Relativistic gases
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We discuss the thermodynamic properties of relativistic gases (classical), based on the partition function of the canonical ensemble. The heat capacity of the non-relativistic gas is $3R/2$, increases with increasing $1/u = 1/(\beta mc^2) = k_B T / mc^2$, it reaches $3R$ in the relativistic limit. This means that the degree of the freedom of the system gradually changes from $f = 3$ to $f = 6$.

1. The partition function

We start with the energy of the particle is given by

$$E_{\boldsymbol{k}} = c\sqrt{p^2 + m^2c^2} - mc^2$$

where the rest mass energy mc^2 of the particles is subtracted, so that there remains only the kinetic energy. First we calculate the part of the one-particle partition function,

$$I_1 = \int_0^\infty p^2 dp \exp[-\beta(c\sqrt{p^2 + m^2c^2}) + \beta mc^2].$$

We put

$$\sinh x = \xi = \frac{p}{mc},$$

or

$$\sqrt{p^2 + m^2c^2} = mc\sqrt{1+\xi^2} = mc\cosh(x).$$

Since

$$p^2 dp = m^3c^3 \sinh^2(x)\cosh(x)dx$$

we get

$$I_1 = m^3c^3e^{\beta mc^2}\int_0^\infty dx \exp[-\beta mc^2\cosh(x)]\sinh^2(x)\cosh(x)$$

$$= m^3c^3e^{\beta mc^2}\int_0^\infty dx \exp[-\beta mc^2\cosh(x)][\cosh^3(x)-\cosh(x)]$$

$$= m^3c^3e^{\beta mc^2}[\int_0^\infty dx \exp[-\beta mc^2\cosh(x)]\cosh^3(x)-\int_0^\infty dx \exp[-\beta mc^2\cosh(x)]\cosh(x)$$

noting that

$$\sinh^2(x)=\cosh^2(x)-1\,.$$

We introduce the modified Bessel function of the second which is defined by

$$K_\nu(u)=\int_0^\infty dx \exp[-u\cosh(x)]\cosh(\nu x)$$

and

$$K_1(u)=\int_0^\infty dx \exp[-u\cosh(x)]\cosh(x)$$

where $u=\beta mc^2$. Note that

$$\frac{d}{du}K_1(u)=-\int_0^\infty dx \exp[-u\cosh(x)]\cosh^2(x)$$

and

$$\frac{d^2}{du^2}K_1(u)=\int_0^\infty dx \exp[-u\cosh(x)]\cosh^3(x)$$

Thus we have

$$I_1 = m^3c^3e^{\beta mc^2}[\frac{d^2}{du^2}K_1(u)-K_1(u)]$$

Using the relation

$$\frac{d^2}{du^2}K_1(u) - K_1(u) = \frac{1}{u}K_2(u)$$

we have

$$I_1 = \frac{1}{u}m^3c^3e^{\beta mc^2}K_2(u)$$

Then the on-particle partition function is obtained as

$$\begin{aligned} Z_{C1} &= \frac{V}{(2\pi\hbar)^3}4\pi\int_0^\infty p^2 dp \exp[-\beta c\sqrt{p^2+m^2c^2} + \beta mc^2] \\ &= \frac{Vm^3c^3}{2\pi^2\hbar^3}e^u \frac{K_2(u)}{u} \end{aligned}$$

with

$$u = \beta mc^2.$$

(i) The nonrelativistic limit

This immediately leads to the nonrelativistic limit with $u = \beta mc^2 \to \infty$ (the mean thermal energy k_BT is very small compared to the rest mass energy mc^2 of the particle).

$$K_2(u) \approx \sqrt{\frac{\pi}{2u}}e^{-u}$$

$$\begin{aligned} Z_{C1} &= \frac{Vm^3c^3}{2\pi^2\hbar^3}\frac{1}{u}e^u\sqrt{\frac{\pi}{2u}}e^{-u} \\ &= \frac{Vm^3c^3}{2\pi^2\hbar^3}\frac{1}{u}\sqrt{\frac{\pi}{2u}} \\ &= V(\frac{2\pi mk_BT}{h^2})^{3/2} \end{aligned}$$

(ii) The high temperature limit; $u = \beta mc^2 \to 0$

$$K_2(u) \approx \frac{2}{u^2}$$

Thus we have

$$Z_{C1} = \frac{Vm^3c^3}{2\pi^2\hbar^3}\frac{2}{u^3} = 8\pi V\left(\frac{k_BT}{hc}\right)^3$$

The N-particle partition function is

$$Z_{CN} = \frac{{Z_{C1}}^N}{N!}$$

taking into account of the identical particles.

