#### Photon gas and ideal gas based on canonical ensemble (classical statistics) Masatsugu Sei Suzuki Department of Physics, SUNY at Binghamton (Date: September 21, 2016)

We consider an ideal gas containing of N particles obeying classical statistics. Suppose that the energy of one particle  $\varepsilon$  is proportional to the magnitude of momentum p. The dispersion relation is expressed by  $\varepsilon = cp$ . We find the thermodynamic functions of this ideal gas without considering the internal structure of the particles, based on the approach of the canonical ensemble. This approach will be also used for an ideal gas.

#### 1. Photon gas

We show the relation given by

$$PV = \frac{1}{3}E,$$

for the photon gas with the dispersion of  $\varepsilon = cp$ , which is different from the form for the ideal gas with the dispersion  $\varepsilon = \frac{1}{2m}p^2$ 

(a) The one particle partition function:

$$Z_{C1} = \frac{V}{h^3} \int d^3 p \exp(-\beta cp)$$
  
=  $\frac{V}{h^3} \int_0^\infty 4\pi p^2 \exp(-\beta cp) dp$   
=  $\frac{4\pi V}{h^3} \int_0^\infty p^2 \exp(-\beta cp) dp$   
=  $\frac{4\pi V}{h^3} \frac{2!}{(\beta c)^3}$   
=  $\frac{8\pi V}{h^3 c^3} \frac{1}{\beta^3}$ 

where

$$\int_{0}^{\infty} p^{2} \exp(-\alpha p) dp = \frac{2!}{\alpha^{3}}$$

(Laplace transformation)

(b) The *N*-particle partition function:

$$Z_{CN} = \frac{1}{N!} (Z_{C1})^{N} = \frac{1}{N!} (\frac{8\pi V}{h^{3}c^{3}} \frac{1}{\beta^{3}})^{N} = \frac{1}{N!} (\frac{8\pi V}{h^{3}c^{3}})^{N} \beta^{-3N}$$
$$\ln Z_{CN} = -\ln N! + N \ln(\frac{8\pi}{h^{3}c^{3}}) + N \ln V - 3N \ln \beta$$

Using the Stirling's law in the limit of large N,

$$\ln Z_{CN} = N[\ln \frac{V}{N} - 3\ln\beta + \ln(\frac{8\pi}{h^3c^3}) + 1]$$

The internal energy

$$U = -\frac{\partial}{\partial\beta} \ln Z_{CN} = \frac{3N}{\beta} = 3Nk_B T \tag{1}$$

The Helmholtz free energy:

$$F = -k_B T \ln Z_{CN} = -Nk_B T [\ln \frac{V}{N} - 3\ln\beta + \ln(\frac{8\pi}{h^3 c^3}) + 1]$$

The entropy S is obtained as

$$S = \frac{1}{T}(E - F) = Nk_{B} \left[ \ln \frac{V}{N} - 3\ln \beta + \ln(\frac{8\pi}{h^{3}c^{3}}) + 4 \right]$$

S can be also derived from the relation as

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V} = Nk_{B}\left[\ln\frac{V}{N} - 3\ln\beta + \ln(\frac{8\pi}{h^{3}c^{3}}) + 4\right]$$

When S = const (isentropic), we have

 $VT^3 = \text{const.}$ 

The pressure *P*:

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T} = \frac{Nk_{B}T}{V} \qquad \text{or} \qquad PV = Nk_{B}T \qquad (2)$$

From Eqs.(1) and (2) we get the relation

$$PV = \frac{1}{3}U$$

or

$$P = \frac{1}{3}\frac{U}{V} = \frac{1}{3}u$$

where u is the energy density of photon gas.

#### ((Summary))

 $VT^3 = \text{const}, \qquad PV^{4/3} = \text{const}$ 

leading to

 $\gamma = \frac{4}{3}$ .

