Thermodynamics of elastic rod

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1. Thermodynamics in elastic rod:

(a) Isothermal Young's modulus

$$\frac{\delta f}{A} = E_T \frac{\delta l}{l}$$

$$E_T = \frac{l}{A}\left(\frac{\partial f}{\partial l}\right)_T$$

(b) Linear expansivity at constant tension

$$\alpha_f = \frac{1}{l}\left(\frac{\partial l}{\partial T}\right)_f$$

((Example-1))

$$\begin{aligned}
\left(\frac{\partial f}{\partial T}\right)_l &= \frac{\partial(f,l)}{\partial(T,l)} \\
&= \frac{\partial(f,T)}{\partial(T,l)}\frac{\partial(f,l)}{\partial(f,T)} \\
&= -\left(\frac{\partial f}{\partial l}\right)_T\left(\frac{\partial l}{\partial T}\right)_f \\
&= -l\alpha_f \frac{AE_T}{l} \\
&= -AE_T\alpha_f
\end{aligned}$$

2. Internal energy

$$dU = TdS - PdV = TdS + fdl$$

The correspondence:

$$P \rightarrow -f\,, \qquad V \rightarrow l$$

$$dU = d(ST) - SdT + fdl$$

or

$$dF = d(U - ST) = -SdT + fdl$$

with

$$S = -\left(\frac{\partial F}{\partial T}\right)_l, \qquad f = \left(\frac{\partial f}{\partial l}\right)_T$$

Maxwell's relation:

$$\left(\frac{\partial S}{\partial l}\right)_T = -\left(\frac{\partial f}{\partial T}\right)_l$$

Since

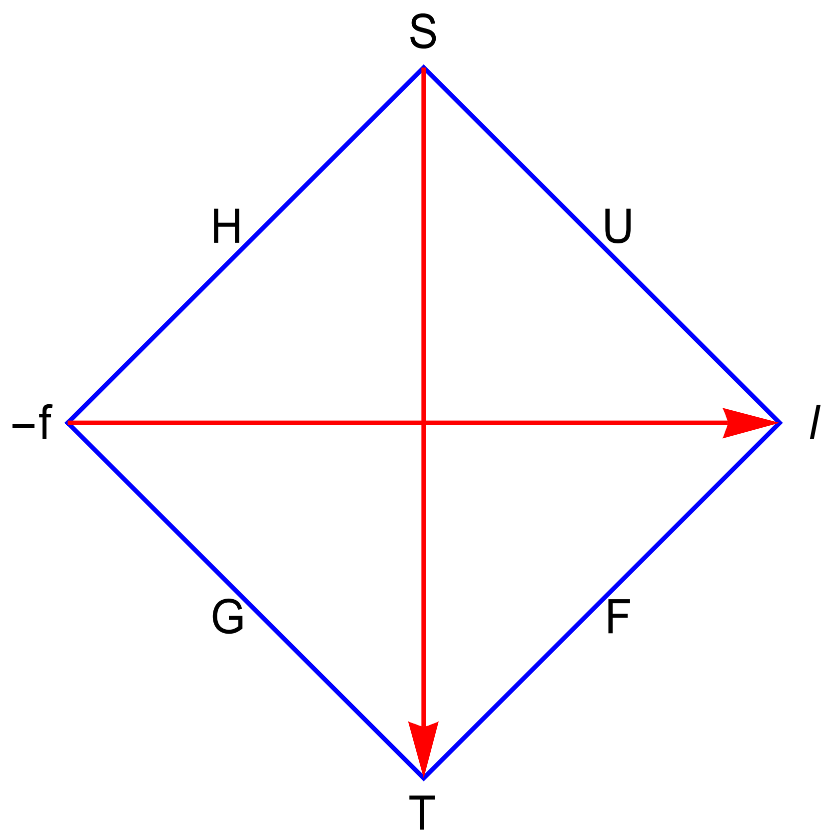
$$\left(\frac{\partial f}{\partial T}\right)_l = -AE_T\alpha_f$$

