

**Langevin function**  
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**(Date: April 17, 2018)**

The classical theory of paramagnetism, which is the limit of quantum theory when  $J \rightarrow \infty$ , was worked out in 1905 by Paul Langevin. Colloidal ferromagnetic minerals, usually magnetite, dispersed in a rock are examples of the systems where the classical theory is expected to apply.

Each atom or particle has a macroscopic magnetic moment  $\mu$ , which can take any orientation relative to the field applied in the direction (the  $z$  axis).

We consider a magnetic moment (vector  $\mu$ ) whose direction is arbitrary. When the angle between the direction of  $\mu$  and the  $z$  axis is an angle  $\theta$ , the Zeeman energy is obtained as

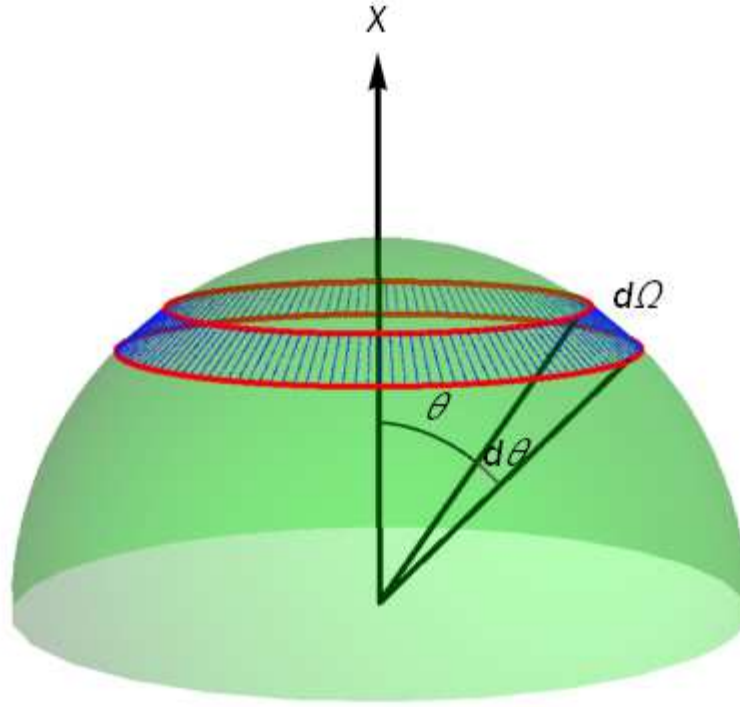
$$H = -\mu B \cos \theta$$

The one-particle partition  $Z_1$  is given by

$$\begin{aligned} Z_1 &= \int_0^\pi e^{\beta \mu B \cos \theta} d\Omega \\ &= \int_0^\pi e^{\beta \mu B \cos \theta} 2\pi \sin \theta d\theta \\ &= 4\pi \frac{\sinh(\beta \mu B)}{\beta \mu B} \end{aligned}$$

where  $\Omega$  is the solid angle. The  $N$ -particle partition function is expressed by

$$Z_N = Z_1^N$$



**Fig.** Solid angle.  $d\Omega = 2\pi \sin \theta d\theta$

We note that the total magnetization is given by

$$\begin{aligned}\langle M \rangle &= k_B T \frac{\partial \ln Z^N}{\partial B} \\ &= N k_B T \frac{\partial \ln Z_1}{\partial B} \\ &= N \mu L(x)\end{aligned}$$