$$\begin{aligned}\ln Z_{CN} &= N\ln Z_{C1} - \ln N! \\ &= N[\ln(\frac{m^3c^3}{2\pi^2\hbar^3}) + \ln V + \ln K_2(u) + u - \ln u] - N\ln N + N \\ &= N[\ln(\frac{m^3c^3}{2\pi^2\hbar^3}) + \ln\frac{V}{N} + \ln K_2(u) + u + 1 - \ln u]\end{aligned}$$

where

$$Z_{CN} = \frac{Vm^3c^3}{2\pi^2\hbar^3}e^u\frac{K_2(u)}{u}$$

2. Helmholtz free energy and Internal energy

The Helmholtz free energy is

$$F = -k_BT\ln Z_{CN} = -Nk_BT[\ln(\frac{m^3c^3}{2\pi^2\hbar^3}) + \ln\frac{V}{N} + \ln K_2(u) + u + 1 - \ln u]$$

The internal energy is given by

$$
\begin{aligned}
U &= -\frac{\partial}{\partial \beta}\ln Z_{CN} \\
&= -mc^2 \frac{\partial}{\partial u}\ln Z_{CN} \\
&= -Nmc^2 \frac{d}{du}[\ln K_2(u) + u - \ln u] \\
&= Nmc^2[\frac{1}{u} - 1 + \frac{K_1(u) + K_3(u)}{2K_2(u)}]
\end{aligned}
$$

where

$$
\frac{d}{du}K_2(u) = -\frac{1}{2}[K_1(u) + K_3(u)]
$$

The heat capacity:

$$
\begin{aligned}
C &= \frac{\partial U}{\partial T} \\
&= -\frac{mc^2}{k_B T^2}\frac{d}{du}U(u) \\
&= -\frac{k_B}{mc^2}u^2 \frac{d}{du}U(u) \\
&= -Nk_B u^2 \frac{d}{du}[\frac{1}{u} - 1 + \frac{K_1(u) + K_3(u)}{2K_2(u)}] \\
&= Nk_B u^2 \{\frac{1}{u^2} + \frac{1}{2} - \frac{[K_1(u) + K_3(u)]^2 - K_2(u)[K_0(u) + K_4(u)]}{4[K_2(u)]^2}\}
\end{aligned}
$$

where we use the Mathematica for this calculation; $K_\nu(x) \to$ BesselK[ν, x].

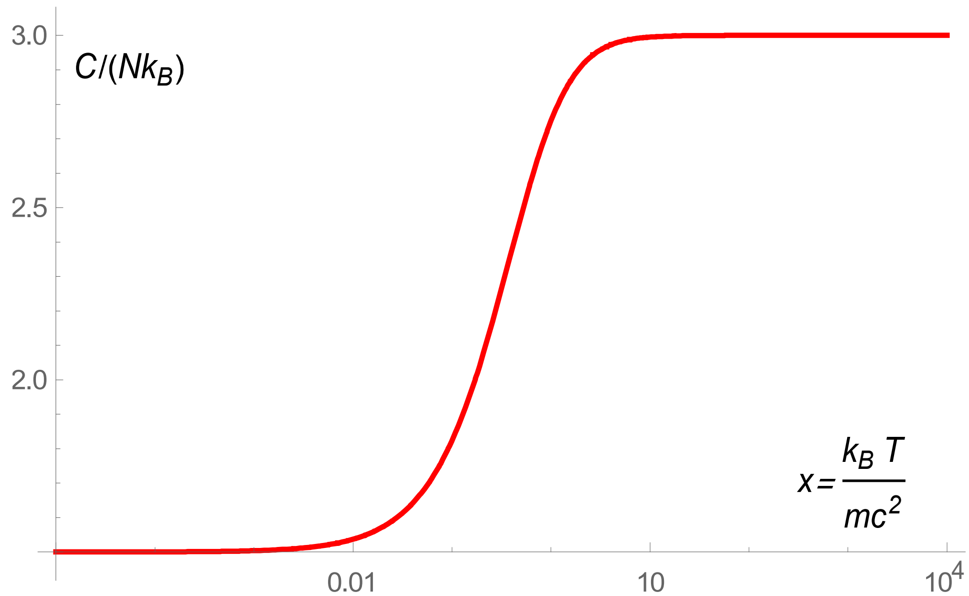

Fig. Heat capacity of the relativistic gas.

