt{e^{2K} \sinh^2(h) + e^{-2k}} + e^K \cosh(h)$$

(ii) The other eigenkets of  $\hat{T}$  is

$$|-\boldsymbol{n}\rangle = \hat{U}|-z\rangle = \begin{pmatrix} -\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2} \end{pmatrix},$$

with the eigenvalue

$$\lambda_{-} = -\sqrt{e^{2K}\sinh^2(h) + e^{-2k}} + e^K\cosh(h)$$

The unitary matrix  $\hat{U}$  is obtained as

$$\hat{U} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

where

$$\hat{U}^{+}\hat{U}=\hat{1}$$

Thus we have

$$\hat{U}^{+}\hat{T}\hat{U} = \begin{pmatrix} \lambda_{+} & 0\\ 0 & \lambda_{-} \end{pmatrix}$$
 (diagonal matrix)

Then the partition function is rewritten as

$$Z = Tr[\hat{T}^{N}]$$
  
=  $Tr[(\hat{U}^{+}\hat{T}\hat{U})(\hat{U}^{+}\hat{T}\hat{U})\cdots(\hat{U}^{+}\hat{T}\hat{U})]$   
=  $Tr[(\hat{U}^{+}\hat{T}\hat{U})^{N}]$   
=  $\lambda_{+}^{N} + \lambda_{-}^{N}$ 

The free energy per spin is

$$\frac{F}{N} = -\frac{k_B T}{N} \ln Z = -\frac{k_B T}{N} \ln \left(\lambda_+^N + \lambda_-^N\right).$$

or

$$\frac{\beta F}{N} = -\frac{1}{N} \ln(\lambda_{+}^{N} + \lambda_{-}^{N})$$

The energy per spin is

$$\frac{E}{N} = -\frac{1}{N} \frac{\partial}{\partial \beta} \ln Z = \frac{\partial}{\partial \beta} (\frac{\beta F}{N}) = -k_B T^2 \frac{\partial}{\partial T} (\frac{\beta F}{N}).$$

The heat capacity per spin

$$\frac{C}{N} = \frac{d}{dT} (\frac{E}{N})$$

The entropy per spin

$$\frac{S}{N} = \frac{1}{T} \left(\frac{E}{N} - \frac{F}{N}\right)$$

The magnetization M is given by

$$\frac{M}{N} = \frac{\mu_B}{N} \sum_i \left\langle \sigma_i \right\rangle = -\frac{\partial}{\partial B} \left(\frac{F}{N}\right)$$

# **Heat capacity** In the limit of sufficiently large *N*, 2.

$$Z = \lambda_{+}^{N}$$

since  $\lambda_{_+} > \lambda_{_-}$ . The Helmholtz free energy per spin is

$$\frac{F}{N} = -\frac{k_B T}{N} \ln \lambda_+^N$$
$$= -k_B T \ln[\sqrt{e^{2K} \sinh^2(h) + e^{-2k}} + e^K \cosh(h)]$$

When h = 0,



**Fig.** Heat capacity vs temperature for the 1D Ising model.  $\frac{1}{K} = \frac{k_B T}{J}$ .

# 3. Magnetization and magnetic susceptibility

The magnetization M per spin is given by

$$\frac{M}{N} = -\frac{\partial}{\partial B} \left(\frac{F}{N}\right) = \mu_B \frac{e^{2K} \sinh(h)}{\sqrt{e^{4K} \sinh^2(h) + 1}}$$

We make a plot of  $y = \frac{M}{N\mu_B}$  as s function of x = ah, where  $a = \frac{1}{K} = \frac{k_B T}{J}$  is changed as a parameter.

$$y = \frac{e^{2/a}\sinh(x/a)}{\sqrt{e^{4/a}\sinh^2(x/a) + 1}}$$

where



**Fig.** Magnetization vs magnetic field for the 1D Ising model.  $y = \frac{M}{N\mu_B}$  as a function of

$$x = ah = \frac{\mu_B B}{J}$$
, where  $a = \frac{1}{K} = \frac{k_B T}{J}$ 

### 4. Magnetic susceptibility

The magnetic susceptibility is obtained as

$$\frac{\chi}{N} = \frac{\partial}{\partial B} \frac{M}{N} = \frac{\mu_B^2}{k_B T} \frac{e^{2K} \cosh(h)}{\left[1 + e^{4K} \sinh^2(h)\right]^{3/2}}$$

At h = 0, we have

$$\frac{\chi}{N} = \frac{\mu_B^2}{k_B T} e^{2K} = \frac{\mu_B^2}{J} K e^{2K} = \frac{\mu_B^2}{k_B T} e^{\frac{2J}{k_B T}}$$

This susceptibility exponentially diverges as *T* tends to zero, as shown below.



$$\langle \sigma_i \sigma_j \rangle = \frac{1}{Z} Tr[\sigma_i \sigma_j e^{-\beta H}]$$

where j > i. We calculate the trace using the techniques of quantum mechanics,

$$Tr[\sigma_{i}\sigma_{j}e^{-\beta H}] = Tr[T(\sigma_{1},\sigma_{2})T(\sigma_{2},\sigma_{3})...T(\sigma_{i-1},\sigma_{i})\sigma_{i}T(\sigma_{i},\sigma_{i+1})T(\sigma_{i+1},\sigma_{i+2})$$
  
