The Boltzmann factor in the potential energy

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Here we consider the case for the gas consisting of particles with mass *m* under the gravity. We use the Boltzmann factor in the potential energy.

1 Probability

The potential energy under the gravitation is given by

$$U(z) = mgz$$

For the canonical ensemble, the probability of finding a particle with mass at the height *z* is given by

$$p(z) = \frac{\exp(-\beta mgz)}{\int_0^\infty dz \exp(-\beta mgz)} = \beta mg \exp(-\beta mgz)$$

with

$$\int_0^\infty p(z)dz = 1$$

The average $\langle z \rangle$;

$$\begin{aligned}
\langle z \rangle &= \int_0^\infty zp(z)dz \\
&= \beta mg \int_0^\infty z \exp(-\beta mgz)dz \\
&= \frac{\beta mg}{(\beta mg)^2} \\
&= \frac{k_B T}{mg}
\end{aligned}$$

The average of potential energy:

$$\langle U(z)\rangle = mg\langle z\rangle = k_B T$$

1.2. Pressure

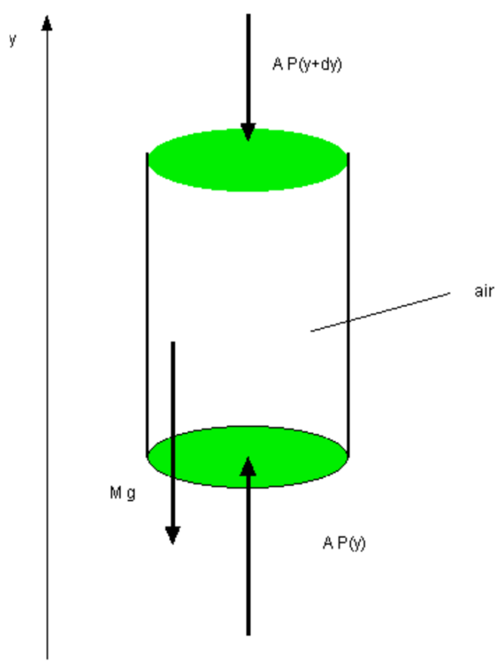
Density is denoted by

$$\rho = \frac{M}{V} = \frac{M}{N}\frac{N}{V} = mn$$

Weight:

$$Mg = \rho(Adz)g = mngAdz$$

$$AP(z) - AP(z+dz) - mngAdz = 0$$



where Mg is the weight of gas.

or

$$\frac{P(z+dz) - P(z)}{dz} = -mng$$

In the limit of $dz \to 0$,

$$\frac{dP(z)}{dz} = -mng$$

Boyle's law:

$$PV = Nk_B T\,, \qquad P = \frac{N}{V}k_B T = nk_B T$$

Thus we get

$$\frac{dP(z)}{dz} = -\frac{mg}{k_B T}P(z) = -\beta mgP(z)$$

or

$$\int \frac{1}{P(z)}dP(z) = -\int \beta mgdz$$

$$\ln P(z) = -\beta mgz + \text{constant}$$

or

$$P(z) = P(z=0)\exp(-\beta mgz)$$

We note that

$$\rho = nm = \frac{P}{k_B T}m\,, \qquad m\beta = \frac{\rho}{P}$$

We note that

$$m\beta = \frac{\rho_0}{P_0} = \frac{\rho}{P} \qquad \text{(Boyle's law; we assume that T is independent of the height } z\text{)}$$

where ρ_0 (= 1.225 kg/m^3) is the density of air at room temperature ($P = P_0$ = 1 atm) at z = 0.

$$P(z) = P_0 \exp(-\frac{\rho_0 g z}{P_0})$$

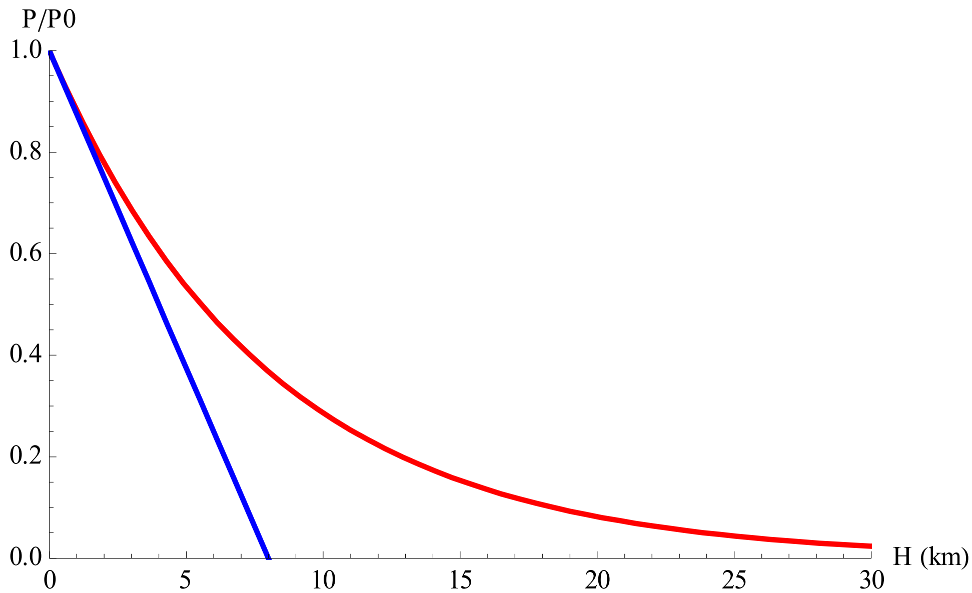


Fig. Red curve for the present case. The straight line (blue) is a tangential line of the red curve at $H = 0$. In the blue line, P becomes zero at $z = \frac{P_0}{\rho_0 g} = 8432$ m.

3. Number density *n*(*z*)

We note that

$$n(z) = \frac{P(z)}{k_B T}, \qquad n(0) = \frac{P(0)}{k_B T}$$

Thus we have

$$n(z) = n(z=0)\exp(-\beta m g z)$$

The probability can be obtained from

$$p(z) = \frac{n(z=0)\exp(-\beta m g z)}{n(z=0)\int_0^\infty \exp(-\beta m g z)dz} = \beta m g \exp(-\beta m g z)$$

REFERENCES

D.J. Amit and Y. Verbin, Statistical Physics, An Introductory Course (World Scientific, 1995).