3. Pressure

The pressure P is

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{Nk_B T}{V}.$$

4. Entropy

The entropy is calculated as

$$\begin{aligned} S &= \frac{U}{T} - \frac{F}{T} \\ &= Nk_B u[\frac{1}{u} - 1 + \frac{K_1(u) + K_3(u)}{2K_2(u)}] + Nk_B[\ln(\frac{m^3 c^3}{2\pi^2\hbar^3}) + \ln\frac{V}{N} + \ln K_2(u) + u + 1 - \ln u] \end{aligned}$$

where $$\frac{mc^2}{T} = k_B u$$

S can be rewritten as

$$\frac{S}{Nk_B} = 1 - u + u\left[\frac{K_1(u) + K_3(u)}{2K_2(u)}\right] + \ln(\frac{m^3c^3}{2\pi^2\hbar^3}) + \ln\frac{V}{N} + \ln K_2(u) + u + 1 - \ln u$$
$$= -u + u\left[\frac{K_1(u) + K_3(u)}{2K_2(u)}\right] + \ln\frac{K_2(u)}{u} + 2 + \ln\frac{V}{N} + \ln(\frac{m^3c^3}{2\pi^2\hbar^3})$$

(i) $u >> 1$ (non-relativistic case)

$$K_\nu(u) = \sqrt{\frac{\pi}{2u}}e^{-u}[1 + \frac{(4\nu^2 - 1^2)}{1!8u} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2!(8u)^2} + ...]$$

$$\frac{S}{Nk_B} = -u + u[\frac{1 + \frac{3}{8u} + 1 + \frac{35}{8u}}{2(1 + \frac{15}{8u})}] + \ln(\frac{1}{u}\sqrt{\frac{\pi}{2u}}e^{-u}) + 2 + \ln\frac{V}{N} + \ln(\frac{m^3c^3}{2\pi^2\hbar^3})$$
$$= -u + u(1 + \frac{1}{2u}) + \ln(\frac{1}{u}\sqrt{\frac{\pi}{2u}}e^{-u}) + 2 + \ln\frac{V}{N} + \ln(\frac{m^3c^3}{2\pi^2\hbar^3})$$
$$= \frac{1}{2}\ln(\frac{\pi}{2}) - \frac{3}{2}\ln(\frac{mc^2}{k_BT}) + \frac{5}{2} + \ln\frac{V}{N} + \ln(\frac{m^3c^3}{2\pi^2\hbar^3})$$
$$= \frac{3}{2}\ln T + \ln\frac{V}{N} + \frac{5}{2} + \frac{1}{2}\ln(\frac{\pi}{2}) - \frac{3}{2}\ln(\frac{mc^2}{k_B}) + \ln(\frac{m^3c^3}{2\pi^2\hbar^3})$$
$$= \frac{3}{2}\ln T + \ln\frac{V}{N} + \frac{5}{2} + \frac{3}{2}\ln(\frac{2\pi mk_B}{h^2})$$

which is the same as the Sackur-Tetrode equation.

The heat capacity is

$$C = T\left(\frac{\partial S}{\partial T}\right) = \frac{3}{2}Nk_B$$

(ii) $u << 1$ (relativistic case)

$$K_1(u) = \frac{1}{u}, \qquad K_2(u) \approx \frac{2}{u^2}, \qquad K_3(u) \approx \frac{8}{u^3}$$

where

$$K_\nu(u) = \frac{(\nu-1)!2^{\nu-1}}{u^\nu}$$

$$\begin{aligned}
\frac{S}{Nk_B} &= u\left[\frac{K_1(u)+K_3(u)}{2K_2(u)}\right] + \ln\frac{K_2(u)}{u} + 2 + \ln\frac{V}{N} + \ln(\frac{m^3c^3}{2\pi^2\hbar^3}) \\
&= \frac{1}{4}(u^2+8) - \ln\frac{u^3}{2} + 2 + \ln\frac{V}{N} + \ln(\frac{m^3c^3}{2\pi^2\hbar^3}) \\
&\approx 4 + 3\ln T + \ln\frac{V}{N} + \ln(\frac{k_B{}^3}{\pi^2\hbar^3c^3})
\end{aligned}$$

leading to

$$C = T\left(\frac{\partial S}{\partial T}\right) = 3Nk_B$$

5. Adiabatic expansion

(i) U-relativistic gas

For an adiabatic process (S = constant),

$$S = Nk_B[\ln\frac{V}{N} + 3\ln T + 4 + \ln(\frac{k_B{}^3}{\pi^2c^3\hbar^3})] = \text{constant},$$

leading to

$$VT^3 = \text{constant}.$$

Since $PV = Nk_BT$, we have

$$V(PV)^3 = P^3V^4 = (PV^{4/3})^3 = \text{constant},$$

or

$$PV^\gamma = \text{const} \qquad \text{with} \qquad \gamma = \frac{4}{3}.$$

(ii) Non-relativistic case

For an adiabatic process (S = constant),

$$\frac{S}{Nk_B} = \frac{3}{2}\ln T + \ln\frac{V}{N} + \frac{5}{2} + \frac{3}{2}\ln(\frac{2\pi m k_B}{h^2}) = \text{constant}$$

leading to

$VT^{3/2}$ =constant.