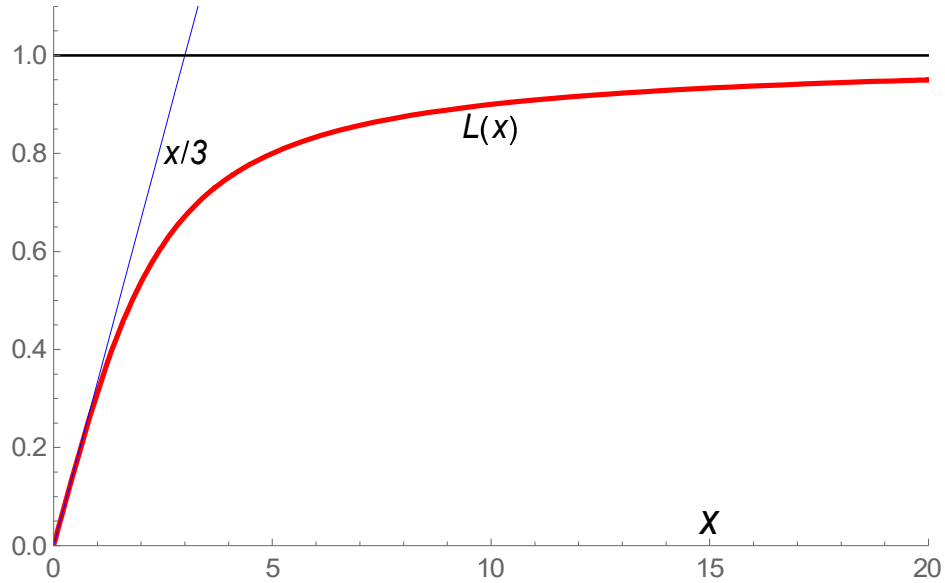
where  $x = \beta \mu B$  and  $N$  is the total number of atoms,

$$L(x) = \coth(x) - \frac{1}{x} \quad (\text{Langevin function}).$$

In the limit of  $x \rightarrow 0$ , we have

$$L(x) \approx \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} + O(x^9).$$

In the limit of  $x \rightarrow \infty$ , we have  $L(\infty) = 1$



**Fig.** Langevin function  $L(x)$  as a function of  $x = \beta\mu B$ .

((Note)) Derivation of magnetization using the partition function

$$Z = \text{Tr}[e^{-\beta H}]$$

$$\begin{aligned} \frac{\partial Z}{\partial B} &= \text{Tr}\left[(-\beta) \frac{\partial H}{\partial B} e^{-\beta H}\right] \\ &= \beta \text{Tr}[M e^{-\beta H}] \end{aligned}$$

$$\frac{1}{Z} \frac{\partial Z}{\partial B} = \beta \frac{\text{Tr}[M e^{-\beta H}]}{\text{Tr}[e^{-\beta H}]} = \beta \langle M \rangle$$

or

$$\langle M \rangle = k_B T \frac{\partial \ln Z}{\partial B}$$

## 2. Another method

$$\begin{aligned}\langle \cos \theta \rangle &= \frac{\int_0^\pi e^{\beta \mu B \cos \theta} \cos \theta d\Omega}{\int_0^\pi e^{\beta \mu B \cos \theta} d\Omega} = \frac{\int_0^\pi e^{\beta \mu B \cos \theta} 2\pi \sin \theta \cos \theta d\theta}{\int_0^\pi e^{\beta \mu B \cos \theta} 2\pi \sin \theta d\theta} \\ &= L(x)\end{aligned}$$

where  $d\Omega = 2\pi \sin \theta d\theta$ . Note that the probability  $P(\theta)$  is expressed by

$$P(\theta)d\theta = A e^{\beta \mu B \cos \theta} 2\pi \sin \theta d\theta$$

where  $A$  is constant,

$$\int_0^\pi P(\theta) d\theta = 1$$

((**Mathematica**))

```
Clear["Global`*"];
```

```
f1 = Exp[ $\beta \mu B \cos[\theta]$ ] 2  $\pi$  Sin[ $\theta$ ];
```

```
Z1 = Integrate[f1, { $\theta$ , 0,  $\pi$ }]
```

$$\frac{4 \pi \sinh[B \beta \mu]}{B \beta \mu}$$

```
 $\frac{N1}{\beta}$  D[Log[Z1], B] // FullSimplify
```

