# Planck distribution function <br> Simple harmonics <br> Masatsugu Sei Suzuki <br> Department of Physics, SUNY at Binghamton 

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Here we discuss the Planck distribution function. It describes the spectrum of the electromagnetic radiation (photons) in thermal equilibrium within a cavity (black body). It is often called the black body radiation. The Planck distribution was the first application of quantum thermal physics.

## 1. Simple harmonics system with $\boldsymbol{f}$ oscillators: canonical ensemble

We consider a system consists of $f$ harmonic oscillators (with the same $\omega$ )

For each oscillator, the energy is quantized as

$$
\hat{H}|n\rangle=\hbar \omega\left(n+\frac{1}{2}\right)|n\rangle=\varepsilon_{n}|n\rangle
$$

where $|n\rangle$ is the eigenstate of the Hamiltonian with the energy eigenvalue $\varepsilon_{n}=\hbar \omega\left(n+\frac{1}{2}\right) . \mathrm{n}$ is the integer; $n=0,1,2, \ldots$.

Suppose that $m_{\mathrm{i}}$ is the number of oscillators with the energy $\varepsilon_{i}$.
The total number of simple harmonic oscillators is

$$
f=m_{0}+m_{1}+m_{2}+\ldots
$$

State $\left|\varepsilon_{0}\right\rangle$ with energy $\varepsilon_{0}$ level $\quad m_{0}$ oscillators
State $\left|\varepsilon_{1}\right\rangle$ with energy $\varepsilon_{1}$ level $\quad m_{1}$ oscillators


The total energy for this configuration is

$$
E=\varepsilon_{0} m_{0}+\varepsilon_{1} m_{1}+\varepsilon_{2} m_{2}+\ldots .
$$

The way to choose $m_{\mathrm{i}}$ oscillators in the same state $\left|\varepsilon_{i}\right\rangle$ is evaluated as

$$
\frac{f!}{m_{0}!m_{1}!m_{2}!\ldots} \delta_{m_{0}+m_{1}+m_{2}+\ldots+, f}
$$

where these oscillators are distinguishable. Then the partition function for the simple harmonics can be obtained as

$$
\begin{aligned}
Z_{C}(\beta) & =\sum_{m_{0}} \sum_{m_{1}} \cdots \sum_{m_{s}} \frac{f!}{m_{0}!m_{1}!\ldots} \exp \left[-\beta\left(\varepsilon_{0} m_{0}+\varepsilon_{1} m_{1}+\varepsilon_{2} m_{2}+\ldots . .\right)\right] \delta_{m_{0}+m_{1}+m_{2}+\ldots+, f} \\
& =\sum_{m_{0}} \sum_{m_{1}} \cdots \sum_{n_{s}} \frac{f!}{m_{0}!m_{1}!\ldots}\left[\left(e^{-\beta \varepsilon_{0}}\right)^{m_{0}}\left(e^{-\beta \varepsilon_{1}}\right)^{m_{1}} . .\right] \delta_{m_{0}+m_{1}+m_{2}+\ldots+, f} \\
& =\left(e^{-\beta \varepsilon_{0}}+e^{-\beta \varepsilon_{1}}+e^{-\beta \varepsilon_{2}}+\ldots\right)^{f} \\
& =\left[Z_{C 1}(\beta)\right]^{f}
\end{aligned}
$$

where $\beta=\frac{1}{k_{B} T} . \delta_{m_{0}+m_{1}+m_{2}+\ldots+, f}$ means the condition of total oscillator number being kept constant.
We note that $Z_{C 1}(\beta)$ is the partition function for the one oscillator system.