Since $PV = Nk_B T$, we have

$V(PV)^{3/2} = P^{3/2}V^{5/2} = (PV^{5/3})^{3/2}$ =constant,

or

PV^{γ} =const with $\gamma = \frac{5}{3}$.

6 Relativistic case (simpler method)

Here we discuss the relativistic case in a simpler way. The energy dispersion of the relativistic gases is given by

$$\varepsilon = c\sqrt{p^2 + m^2c^2} - mc^2$$

For $p >> mc$

$$\varepsilon = c\hbar k = cp$$

which is the same as that of photon. The One-particle partition function of the canonical ensemble is given by

$$Z_{C1} = \sum_{\boldsymbol{k}} \exp(-\beta c\hbar k)$$

where $\boldsymbol{k}$ is the wave vector and $\boldsymbol{p} = \hbar\boldsymbol{k}$. Noting that

$$\sum_{\boldsymbol{k}} \rightarrow \frac{V}{(2\pi)^3} 4\pi k^2 dk = \frac{V}{2\pi^2} k^2 dk$$

we have

$$Z_{C1} = \frac{V}{2\pi^2}\int_0^\infty dk k^2 \exp(-\beta c\hbar k).$$

We introduce new parameter, $x = \beta c\hbar k$. Thus we have

$$\begin{aligned}
Z_{C1} &= \frac{V}{2\pi^2}\frac{1}{(\beta c\hbar)^3}\int_0^\infty dx x^2 \exp(-x) \\
&= \frac{V}{\pi^2}\frac{1}{(\beta c\hbar)^3} \\
&= \frac{V}{\pi^2}\frac{{k_B}^3 T^3}{c^3\hbar^3} \\
&= \frac{V}{\Lambda^3}
\end{aligned}$$

with the thermal wavelength as

$$\Lambda = \frac{\hbar c \pi^{2/3}}{k_B T},$$

since

$$\int_0^\infty dx x^2 \exp(-x) = 2.$$

For the N particle system (identical), the partition function is given by

$$Z_{CN} = \frac{1}{N!}(Z_{C1})^N.$$

The Helmholtz free energy is obtained as

$$\begin{aligned}
F &= -k_B T \ln Z_{CN} \\
&= -Nk_B T[\ln V + 3\ln T - \ln N + 1 + \ln(\frac{{k_B}^3}{\pi^2 c^3\hbar^3})]
\end{aligned}$$

where

$$\begin{aligned}\ln Z_{CN} &= N\ln Z_{C1} - \ln N! \\ &= N\ln\left(\frac{V}{\pi^2}\frac{k_B{}^3}{c^3\hbar^3}T^3\right) - N\ln N + N \\ &= N\left[\ln\frac{V}{N} + 3\ln T + 1 + \ln\left(\frac{k_B{}^3}{\pi^2 c^3\hbar^3}\right)\right]\end{aligned}$$

The internal energy:

$$\begin{aligned}U &= -\frac{\partial}{\partial\beta}\ln Z_{CN} \\ &= k_B T^2\frac{\partial}{\partial T}\ln Z_{CN} \\ &= 3Nk_B T\end{aligned}$$

The heat capacity C:

$$C = \frac{\partial U}{\partial T} = 3Nk_B\,.$$

The entropy S;

$$\begin{aligned}S &= \frac{U}{T} - \frac{F}{T} \\ &= Nk_B\left[\ln\frac{V}{N} + 3\ln T + 4 + \ln\left(\frac{k_B{}^3}{\pi^2 c^3\hbar^3}\right)\right]\end{aligned}$$

The pressure P;

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{Nk_B T}{V}$$

The enthalpy H:

$$H = U + PV = 4Nk_B T$$

The Gibbs free energy G:

$$G = F + PV$$
$$= -Nk_B T[\ln V + 3\ln T - \ln N + 1 + \ln(\frac{k_B{}^3}{\pi^2 c^3 \hbar^3})] + Nk_B T$$
$$= -Nk_B T[\ln V + 3\ln T - \ln N + \ln(\frac{k_B{}^3}{\pi^2 c^3 \hbar^3})]$$

We note that

$$PV = Nk_B T\,, \qquad U = 3Nk_B T$$

Then we have

$$PV = \frac{U}{3}\,, \qquad P = \frac{1}{3}\frac{U}{V} = \frac{1}{3}u$$

7. Summary

Non-relativistic limit	Relativistic ;limit
$u = \beta mc^2 >> 1$	$u = \beta mc^2 << 1$
$C = 3R/2$	$C = 3R$
$P = \frac{2}{3}u$	$P = \frac{1}{3}u$
$PV = Nk_B T$	$PV = Nk_B T$
$\gamma = \frac{5}{3}$	$\gamma = \frac{4}{3}$

8. Maxwell-Boltzmann distribution for a relativistic gas

Problem and solution (Huang, Introduction to Statistical Mechanics)
Problem 6-7

6.7 The Maxwell-Boltzmann distribution for a relativistic gas is

$$f(\boldsymbol{p}) = C\exp(-\beta c\sqrt{\boldsymbol{p}^2 + m^2 c^2})$$

where we use the units in which the velocity of light is $c = 1$.