$$-\frac{N1}{B \beta} + N1 \mu \coth[B \beta \mu]$$

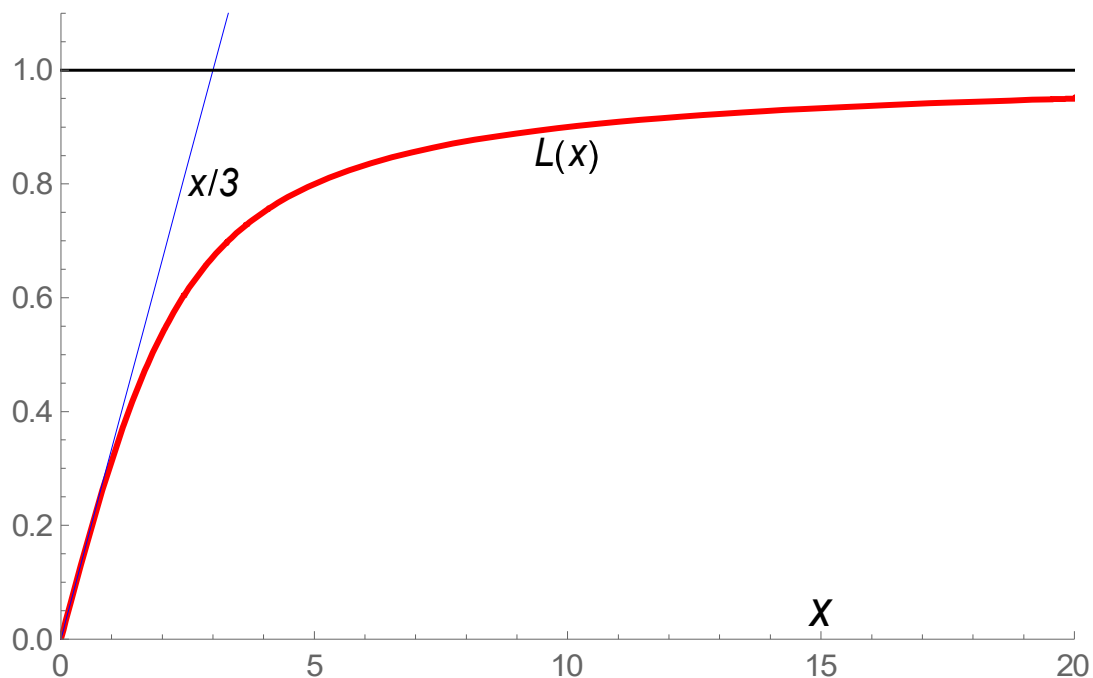
```
L1[ $x_$ ] := Coth[ $x$ ] -  $\frac{1}{x}$ ; Series[L1[ $x$ ], { $x$ , 0, 10}]
```

$$\frac{x}{3} - \frac{x^3}{45} + \frac{2 x^5}{945} - \frac{x^7}{4725} + \frac{2 x^9}{93 555} + O[x]^{11}$$

```

h1 = Plot[{L1[x], x/3}, {x, 0, 20},
  PlotStyle -> {{Red, Thick}, {Blue, Thin}},
  PlotRange -> {{0, 20}, {0, 1.1}}];
h2 = Graphics[{Line[{{0, 1}, {20, 1}}],
  Text[Style["x/3", Black, 12, Italic], {3, 0.8}],
  Text[Style["x", Black, 15, Italic], {15, 0.05}],
  Text[Style["L(x)", Black, 12, Italic],
    {10, 0.85}]}];
Show[h1, h2]

```



`Limit[L1[x], x → ∞]`

`1`

`f11 = Exp[β μ B Cos[θ]] 2 π Sin[θ];`

`f12 = Exp[β μ B Cos[θ]] 2 π Sin[θ] Cos[θ];`

`Integrate[f12, {θ, 0, π}]`  

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`// Simplify`  
`Integrate[f11, {θ, 0, π}]`

$$-\frac{1}{B \beta \mu} + \text{Coth}[B \beta \mu]$$