$$
\begin{aligned}
Z_{C 1}(\beta) & =\sum_{n} e^{-\beta \varepsilon_{n}} \\
& =\sum_{n} e^{-\beta \hbar \omega\left(n+\frac{1}{2}\right)} \\
& =e^{-\frac{\beta \hbar \omega}{2}}\left(1+e^{-\beta \hbar \omega}+e^{-2 \beta \hbar \omega}+\ldots\right) \\
& =e^{-\frac{\beta \hbar \omega}{2}}\left(\frac{1}{1-e^{-\beta \hbar \omega}}\right)
\end{aligned}
$$

or

$$
Z_{C 1}(\beta)=e^{-\frac{\beta \hbar \omega}{2}}\left(\frac{1}{1-e^{-\beta \hbar \omega}}\right)=^{\frac{\beta \hbar \omega}{2}}\left(\frac{1}{e^{\beta \hbar \omega}-1}\right)
$$

and

$$
Z_{C}(\beta)=\left[Z_{C 1}(\beta)\right]^{f}==^{\frac{f \beta \hbar \omega}{2}}\left(\frac{1}{e^{\beta \hbar \omega}-1}\right)^{f}
$$

The internal energy is

$$
\frac{U}{f}=\frac{1}{2} \hbar \omega+\frac{\hbar \omega}{e^{\beta \hbar \omega}-1}
$$

The first term is the zero-point energy.
2. Method originally proposed by Planck: micro-canonical ensemble We consider the $f$-independent oscillators. The total energy is given by

$$
\begin{aligned}
& E=\frac{f}{2} \hbar \omega+\hbar \omega\left(n_{1}+n_{2}+\ldots+n_{f}\right)=\frac{f}{2} \hbar \omega+\hbar \omega M \\
& \hat{H}\left|n_{i}\right\rangle=\hbar \omega\left(n_{i}+\frac{1}{2}\right)\left|l_{i}\right\rangle
\end{aligned}
$$

where $i=1,2, \ldots, f$

For example,

$$
f=5, M=11
$$

$n_{1}=2 ; n_{2}=5, n_{3}=0, n_{4}=3, n_{5}=1$ corresponding to $|2\rangle,|5\rangle,|0\rangle,|3\rangle,|1\rangle$.

which is equivalent to
by replacing the bars by the black dots. Another example is

$$
\begin{aligned}
& n_{1}=2 ; n_{2}=3, n_{3}=3, n_{4}=0, n_{5}=3 \text { corresponding to }|2\rangle,|3\rangle,|3\rangle,|0\rangle,|3\rangle, \text { where } M=11 \\
& n_{1}=1 ; n_{2}=7, n_{3}=0, n_{4}=3, n_{5}=0 \text { corresponding to }|1\rangle,|7\rangle,|0\rangle,|3\rangle,|0\rangle, \text { where } M=11
\end{aligned}
$$

Thus the number of ways is

$$
W(f, M)=\frac{(M+f-1)!}{M!(f-1)!} \approx \frac{(M+f)!}{M!f!}
$$

Here we apply the Stirling's approximation,

$$
\begin{aligned}
\ln W(f, M) & =\ln (M+f)!-\ln M!-\ln f! \\
& =(M+f)[\ln (M+f)-1]-M(\ln M-1)-f(\ln f-1) \\
& =(M+f) \ln (M+f)-M \ln M-f \ln f
\end{aligned}
$$

The entropy of this system is

$$
\begin{aligned}
S(f, E) & =k_{B} \ln W(f, M) \\
& \left.=k_{B}[(M+f) \ln (M+f)-M \ln M-f \ln f]\right]
\end{aligned}
$$

where

$$
E=\frac{f}{2} \hbar \omega+\hbar \omega M
$$

The temperature $T$ is defined by

$$
\begin{aligned}
\frac{1}{T} & =\frac{\partial S(f, M)}{\partial E} \\
& =\frac{\partial M}{\partial E} \frac{\partial S(f, M)}{\partial M} \\
& =\frac{k_{B}}{\hbar \omega} \ln \left(\frac{M+f}{M}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{\hbar \omega}{k_{B} T} & =\ln \left(\frac{M+\frac{f}{2}+\frac{f}{2}}{M+\frac{f}{2}-\frac{f}{2}}\right) \\
& =\ln \left(\frac{E+\frac{f}{2} \hbar \omega}{E-\frac{f}{2} \hbar \omega}\right)
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{E}{f} & =\frac{1}{2} \hbar \omega\left(1+\frac{2}{e^{\beta \hbar \omega}-1}\right) \\
& =\frac{1}{2} \hbar \omega+\frac{\hbar \omega}{e^{\beta \hbar \omega}-1}
\end{aligned}
$$