(a) Find the most probable velocity. Obtain its nonrelativistic $(k_B T << m)$ limits and ultrarelativistic $(k_B T >> m)$ limits, both with first-order corrections.

(b) Set up an expression for the pressure. Show that $PV = \dfrac{U}{3}$ in the ultra-relativistic limit, where U is the average energy.

(c) Find the velocity distribution function, $f(\boldsymbol{v})$, such that $f(\boldsymbol{v})d\boldsymbol{v}$ is the density of particles whose velocity lies in the volume element $d\boldsymbol{v}$. Find the nonrelativistic limit to the first order in v/c.

(d) At what temperatures would the relativistic effect be important for a gas of H_2 molecules?

((**Solution**))

$N(\boldsymbol{p})d^3\boldsymbol{p}$ is the number of particles having the momentum with

$$p_x - p_x + dp_x,$$
$$p_y - p_y + dp_y,$$
$$p_z - p_z + dp_z$$

$f(\boldsymbol{p})d^3\boldsymbol{p}$ is the number density (the number per unit volume) of particles having the momentum with

$$p_x - p_x + dp_x,$$
$$p_y - p_y + dp_y,$$
$$p_z - p_z + dp_z$$

Note that

$$N(\boldsymbol{p}) = Vf(\boldsymbol{p}), \qquad f(\boldsymbol{p}) = C_0 \exp(-\beta\varepsilon(\boldsymbol{p}))$$

(a)

Here we use $N(\boldsymbol{p})$ instead of $f(\boldsymbol{p})$ such that

$$\frac{1}{V}\int d^3\boldsymbol{p}\, N(\boldsymbol{p}) = \frac{N}{V}, \qquad \int d^3\boldsymbol{p}\, f(\boldsymbol{p}) = n = \frac{N}{V}$$

or

$$\frac{1}{V}\int d^3\boldsymbol{p}\,N(\boldsymbol{p}) = \int d^3\boldsymbol{p}\,f(\boldsymbol{p})$$

or

$$\frac{1}{V}N(\boldsymbol{p}) = f(\boldsymbol{p}) = C_0\exp(-\beta\,\varepsilon_p)$$

with

$$\varepsilon_{\boldsymbol{p}} = \sqrt{m^2c^4 + c^2p^2}$$

(b)

The pressure P is given by

$$F_x = \int_{v_x>0} \frac{2p_x}{\Delta t}(Av_x\Delta t)\frac{1}{V}N(\boldsymbol{p})d^3\boldsymbol{p} = \int_{v_x>0} 2p_x(Av_x)\frac{1}{V}N(\boldsymbol{p})d^3\boldsymbol{p}$$

or

$$P = \frac{F_x}{A} = \frac{1}{V}\int_{v_x>0} 2p_xv_xN(\boldsymbol{p})d^3\boldsymbol{p}$$

Thus we have

$$\begin{aligned}P &= \frac{1}{V}\int_{v_x>0} 2p_xv_xN(\boldsymbol{p})d^3\boldsymbol{p} \\ &= \int p_xv_xf(\boldsymbol{p})d^3\boldsymbol{p} \\ &= \int \frac{p_x^{\ 2}}{\sqrt{m^2 + \frac{p^2}{c^2}}} f(\boldsymbol{p})d^3\boldsymbol{p}\end{aligned}$$

or

$$P = \frac{1}{3}\int \frac{p^2}{\sqrt{m^2 + \frac{p^2}{c^2}}} f(\boldsymbol{p}) d^3\boldsymbol{p}$$

Note that

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}}$$

or

$$\boldsymbol{p}^2 = \frac{m^2\boldsymbol{v}^2}{1 - \frac{\boldsymbol{v}^2}{c^2}}$$

leading to the expression of $\boldsymbol{v}$ as

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{\sqrt{m^2 + \frac{\boldsymbol{p}^2}{c^2}}} .$$

We now calculate the pressure P as follows.

$$P = \frac{1}{3}\int \frac{p^2}{\sqrt{m^2 + \frac{p^2}{c^2}}} f(\boldsymbol{p}) d^3\boldsymbol{p}$$

Note that

$$\begin{aligned}
\frac{p^2}{\sqrt{m^2 + \frac{p^2}{c^2}}} &= c\frac{p^2}{\sqrt{m^2c^2 + p^2}} \\
&= c[\frac{p^2 + m^2c^2 - m^2c^2}{\sqrt{m^2c^2 + p^2}}] \\
&= c[\sqrt{m^2c^2 + p^2} - \frac{m^2c^2}{\sqrt{m^2c^2 + p^2}}]
\end{aligned}$$