we have the relation

$$\left(\frac{\partial S}{\partial l}\right)_T = AE_T\alpha_f$$

3. Maxwell's relation

$$P \rightarrow -f\,, \qquad V \rightarrow l$$



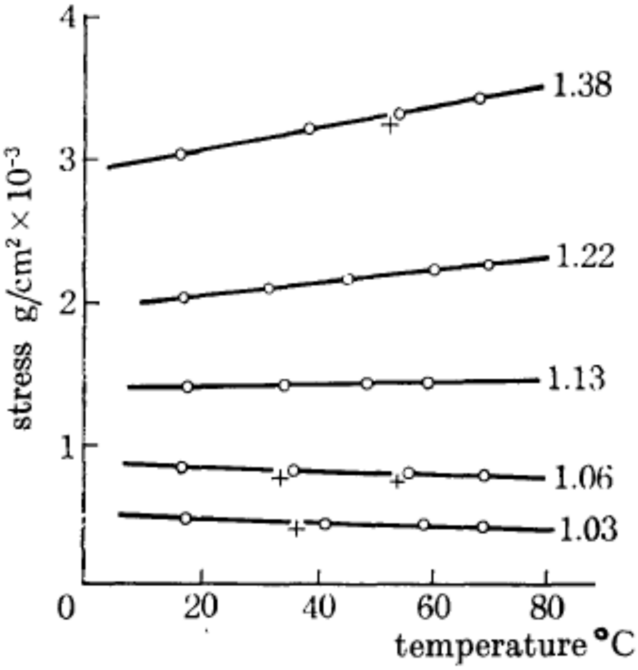
4. Problem and solution

R. Kubo, Thermodynamics An Advanced Course with Problems and Solutions (North-Holland, 1968). Problem 3-34

34. The figure shows the experimental data for the tension of a suitably vulcanized rubber band maintained at a constant length plotted against the temperature. Let l_0 be the natural length of the rubber band at temperature T_0 and l the actual length. The total tension (stress × cross-section) is related to l as

$$X = AT\left[\frac{l}{l_0} - \{1 + \alpha(T - T_0)\}\left(\frac{l_0}{l}\right)^2\right],$$

where α is the thermal expansion coefficient and is approximately equal to $7 \times 10^{-4}\ \mathrm{deg}^{-1}$. Obtain the change in temperature T, when the rubber band at temperature T_0 is stretched suddenly (adiabatically) from its natural length l_0 to L times that. (This is called the Joule effect.) Plot ΔT as a function of L.



stress g/cm² × 10⁻³
4
3
2
1
0
20
40
60
80
temperature °C
1.38
1.22
1.13
1.06
1.03

((**Solution**))

34. The rate of temperature change due to the adiabatic elongation of a rubber band is given as

$$\left(\frac{\partial T}{\partial l}\right)_S = -\left(\frac{\partial S}{\partial l}\right)_T \Big/ \left(\frac{\partial S}{\partial T}\right)_l = \frac{T}{C_l}\left(\frac{\partial X}{\partial T}\right)_l, \qquad (1)$$

where C_l is the heat capacity at constant length. Here the Maxwell relation, $(\partial S/\partial l)_T = -(\partial X/\partial T)_l$ obtained from $\mathrm{d}F = -S\mathrm{d}T + X\mathrm{d}l$, was used. For $T = T_0$ the given equation becomes

$$\left(\frac{\partial X}{\partial T}\right)_l = A(L - L^{-2}) - \alpha A T_0 L^{-2},$$

or

$$\left(\frac{\partial T}{\partial l}\right)_S = \frac{T_0}{C_l} A\{L - (1 + \alpha T_0) L^{-2}\}.$$

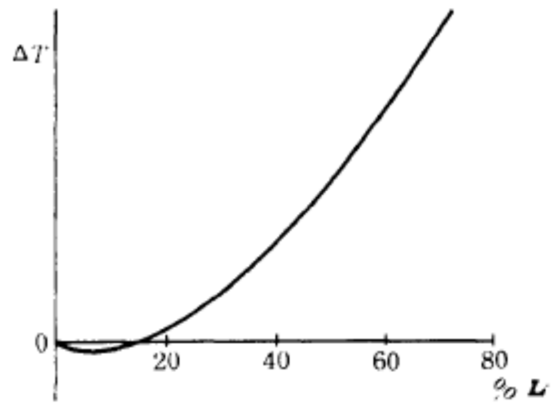
The temperature change due to an adiabatic elongation from $L = 1$ to L is

$$\begin{aligned}\Delta T &= \int_{l_0}^{Ll_0} \left(\frac{\partial T}{\partial l}\right)_S \mathrm{d}l = \frac{T_0 A l_0}{C_l} \int_1^L \{L - (1 + \alpha T_0) L^{-2}\}\, \mathrm{d}L \\ &= \frac{Al_0}{2C_l} T_0 \frac{L-1}{L} \{L^2 + L - 2(1 + \alpha T_0)\}. \qquad (2)\end{aligned}$$

If $T_0 = 300\ ^\circ\mathrm{K}$, $\delta = \alpha T_0 = 0.21$. The expression in parentheses in (2) vanishes when $L = 1 + \varepsilon$, where ε is determined by $3\varepsilon + \varepsilon^2 = 2\delta$, or

$$\varepsilon = \tfrac{2}{3}\delta - \tfrac{1}{3}\varepsilon^2 \fallingdotseq 0.14 - \tfrac{1}{3}\cdot 0.020 \fallingdotseq 0.13.$$

Equation (2) is plotted against L in the figure.



NOTE: The relation given in this problem between the tension and the strain can be derived from the statistical mechanical theory of rubber elasticity and actually applies quite well to moderately vulcanized rubber and also to some other kinds of rubber-like materials. The thermal effect that is considered here is called the Joule effect. As seen in the figure, the fact that $\Delta T<0$ for small strains is due to thermal expansion. Ignoring this, such rubber-like materials are heated up ($\Delta T>0$) when they are elongated adiabatically. This is a characteristic of rubber-like elasticity. One can easily verify this with a rubber band.