((Note))
This method was originally discussed by Planck in the following paper;
M. Planck, Ann. Phys. 4, 553 (1901)

On the Law of the Energy Distribution in the Normal Spectrum
3. Canonical ensemble for the system with one oscillator

For the simple harmonic oscillator with the angular frequency $\omega$

$$
\hat{H}|n\rangle=\left(n+\frac{1}{2}\right) \hbar \omega|n\rangle,
$$

with $\quad n=0,1,2, \ldots$ The Hamiltonian is defined by

$$
\hat{H}=\frac{1}{2 m} \hat{p}^{2}+\frac{1}{2} m \omega^{2} \hat{x}^{2}
$$

The partition function is

$$
Z_{C 1}=\operatorname{Tr}\left[e^{-\beta \hat{H}}\right]=\sum_{n=0}^{\infty} e^{-\beta \varepsilon_{n}}=\sum_{n=0}^{\infty} e^{-\beta \hbar \omega\left(n+\frac{1}{2}\right)}
$$

or

$$
\begin{aligned}
Z_{C 1} & =\sum_{n=0}^{\infty} e^{-\beta \hbar \omega\left(n+\frac{1}{2}\right)} \\
& =e^{-\frac{\beta \hbar \omega}{2}} \sum_{n=0}^{\infty}\left(e^{-\beta \hbar \omega}\right)^{n} \\
& =\frac{e^{-\frac{\beta \hbar \omega}{2}}}{1-e^{-\beta \hbar \omega}} \\
& =\frac{e^{\frac{\beta \hbar \omega}{2}}}{e^{\beta \hbar \omega}-1}
\end{aligned}
$$

or

$$
\begin{aligned}
Z_{C 1} & =\frac{e^{\frac{\beta \hbar \omega}{2}}}{e^{\beta \hbar \omega}-1} \\
& =\frac{1}{2 \sinh \left(\frac{\beta \hbar \omega}{2}\right)} \\
& =\frac{1}{2} \operatorname{cosech}\left(\frac{\beta \hbar \omega}{2}\right)
\end{aligned}
$$

Note that

$$
\ln Z_{C 1}=\frac{1}{2} \beta \hbar \omega-\ln \left(e^{\beta \hbar \omega}-1\right) .
$$

The average energy of a single oscillator is

$$
\begin{aligned}
\langle E\rangle & =-\frac{\partial}{\partial \beta} \ln Z_{C 1} \\
& =\frac{1}{2} \hbar \omega \operatorname{coth}\left(\frac{\beta \hbar \omega}{2}\right) \\
& =\hbar \omega\left(\frac{1}{2}+\frac{1}{e^{\beta \hbar \omega}-1}\right)
\end{aligned}
$$

The heat capacity:

$$
C=\frac{d\langle E\rangle}{d T}=\frac{d \beta}{d T} \frac{d\langle E\rangle}{d \beta}=-k_{B} \beta^{2} \frac{d\langle E\rangle}{d \beta}
$$

or

$$
C=-k_{B} \beta^{2} \frac{d\langle E\rangle}{d \beta}=k_{B} \frac{(\beta \hbar \omega)^{2} e^{\beta \hbar \omega}}{\left(e^{\beta \hbar \omega}-1\right)^{2}}
$$

For $N$ oscillators with the same $\omega$, we have

$$
Z_{C N}=Z_{C 1}{ }^{N}=\left(\frac{1}{2 \sinh \left(\frac{\beta \hbar \omega}{2}\right)}\right)^{N}
$$

Note that there is no additional factor such as $(1 / N!)$ for the system such as simple harmonics. The reason is as follows. The element of the system is localized by its mechanics. For the free particles, the element of the system is not localized by its mechanics.

