Understanding of the ergodic hypothesis using a simple example

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Using the following example, we can understand the validity of the ergodic hypothesis; the time average is equal to the ensemble average. I found this example from a book written by Prof. Yasushi Takahashi (Professor of the Department of Physics, University of Alberta). This book is an excellent book for students who want to learn the principle of statistical mechanics. Unfortunately it was written in Japanese. The content of the example is as follows.

((**Example of ergodic principle**))

How can we guess the temperature of the outside in a very cold day of winter without measuring the temperature with thermometer? Suppose that there are many houses in your neighbors. Each house has one chimney. You wake up early in the morning and see the chimneys outside from your bed in the second floor. You may see white smokes coming out from chimneys. Roughly, you may guess the temperature of the outside from the number of chimneys where white smoke comes out. It is common knowledge that the number of chimneys with smokes increases with decreasing temperature of the outside. Suppose that each house has the same type of heating system. The temperature inside the house is kept at the same one. To this end, the heating system should be on for the time Δt over the finite time period t_0, and should be off for the remaining time $t_0 - \Delta t$. The probability of the heating system with the on-state is

$$g = \frac{\Delta t}{t_0} \qquad \text{(on-state for the heating system).}$$

The probability (in time) of the heating system with the off-state is

$$1 - g = 1 - \frac{\Delta t}{t_0} \qquad \text{(off-state for the heating system)}$$

Suppose that there are N houses surrounding your home. So you can see all the chimneys through your windows. Among them, there are n houses where smokes come out from the chimneys, and (N-n) houses with no smoke coming out. The probability for this event is given by

$$P(n) = \frac{N!}{n!(N-n)!} g^n (1-g)^{N-n}$$

where

$$\sum_{n=0}^{N} P(n) = (g + 1 - g)^N = 1$$

$$\begin{aligned}
\frac{S}{k_B} &= \ln P(n) \\
&= \ln N! - \ln n! - \ln(N-n)! + n\ln g + (N-n)\ln(1-g) \\
&= N\ln N - N - n\ln n + n - (N-n)\ln(N-n) + N - n + n\ln g + (N-n)\ln(1-g) \\
&= N\ln N - n\ln n - (N-n)\ln(N-n) + n\ln g + (N-n)\ln(1-g) \\
&= f(n)
\end{aligned}$$

((**Average and standard deviation**))

Note that the average number $\langle n \rangle$ and the standard deviation are evaluated as follows.

$$\begin{aligned}
\langle n \rangle &= \sum_{n=0}^{N} nP(n) \\
&= \sum_{n=0}^{N} n \frac{N!}{n!(N-n)!} g^n (1-g)^{N-n} \\
&= Ng \sum_{n=0}^{N} n \frac{(N-1)!}{(n-1)!(N-1-n+1)!} g^{n-1} (1-g)^{N-n} \\
&= Ng \sum_{n'=0}^{N-1} n \frac{(N-1)!}{(n')!(N-1-n')!} g^{n'} (1-g)^{N-1-n'} \\
&= Ng(g+1-g)^{N-1} \\
&= Ng
\end{aligned}$$

$$\begin{aligned}
\langle n(n-1) \rangle &= \sum_{n=0}^{N} n(n-1)P(n) \\
&= \sum_{n=0}^{N} \frac{N!}{(n-2)!(N-n)!} g^n (1-g)^{N-n} \\
&= N^2 g^2 \sum_{n=0}^{N} \frac{(N-2)!}{(n-2)!(N-2-n+2)!} g^{n-2} (1-g)^{N-n} \\
&= N^2 g^2 \sum_{n'=0}^{N-2} \frac{(N-2)!}{(n')!(N-2-n')!} g^{n'} (1-g)^{N-2-n'} \\
&= N^2 g^2 (g+1-g)^{N-2} \\
&= N^2 g^2
\end{aligned}$$

where $n' = n - 2$. The fluctuation (the standard deviation) is evaluated as

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = Ng + N^2 g^2 - N^2 g^2 = Ng$$

The ratio of Δn to $\langle n \rangle$ is evaluated as

$$\frac{\Delta n}{\langle n \rangle} = \frac{\sqrt{Ng}}{Ng} = \frac{1}{\sqrt{Ng}}$$

So the ratio becomes zero in the limit of large *N*.

We now calculate $\ln P(n)$ (proportional to the entropy) with the use of Stirling's law. We find the condition that the entropy *S* takes a maximum. To this end, we take a derivative of $f(n)$ with respect to *n*,

$$\begin{aligned} f'(n) &= -n \ln n - (N-n)\ln(N-n) + n \ln g + (N-n)\ln(1-g) \\ &= \ln(N-n) - \ln n + \ln g - \ln(1-g) \end{aligned}.$$

which becomes zero when

$$\frac{N - n^*}{n^*} = \frac{1-g}{g}$$

or

$$n = n^* = gN$$

or

$$g = \frac{n^*}{N} = \frac{\Delta t}{t_0}.$$

This means that the ratio ($\frac{n^*}{N}$) for the chimneys (the ensemble) is equal to the ratio $\left(\frac{\Delta t}{t_0}\right)$ for the time ratio for the heating system. So this example supports the validity of the ergodic hypothesis in the statistical mechanics.

REFERENCE

Y. Takahashi, Introduction to Statistical Mechanics Approach from (in Japanese) (Kodansha Scientific, 1984). The Author is a professor of Physics, University of Alberta. I think that this book is read by many Japanese students who begin to study the principle of statistical mechanics.