((**Solution**))

$$\left(\frac{\partial T}{\partial l}\right)_S=\frac{\dfrac{\partial(T,S)}{\partial(l,T)}}{\dfrac{\partial(l,S)}{\partial(l,T)}}=-\frac{\left(\dfrac{\partial S}{\partial l}\right)_T}{\left(\dfrac{\partial S}{\partial T}\right)_l}=\frac{T}{C_l}\left(\frac{\partial f}{\partial T}\right)_l$$

where we use the Maxwell's relation

$$\left(\frac{\partial S}{\partial l}\right)_T=-\left(\frac{\partial f}{\partial T}\right)_l$$

and the heat capacity is

$$C_l = T\left(\frac{\partial S}{\partial T}\right)_l$$

The force f is given by

$$f = X = AT[\frac{l}{l_0} - \{1+\alpha(T-T_0)\}\left(\frac{l_0}{l}\right)^2$$

Thus we get

$$\left(\frac{\partial X}{\partial T}\right)_l = A[\frac{l}{l_0} - \{1+\alpha(T-T_0)\}\left(\frac{l_0}{l}\right)^2] - AT\alpha\left(\frac{l_0}{l}\right)^2$$

At $T = T_0$

$$\left(\frac{\partial X}{\partial T}\right)_l = A[\frac{l}{l_0} - (\alpha T_0 + 1)\left(\frac{l_0}{l}\right)^2]$$

$$\left(\frac{\partial T}{\partial l}\right)_S = \frac{T_0}{C_l}\left(\frac{\partial X}{\partial T}\right)_l = \frac{T_0}{C_l}A[\frac{l}{l_0} - (\alpha T_0 + 1)\left(\frac{l_0}{l}\right)^2]$$

The temperature change due to an adiabatic elongation;

$$\begin{aligned}
\Delta T &= \int_{l_0}^{Ll_0}\left(\frac{\partial T}{\partial l}\right)_S dl \\
&= \frac{T_0}{C_l}A\int_{l_0}^{Ll_0}[\frac{l}{l_0} - (\alpha T_0 + 1)\left(\frac{l_0}{l}\right)^2]dl \\
&= \frac{T_0}{C_l}Al_0\int_1^L[x - (\alpha T_0 + 1)\frac{1}{x^2}]dx \\
&= \frac{T_0}{2C_l}Al_0\frac{L-1}{L}[L^2 + L - 2(\alpha T_0 + 1)]
\end{aligned}$$

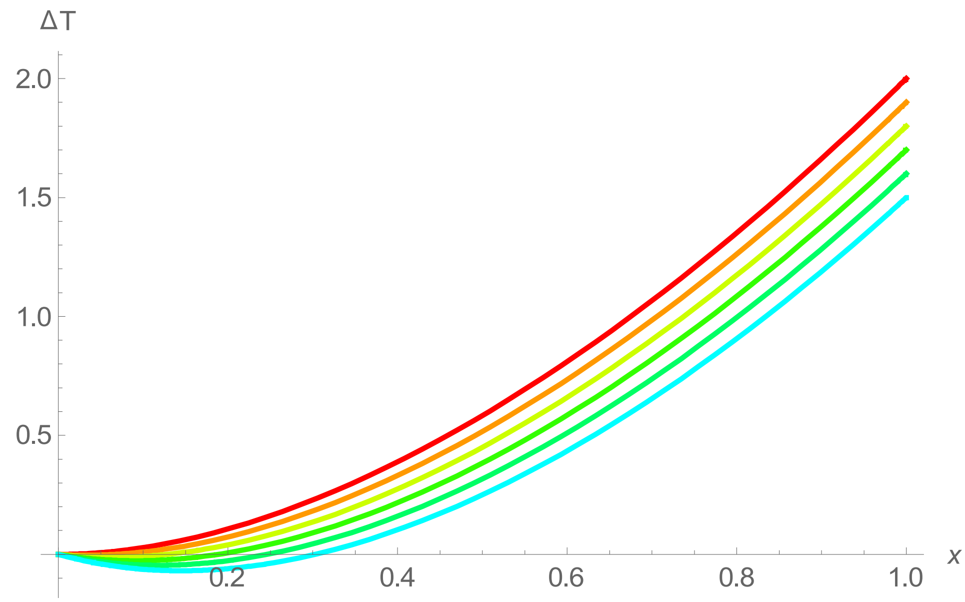


Fig. Plot of $y = \dfrac{\Delta T}{\dfrac{T_0}{2C_l} A l_0} = \dfrac{L-1}{L}[L^2 + L - 2(\alpha T_0 + 1)]$, as a function of $x = (L-1)$. The parameter $a = \alpha T_0$ is changed for a = 0 (red), 0.1, 0.2, 0.3, 0.4, and 0.5 (blue).