## 4. Thermodynamic properties

The Helmholtz free energy is given by

$$
F=-k_{B} T \ln Z_{C 1}=k_{B} T \ln \left[2 \sinh \left(\frac{\hbar \omega}{2 k_{B} T}\right)\right]
$$

using the partition function $Z_{C 1}$. The internal energy is

$$
U=\frac{\partial}{\partial \beta}(\beta F)=\frac{N \hbar \omega}{2} \operatorname{coth}\left(\frac{\beta \hbar \omega}{2}\right)
$$

or

$$
\frac{U}{N}=\frac{\partial}{\partial \beta}(\beta F)=\frac{\hbar \omega}{2}+\frac{\hbar \omega}{e^{\beta \hbar \omega}-1} .
$$

The entropy $S$ :

$$
S=\frac{U}{T}-\frac{F}{T}=k_{B}(\beta U-\beta F)
$$

The entropy $S$ can be expressed as

$$
S=k_{B}\left(\ln Z_{C 1}-\beta \frac{\partial}{\partial \beta} \ln Z_{C 1}\right)=-k_{B} \beta^{2} \frac{\partial}{\partial \beta}\left(\frac{1}{\beta} \ln Z_{C 1}\right)
$$

or

$$
S=k_{B} N\left\{\frac{\beta \hbar \omega}{2} \operatorname{coth}\left(\frac{\beta \hbar \omega}{2}\right)-\ln \left[2 \sinh \left(\frac{\beta \hbar \omega}{2}\right)\right]\right\}
$$

The heat capacity:

$$
C=\frac{d U}{d T}=N k_{B}(\beta \hbar \omega)^{2} \frac{e^{\beta \hbar \omega}}{\left(e^{\beta \hbar \omega}-1\right)^{2}}
$$



Fig. Heat capacity $\frac{C}{N k_{B}}$ vs $x=\frac{k_{B} T}{\hbar \omega}$ (quantum mechanics). $\frac{C}{N k_{B}}$ tends to 1.0 as $T \rightarrow \infty$, as is predicted from the energy partition theorem.

## Energy fluctuation

$$
\left.\langle\Delta E)^{2}\right\rangle=\left\langle E^{2}\right\rangle-\langle E\rangle^{2}=\frac{\partial^{2}}{\partial \beta^{2}} \ln Z_{C 1}=k_{B} T^{2} \frac{\partial}{\partial T}\langle E\rangle=k_{B} T^{2} C
$$

## ((Mathematica))

Clear["Global **"] ; F1 $=\mathrm{NkB} T \log \left[\frac{1}{2} \operatorname{Sinh}\left[\frac{\hbar \omega}{2 \mathrm{kBT}}\right]\right] ;$
$E 1=-k B T^{2} D\left[\frac{F 1}{k B T}, T\right] / /$ Simplify
S1 = -D [F1, T] // Simplify
$\frac{\mathrm{N} \omega \hbar \operatorname{Coth}\left[\frac{\omega \hbar}{2 \mathrm{kBT}}\right]}{2 \mathrm{~T}}-\mathrm{kBN} \log \left[\frac{1}{2} \operatorname{Sinh}\left[\frac{\omega \hbar}{2 \mathrm{kBT}}\right]\right]$

C1 = D[E1, T] // Simplify
$\underline{N} \omega^{2} \hbar^{2} \operatorname{Csch}\left[\frac{\omega \hbar}{2 \mathrm{kBT}}\right]^{2}$ $4 \mathrm{kB} \mathrm{T}{ }^{2}$

## 5. Quantized electromagnetic field of photon

We consider a system with photon modes with the angular frequency $\omega_{k}$. The state of photon is given by the mode $|\boldsymbol{k}\rangle$, where $\boldsymbol{k}$ is the wave vector. The system consists of many modes denoted by $\boldsymbol{k}$. The value of $\omega_{\boldsymbol{k}}$ is different for different $\boldsymbol{k}$.