In the limit of $p^2 >> m^2c^2$, the second term can be neglected.

$$P = \frac{1}{3}\int \varepsilon_{\boldsymbol{p}} f(\boldsymbol{p}) d^3\boldsymbol{p} = \frac{1}{3} n\langle\varepsilon\rangle = \frac{N}{3V}\langle\varepsilon\rangle = \frac{1}{3}\frac{E}{V}$$

or

$$PV = \frac{1}{3}U$$

where

$$U = N < \varepsilon > = \frac{1}{n}\int \varepsilon_{\boldsymbol{p}} f(\boldsymbol{p}) d^3\boldsymbol{p}$$

(c)

$$1 = \frac{1}{N}\int N(\boldsymbol{p}) d^3\boldsymbol{p} = \frac{V}{N}\int f(\boldsymbol{p}) d^3\boldsymbol{p} = \frac{1}{n}\int f(\boldsymbol{p}) d^3\boldsymbol{p} = \frac{1}{n}\int f(\boldsymbol{v}) 4\pi v^2 dv$$

The probability distribution function

$$\begin{aligned} f(v)4\pi v^2 dv &= f(\boldsymbol{p})4\pi p^2 dp \\ &= 4\pi C_0 p^2 dp \exp(-\beta\sqrt{m^2c^4 + c^2p^2}) \end{aligned}$$

where

$$\varepsilon_{\boldsymbol{p}} = \sqrt{m^2c^4 + c^2p^2} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$p = \frac{mv}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}}$$

$$p^2 dp = \frac{m^2 v^2}{(1-\frac{v^2}{c^2})^{5/2}} m(1+\frac{v^2}{c^2})dv \approx \frac{m^3 v^2}{(1-\frac{v^2}{c^2})^{5/2}} dv$$

since

$$p = \frac{mv}{\sqrt{1-\frac{p^2}{c^2}}}$$

Then we have

$$f(v) 4\pi v^2 dv = 4\pi C_0 \frac{m^3 v^2}{(1-\frac{v^2}{c^2})^{5/2}} dv \exp(-\beta \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}})$$

leading to

$$f(v) = C_0 \frac{m^3}{(1-\frac{v^2}{c^2})^{5/2}} \exp(-\beta \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}})$$

We consider the probability $P(v)dv$, where

$$P(v) = 4\pi m^3 c^2 C_0 \frac{\frac{v^2}{c^2}}{(1-\frac{v^2}{c^2})^{5/2}} \exp(-\beta \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}})$$

and

$$F(v) = \frac{\frac{v^2}{c^2}}{(1-\frac{v^2}{c^2})^{5/2}} \exp(-\beta \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}).$$

For simplicity, we put $x = \frac{v}{c}$, $a = \beta mc^2$.

$$F(x) = \frac{x}{(1-x)^{5/2}} \exp(-\beta \frac{mc^2}{\sqrt{1-x^2}})$$

The most probable velocity can be obtained from the condition that

$$\frac{dF(x)}{dx} = 0$$

Using the Mathematica, we get the equation

$$2 - 3x^4 + x^2(1 - a\sqrt{1-x^2}) = 0$$

or

$$a = \frac{2 + x^2 - 3x^4}{x^2\sqrt{1-x^2}}$$

Suppose that we have

$$y_1 = a, \qquad y_2 = \frac{2 + x^2 - 3x^4}{x^2\sqrt{1-x^2}}$$

We make a plot of y_1 and y_2, as a function of x ($0<x<1$).

