Permutation and combination

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1. Permutation

Suppose that we are given n distinct objects and wish to *arrange* r of these objects in a line. Since there are n ways of choosing the 1st object, and after this is done, $n-1$ ways of choosing the 2nd object, …, and finally $n-r+1$ ways of choosing the r-th object, it follows by the fundamental principle of counting that the number of different *arrangements*, or *permutations* as they are often called, is given by

$$ {}_nP_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!} $$

where it is noted that the product has r factors. We call ${}_nP_r$ the *number of permutations of n objects taken r at a time*.

((**Example**))

The number of different arrangements, or permutations, consisting of 2 letters each that can be formed from the 4 letters *a, b, c, d,* is

$$ {}_4P_2 = \frac{4!}{2!} = 12 $$

Suppose that a set consists of n objects of which n_1 are of one type (i.e., indistinguishable from each other), n_2 are of a second type, . . . , n_k are of a k-th type. Here, of course,

$$ n = n_1 + n_2 + \cdots + n_k $$

Then the number of different permutations of the objects is

$$ \frac{n!}{n_1!n_2!\cdots n_k!} $$

2. Combinations

In a permutation we are interested in the order of arrangement of the objects. For example, *abc* is a different permutation from *bca*. In many problems, however, we are interested only in selecting or choosing objects without regard to order. Such selections are called *combinations*.

For example, *abc* and *bca* are the same combination. The total number of combinations of r objects selected from n (also called the *combinations of n things taken r at a time*) is denoted

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!} $$

where

$$ {}_nC_r = {}_nC_{n-r} $$

3. Bionomial coefficient

$$ {}_nC_r = \frac{n!}{(n-r)!r!} $$

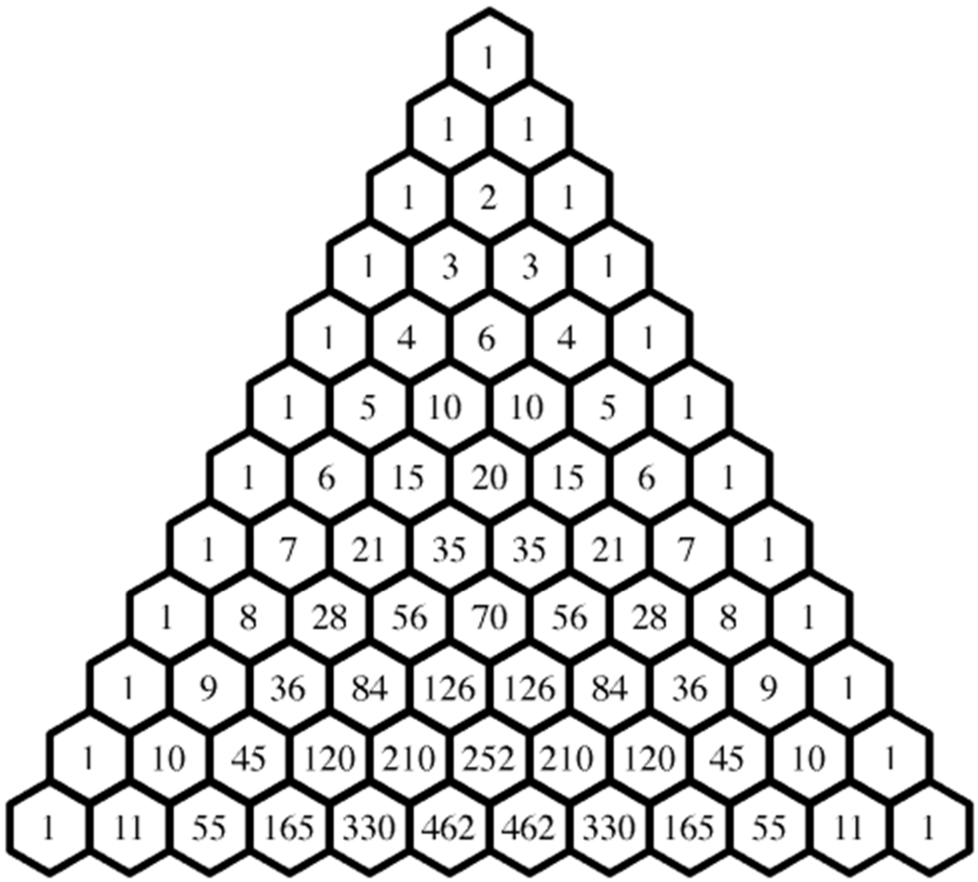
are often called binomial coefficients because they arise in the binomial expansion