....T(\sigma\_{j-1},\sigma\_{j})\sigma\_{j}T(\sigma\_{j},\sigma\_{j+1})T(\sigma\_{j+1},\sigma\_{j+2})....T(\sigma\_{N-2},\sigma\_{N-1})T(\sigma\_{N-1},\sigma\_{N})]  
= Tr[T^{i-1}\sigma\_{z}T^{j-i}\sigma\_{z}T^{N-j+1}]  
= Tr[\begin{array}{c} T^{N+i-j}\sigma\_{z}b^{j-i}\sigma\_{z}

where

$$\left\langle \sigma_{j-1} \left| \hat{\sigma}_{z} \right| \sigma_{j} \right\rangle = \sigma_{j} \left\langle \sigma_{j-1} \left| \sigma_{j} \right\rangle$$

$$\sum_{\sigma_{2}=\pm 1} T(\sigma_{1}, \sigma_{2}) T(\sigma_{2}, \sigma_{3}) = \sum_{\sigma_{2}=\pm 1} \left\langle \sigma_{1} \left| \hat{T} \right| \sigma_{2} \right\rangle \left\langle \sigma_{2} \left| \hat{T} \right| \sigma_{3} \right\rangle$$

$$= \left\langle \sigma_{1} \left| \hat{T}^{2} \right| \sigma_{3} \right\rangle$$

and

$$\sum_{\sigma_{j}=\pm 1} T(\sigma_{j-1},\sigma_{j})\sigma_{j}T(\sigma_{j},\sigma_{j+1}) = \sum_{\sigma_{j}=\pm 1} \langle \sigma_{j-1} | \hat{T}\hat{\sigma}_{z} | \sigma_{j} \rangle \langle \sigma_{j} | \hat{T} | \sigma_{j+1} \rangle$$
$$= \langle \sigma_{j-1} | \hat{T}\hat{\sigma}_{z}\hat{T} | \sigma_{j+1} \rangle$$

Then we have

$$Tr[(\hat{U}^{+}\hat{T}\hat{U})^{N+i-j}(\hat{U}^{+}\hat{\sigma}_{z}\hat{U})(\hat{U}^{+}\hat{T}\hat{U})^{j-i}(\hat{U}^{+}\hat{\sigma}_{z}\hat{U})]$$

$$= Tr[\begin{pmatrix}\lambda_{+}^{N+i-j} & 0\\ 0 & \lambda_{-}^{N+i-j}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\lambda_{+}^{j-i} & 0\\ 0 & \lambda_{-}^{j-i}\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & -\cos\theta\end{pmatrix}]$$

$$= \lambda_{+}^{N+i-j}\lambda_{+}^{j-i}\cos^{2}\theta + \lambda_{+}^{N+i-j}\lambda_{-}^{j-i}\sin^{2}\theta + \lambda_{-}^{N+i-j}\lambda_{+}^{j-i}\sin^{2}\theta + \lambda_{-}^{N+i-j}\lambda_{-}^{j-i}\cos^{2}\theta$$

where

$$(\hat{U}^{+}\hat{T}\hat{U})^{N+i-j} = \begin{pmatrix} \lambda_{+} & 0 \\ 0 & \lambda_{-} \end{pmatrix}^{N+i-j} = \begin{pmatrix} \lambda_{+}^{N+i-j} & 0 \\ 0 & \lambda_{-}^{N+i-j} \end{pmatrix}$$

$$\hat{U} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \qquad \hat{U}^{+} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

and

 $\hat{U}^{\scriptscriptstyle +}\hat{U}=\hat{1}$ 

$$\hat{U}^{+}\hat{\sigma}_{z}\hat{U} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$
$$= \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$$

We consider the special case in the presence of a magnetic field. When B = 0, we have  $\theta = \frac{\pi}{2}$ .

$$\left\langle \sigma_{i}\sigma_{j}\right\rangle = \frac{1}{Z}Tr[\sigma_{i}\sigma_{j}e^{-\beta H}]$$
$$= \frac{\left(\frac{\lambda_{+}}{\lambda_{-}}\right)^{N+i-j} + \left(\frac{\lambda_{+}}{\lambda_{-}}\right)^{j-i}}{\left(\frac{\lambda_{+}}{\lambda_{-}}\right)^{N} + 1}$$

In the limit of  $N \to \infty$  where (j-i) is fixed,

$$\left\langle \sigma_{i}\sigma_{j}\right\rangle = \frac{1}{Z}Tr[\sigma_{i}\sigma_{j}e^{-\beta H}]$$

$$= \frac{\left(\frac{\lambda_{+}}{\lambda_{-}}\right)^{N+i-j} + \left(\frac{\lambda_{+}}{\lambda_{-}}\right)^{j-i}}{\left(\frac{\lambda_{+}}{\lambda_{-}}\right)^{N} + 1}$$

$$\approx \left(\frac{\lambda_{+}}{\lambda_{-}}\right)^{i-j}$$

$$= \left(\frac{\lambda_{-}}{\lambda_{+}}\right)^{j-i}$$

or

$$\langle \sigma_i \sigma_j \rangle \approx (\tanh K)^{j-i} = \exp[-\frac{j-i}{\xi}]$$

since  $\lambda_{+} > \lambda_{-}$ . Note that  $\xi$  is the spin correlation length and is given by

$$\xi = \frac{1}{\left| \ln \tanh(K) \right|}$$

We note that in the limit of  $K \to \infty$   $(T \to 0)$ ,

$$\tanh(K) = \frac{e^{K} - e^{-K}}{e^{K} + e^{-K}} = \frac{1 - e^{-2K}}{1 + e^{-2K}} \approx 1 - 2e^{-2K}$$
$$\left|\ln \tanh(K)\right| = \left|\ln(1 - 2e^{-2K})\right| = 2e^{-2K}$$

Then we have

$$\xi = \frac{1}{2}e^{2K} = \frac{1}{2}e^{2J/k_{B}T}$$

So the correlation length  $\xi$  exponentially increases with decreasing temperature (  $(T \rightarrow 0)$  .



**Fig.** Correlation length as a function of temperature for the 1D Ising model.  $x = J/(k_B T)$ .