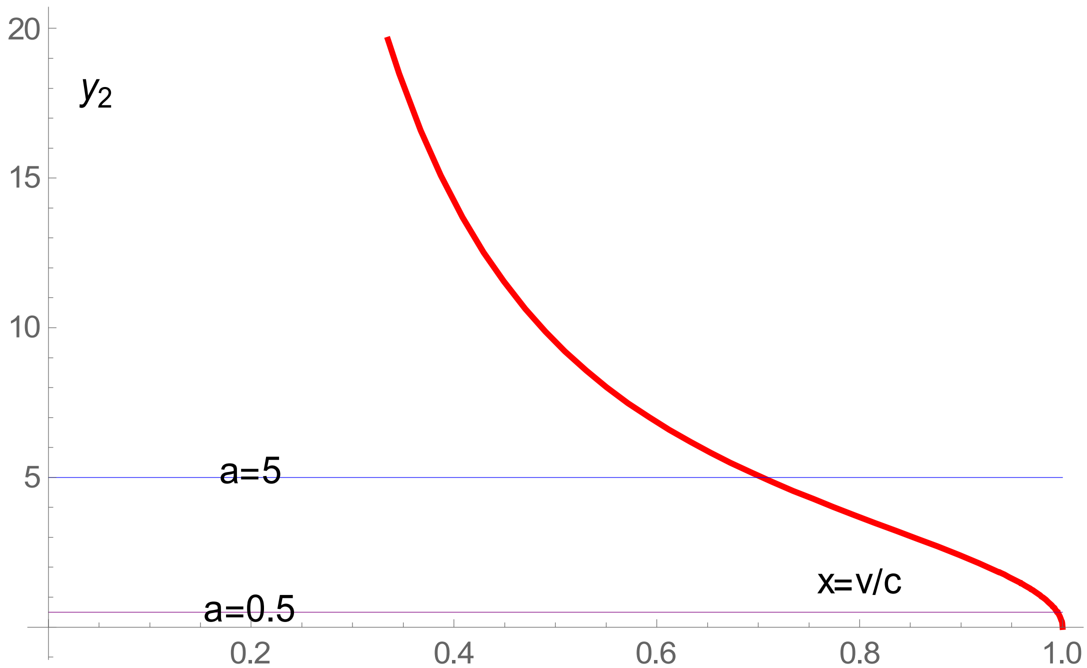

Fig.1 The plot of $y_1 = a$ and y_2 as a function of x. $a = 5$ (blue) and $a = 0.5$ (purple). y_2 vs x (red).

When $y_1 = a = 0.5$ the intersection of y_1 and y_2 occurs around $x = 1$. When $a = 5$, the intersection of y_1 and y_2 occurs at small x. So the value of x decreases rapidly as the value of a. decreases.

((**Example**))
We make a plot of $F(x)$ as a function of x where a is changed as a parameter. We choose $a = 0.5$ and $a = 5$.

(a) $a = 0.5$

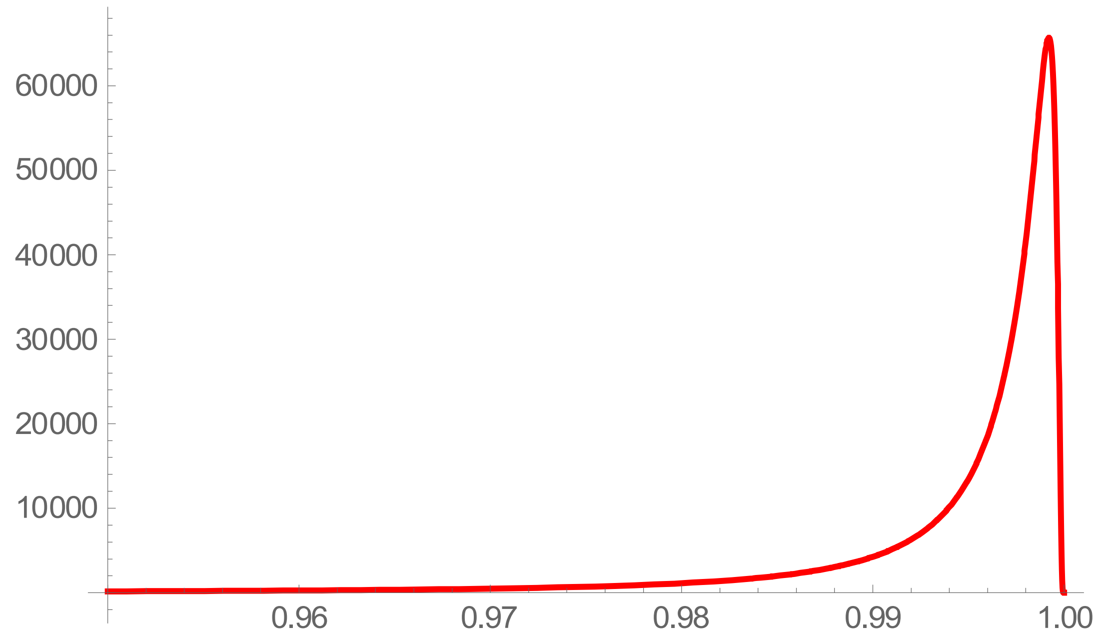

Fig.2 $F(x)$ vs x with $a = 0.5$. The peak appears around $x = v/c = 0.998$.