$$ (x+y)^n = {}_nC_0\, x^n + {}_nC_1\, x^{n-1}y + {}_nC_2\, x^{n-2}y^2 + \cdots + {}_nC_n\, y^n $$

((Example))

$$ \begin{aligned} (1+x)^3 &= 1+3x+3x^2+x^3 \\ &= \sum_{r=0}^{3} {}_3C_r x^r \\ &= {}_3C_0x^0 + {}_3C_1x^1 + {}_3C_2x^2 + {}_3C_3x^3 \end{aligned} $$

Pascal's triangle is a triangular array of the binomial coefficients.



4. The Number of ways W

In studying the probability, one is frequently concerned with the number of ways in which a specified set of events can happen in a group of events, we show what we mean by a set of events by illustrating with two or more throws of a penny. If we were to throw the same penny N (=2, 3, 4,…) times , we could get the following results.

(a) $N = 2$ (two throws)
There are $2^2 = 4$.

First throw	Second throw
1	1
1	2
2	1
2	2

where we use 1 as head and 2 as tail.

(b) N=3 (three throws)
There are $2^3 = 8$ ways.

First throw	Second throw	Third throw
1	1	1
1	1	2
1	2	1
2	1	1
1	2	2
2	1	2
2	2	1
2	2	2

(c) $N = 4$ (four throws)

There are $2^4 = 16$ ways.

First throw	Second throw	Third throw	Fourth throw
1	1	1	1
1	1	2	1
1	2	1	1
2	1	1	1
1	2	2	1
2	1	2	1
2	2	1	1
2	2	2	1
1	1	1	2
1	1	2	2
1	2	1	2
2	1	1	2
1	2	2	2
2	1	2	2
2	2	1	2
2	2	2	2

Restrictions on the ways

$$W(3,1) = {}_4C_3 = \frac{{}_4P_3}{3!} = \frac{4!}{3!1!} = 4\,, \quad W(1,3) = \frac{4!}{3!1!} = 4$$

$$W(2,2) = \frac{4!}{2!2!} = 6\,,$$

Permutaion

(1) Distinguishable case: $3! = 6$

abc, bac, cab, acb, bca, cba

(2) If a = b (indistinguishable)

aac, aac, caa, aca, aca, caa

$$\frac{3!}{2!} = 3$$

(3)

Distinguishable case: $4! = 4 \times 3 \times 2 \times 1 = 24$

Table The permutation of four letters

abcd,	bacd,	cabd,	dabc
abdc	badc	cadb	dacb
acbd	bcad	cbad	dbac
acdb	bcda	cbda	dbca
adbc	bdac	cdab	dcab
adcb	bdca	cdba	dcba

Table: The permutation of four letters when two are alike.

aacd	adca
aadc	daca
adac	caad
daac	cada
acad	cdaa
acda	dcaa

There are only 2 of these arrangements. The reason for the number 12 instead of 24 is that by assigning $b = a$, we lost the permutations of two letters (a and b).

$$W_2 = 2! = 2 .$$

The distinguishable ways of permuting for four letters when two are alike is then

$$W = \frac{W_4}{W_2} = \frac{4!}{2!} = \frac{24}{2} = 12$$

REFERENCES

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