Adiabatic atmosphere
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When we climb a mountain, we notice that the temperature decreases with increasing the altitude. We discuss this in terms of the adiabatic process. The **adiabatic lapse rate** can be evaluated; 9.76 K/km.

Lapse rate is the rate at which Earth's atmospheric temperature decreases with an increase in altitude, or increases with the decrease in altitude. *Lapse rate* arises from the word *lapse*, in the sense of a gradual change. Although this concept is most often applied to Earth's troposphere, lapse rate can be extended to any gravitationally supported parcel of gas.
https://en.wikipedia.org/wiki/Lapse_rate

We start with an equation

$$AP(z) - AP(z+dz) - A\rho g dz = 0$$

or

$$\frac{P(z+dz) - P(z)}{dz} = -\rho g$$

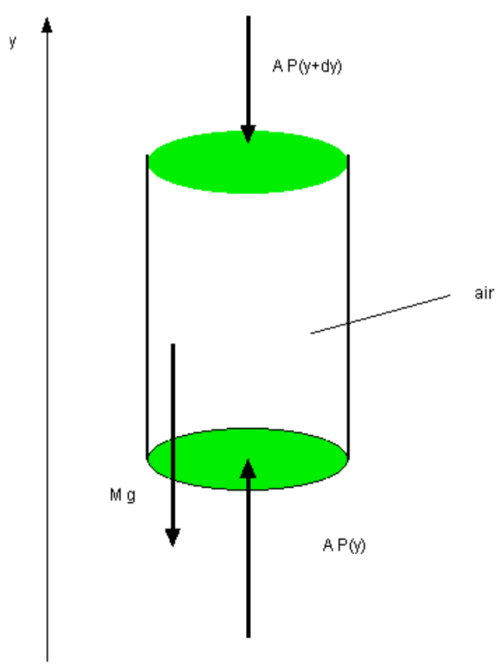

Fig. $y = z$. $Mg = \rho A dz g$, in the present case.

In the limit of $dz \to 0$, we get a differential equation

$$\frac{dP(z)}{dz} = -\rho g$$

Note that

$$\rho = \frac{M}{V} = \frac{M}{N}\frac{N}{V} = mn$$

The Boyle's law:

$$PV = Nk_B T ,$$

or

$$P = \frac{N}{V} k_B T = nk_B T$$

or

$$n = \frac{P}{k_B T}$$

Using this equation, we get

$$\frac{dP}{dz} = -mng = -\frac{mg}{k_B T} P$$

or

$$\frac{dP}{dz} = -\frac{mN_A g}{k_B N_A T} P = -\frac{Mg}{R}\frac{P}{T} \qquad (1)$$

by multiplying the Avogadro number N_A for the numerator and denominator.

In an adiabatic process, we have

$$P^{1-\gamma} T^{\gamma} = \text{const}$$

where $\gamma = \dfrac{C_P}{C_V}$. This can be rewritten as

$$(1-\gamma)\ln P + \gamma \ln T = \text{const}$$

Taking the derivative, we get

$$(1-\gamma)\frac{1}{P}dP + \gamma \frac{1}{T}dT = 0$$

or

$$\frac{dP}{dT} = \frac{\gamma}{\gamma - 1}\frac{P}{T} \qquad (2)$$

From Eq.(1), we have

$$\frac{dP}{P} = -\frac{Mg}{R}\frac{dz}{T} \qquad (3)$$

From Eq.(2), we have

$$\frac{dP}{P} = \frac{\gamma}{\gamma - 1}\frac{dT}{T} \qquad (4)$$

From Eqs.(3) and (4)

$$\frac{\gamma}{\gamma - 1}\frac{dT}{T} = -\frac{Mg}{R}\frac{dz}{T}$$

or

$$\frac{dT}{dz} = -\left(\frac{\gamma - 1}{\gamma}\right)\frac{Mg}{R} = -\frac{R}{C_P}\frac{Mg}{R} = -\frac{Mg}{C_P}$$

or

$$T(z) = T(z=0) - \frac{Mg}{C_P} z \,,$$

where M is the molar mass of air; M = 28.9652 g/mol. C_p is the molar specific heat at constant pressure (air)

$$C_P = 1.003 \text{ J/(g K)} = 1.003 \text{ x } 28.9652 \text{ J/(mol K)}$$

Using these values,

$$\frac{Mg}{C_P} = \frac{28.9652 \times 10^{-3} \times 9.8}{1.003 \text{ x } 28.9652} = 9.76 \text{ K/km} \qquad \text{(adiabatic lapse rate)}$$

where J = Nm.

We can evaluate the pressure P from the relation

$$P^{1-\gamma} T^{\gamma} = \text{const}$$

or

$$P(z) = P(z=0)[\frac{T(z)}{T(z=0)}]^{\gamma/(\gamma-1)}$$

or

$$P(z) = P(z=0)[1 - \frac{\frac{Mg}{C_P}}{T(z=0)} z]^{C_P/R}.$$