(b) $a = 5$

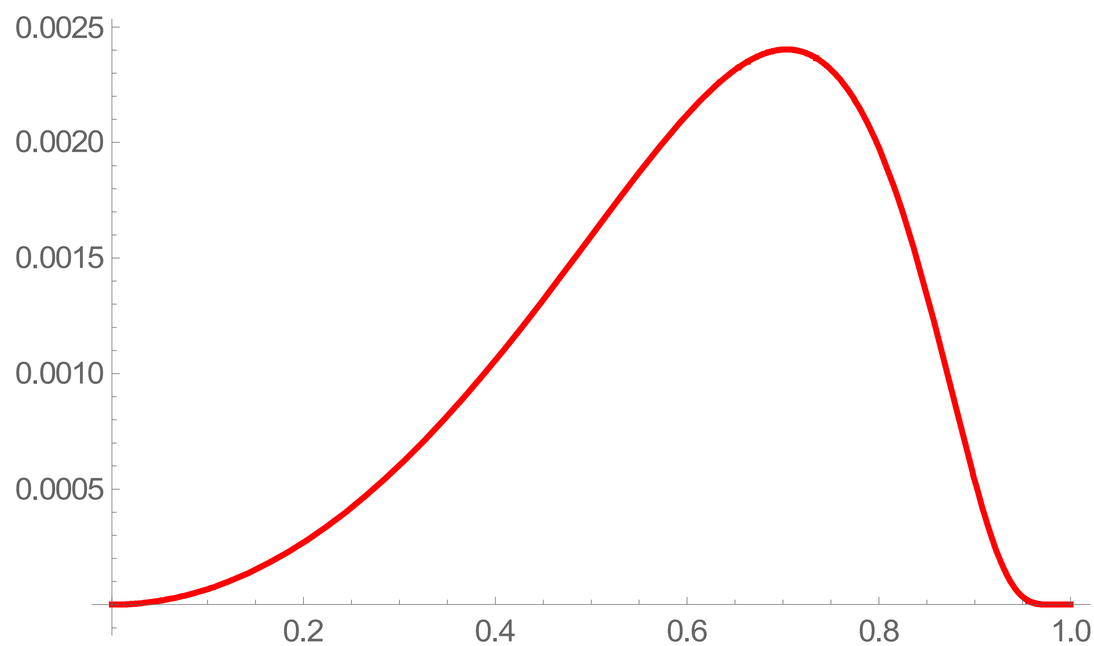

Fig.3 $F(x)$ vs x with $a = 5$. The peak appears around $x = v/c = 0.72$.

Approximation:

We solve the equation

$$y_2 = \frac{2 + x^2 - 3x^4}{x^2\sqrt{1-x^2}} = a$$

(a) a is very small. So the value of x is close to 1.

$$a = \frac{2 + x^2 - 3x^4}{x^2\sqrt{1-x^2}} = \frac{(1-x^2)(2+3x^2)}{x^2\sqrt{1-x^2}} = \frac{\sqrt{1-x^2}}{x^2}(2+3x^2) \approx 5\sqrt{1-x^2}$$

in the limit of $x \to 1$. Then we have

$$\frac{a}{5} = \sqrt{1-x^2}\,, \qquad\qquad x = 1 - \left(\frac{a}{5}\right)^2,$$

or

$$\frac{v}{c} = 1 - \left(\frac{\beta mc^2}{5}\right)^2 = 1 - \left(\frac{mc^2}{5k_B T}\right)^2 \ \text{ for } \beta mc^2 << 1$$

In fact, when $a = 0.5$, $x \approx 0.99$ (see Fig.2). So this approximation is reasonable.

(b) a is large. So the value of x is close to zero.

Suppose that $x \approx 0$.

$$a = \frac{2 + x^2 - 3x^4}{x^2\sqrt{1-x^2}} \approx \frac{2}{x^2}$$

in the limit of $x \to 0$. Then we have

$$x = \sqrt{\frac{2}{a}}\,.$$

or

$$\frac{v}{c} = \sqrt{\frac{2}{\beta mc^2}} = \sqrt{\frac{2k_B T}{mc^2}} \quad \text{for } \beta mc^2 >> 1$$

When $a = 5$, $x = \sqrt{\frac{2}{5}}. = 0.63$ (see Fig.3). So this approximation is reasonable.

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APPENDIX-I

Modified Bessel function of the second

Asymptotic expansion:

For $u << 1$,

$$K_\nu(u) = \frac{(\nu - 1)!2^{\nu-1}}{u^\nu}$$

For $u >> 1$,

$$K_\nu(u) = \sqrt{\frac{\pi}{2u}} e^{-u}[1 + \frac{(4\nu^2 - 1^2)}{1!8u} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2!(8u)^2} + ...]$$

APPENDIX-II Characteristic temperature

For an electron, the characteristic temperature T_e is

$$T_e = \frac{m_e c^2}{k_B} = 5.92987 \text{ x } 10^9 \text{ K},$$

for $\beta m_e c^2 = 1$.

For a proton, the characteristic temperature T_p

$$T_p = \frac{m_p c^2}{k_B} = 1.08881 \text{ x } 10^{13} \text{ K},$$

for $\beta m_p c^2 = 1$.