### 2. Ideal gas (using the canonical ensemble)

We apply the above method to the ideal gas. We show that



### (a) The one particle partition function:

$$Z_{C1} = \frac{V}{h^3} \int d^3 \boldsymbol{p} \exp(-\beta \frac{p^2}{2m})$$
$$= \frac{V}{h^3} \int_0^\infty 4\pi p^2 \exp(-\beta \frac{p^2}{2m}) dp$$
$$= \frac{4\pi V}{h^3} \int_0^\infty p^2 \exp(-\beta \frac{p^2}{2m}) dp$$
$$= \frac{4\pi V}{h^3} \frac{\sqrt{\pi}}{4} \left(\frac{2m}{\beta}\right)^{3/2}$$
$$= V \left(\frac{2\pi m}{h^2 \beta}\right)^{3/2}$$

or

$$Z_{C1} = V \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2}$$

where we use the integral formula

$$\int_{0}^{\infty} p^{2} \exp(-\alpha p^{2}) dp = \frac{\sqrt{\pi}}{4} \alpha^{-3/2}$$

# (b) The *N*-particle partition function:

$$Z_{CN} = \frac{1}{N!} (Z_{C1})^{N} = \frac{1}{N!} \left[ V \left( \frac{2\pi m}{h^{2} \beta} \right)^{3/2} \right]^{N} = \frac{1}{N!} V^{N} \left( \frac{2\pi m}{h^{2}} \right)^{\frac{3N}{2}} \beta^{-\frac{3N}{2}}$$
$$\ln Z_{CN} = -\ln N! + \frac{3}{2} N \ln \left( \frac{2\pi m}{h^{2}} \right) + N \ln V - \frac{3}{2} N \ln \beta$$

Using the Stirling's law in the limit of large N,

$$\ln Z_{CN} = N[\ln \frac{V}{N} - \frac{3}{2} \ln \beta + \frac{3}{2} \ln(\frac{2\pi m}{h^2}) + 1]$$

The internal energy

$$U = -\frac{\partial}{\partial\beta} \ln Z_{CN} = \frac{3N}{2\beta} = \frac{3}{2} N k_B T$$
(3)

The Helmholtz free energy:

$$F = -k_B T \ln Z_{CN} = -Nk_B T \left[ \ln \frac{V}{N} - \frac{3}{2} \ln \beta + \frac{3}{2} \ln \left( \frac{2\pi m}{h^2} \right) + 1 \right]$$

The entropy S is obtained as

$$S = \frac{1}{T}(E - F) = Nk_{B}\left[\ln\frac{V}{N} - \frac{3}{2}\ln\beta + \frac{3}{2}\ln(\frac{2\pi m}{h^{2}}) + \frac{5}{2}\right]$$

S can be also derived from the relation as

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V} = Nk_{B}\left[\ln\frac{V}{N} - \frac{3}{2}\ln\beta + \frac{3}{2}\ln(\frac{2\pi m}{h^{2}}) + \frac{5}{2}\right]$$

The pressure *P*:

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{Nk_BT}{V} \qquad \text{or} \qquad PV = Nk_BT \tag{4}$$

From Eqs.(1) and (2) we get the relation

$$PV = \frac{2}{3}U$$

## 3. Ideal gas (quantum mechanics) We start with an expression given by

$$Z_{1c} = \sum_{k} e^{-\beta \varepsilon_{k}}$$
$$= \frac{V}{(2\pi)^{3}} \int d^{3}k e^{-\beta \varepsilon_{k}}$$
$$= \int D(\varepsilon) e^{-\beta \varepsilon} d\varepsilon$$

where 
$$k = \left(\frac{2m}{\hbar^2}\right)^{1/2} \sqrt{\varepsilon}$$
  
 $D(\varepsilon)d\varepsilon = \frac{V}{(2\pi)^3} 4\pi k^2 dk$   
 $= \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon} d\varepsilon$ 

with the density of states

$$D(\varepsilon) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\varepsilon}$$

Here we do not take into account of the spin factor. Thus we have

$$Z_{1c} = \int D(\varepsilon) e^{-\beta\varepsilon} d\varepsilon$$
$$= \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \sqrt{\varepsilon} e^{-\beta\varepsilon} d\varepsilon$$
$$= \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{\sqrt{\pi}}{2} \beta^{-3/2}$$
$$= V \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2}$$

where

$$\int_{0}^{\infty} \sqrt{\varepsilon} e^{-\beta \varepsilon} d\varepsilon = \frac{\sqrt{\pi}}{2} \beta^{-3/2}$$