The partition function is obtained as

$$
\begin{aligned}
Z_{C}(\beta) & =Z_{C 1}\left(\boldsymbol{k}_{1}, \beta\right) Z_{C 1}\left(\boldsymbol{k}_{2}, \beta\right) Z_{C 1}\left(\boldsymbol{k}_{3}, \beta\right) \ldots . . \\
& =\prod_{k} Z_{C 1}(\boldsymbol{k}, \beta)
\end{aligned}
$$

where $Z_{C 1}(\boldsymbol{k}, \beta)$ is the one-particle partition function for the state $|\boldsymbol{k}\rangle$.


Fig. Energy dispersion of photon with the discrete wave number.
Note that

$$
\begin{aligned}
Z_{C 1}(\boldsymbol{k}, \beta) & =\sum_{n=0}^{\infty} e^{-\beta\left(n+\frac{1}{2}\right) \hbar \omega_{k}} \\
& =e^{-\frac{\beta \hbar \omega_{k}}{2}} \sum_{n=0}^{\infty} e^{-\beta n \hbar \omega_{k}} \\
& =e^{-\frac{\beta \hbar \omega_{k}}{2}} \frac{1}{1-e^{-\beta \hbar \omega_{k}}} \\
& =\frac{e^{\frac{\beta \hbar \omega_{k}}{2}}}{e^{\beta \hbar \omega_{k}}-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\ln Z_{C}(\beta) & =\sum_{k} \ln Z_{C 1}(\boldsymbol{k}, \beta) \\
& =\sum_{k} \ln \left(\frac{e^{\frac{\beta \hbar \omega_{k}}{2}}}{e^{\beta \hbar \omega_{k}}-1}\right) \\
& =\sum_{k}\left[\frac{\beta \hbar \omega_{k}}{2}-\ln \left(e^{\beta \hbar \omega_{k}}-1\right)\right]
\end{aligned}
$$

The internal energy $U$ is given by

$$
\begin{aligned}
U & =\langle E\rangle_{C} \\
& =-\frac{\partial}{\partial \beta} \ln Z_{C}(\beta) \\
& =-\frac{\partial}{\partial \beta} \sum_{k} \ln Z_{C 1}(\boldsymbol{k}, \beta) \\
& =\sum_{k}\left(-\frac{\hbar \omega_{k}}{2}+\frac{\hbar \omega_{k} e^{\beta \hbar \omega_{k}}}{e^{\beta \hbar \omega_{k}}-1}\right) \\
& =\sum_{k}\left(\frac{1}{e^{\beta \hbar \omega_{k}}-1}+\frac{1}{2}\right) \hbar \omega_{k} \\
& =\sum_{k}\left(\left\langle n_{k}\right\rangle+\frac{1}{2}\right) \hbar \omega_{k}
\end{aligned}
$$

The first term is the zero-point energy. Using the Planck's distribution function

$$
\left\langle n_{k}\right\rangle=\frac{1}{e^{\beta \hbar \omega_{k}}-1}
$$

we have

$$
U=\sum_{k}\left[\left\langle n_{k}\right\rangle+\frac{1}{2}\right] \hbar \omega_{k}
$$

We note that photons are bosons. The number $\left\langle n_{k}\right\rangle=\frac{1}{e^{\beta \hbar \omega_{k}}-1}$ is given by the Bose-Einstein distribution with the chemical potential $\mu=0$. The reason for this will be given in the discussion of the chemical potential of photon zero (later).

The entropy is

$$
S=k_{B}\left[\ln Z_{C}(\beta)-\beta \frac{\partial}{\partial \beta} \ln Z_{C}(\beta)\right]
$$

or

$$
\begin{aligned}
\frac{S}{k_{B}} & =\beta U+\ln Z_{C}(\beta) \\
& =\ln Z_{C}(\beta)-\beta \frac{\partial}{\partial \beta} \ln Z_{C}(\beta) \\
& =\sum_{k}\left[\ln Z_{C 1}(\boldsymbol{k}, \beta)-\beta \frac{\partial}{\partial \beta} \ln Z_{C 1}(\boldsymbol{k}, \beta)\right] \\
& =\sum_{k}\left[\frac{\beta \hbar \omega_{k}}{2}-\ln \left(e^{\beta \hbar \omega_{k}}-1\right)\right]-\beta \frac{\partial}{\partial \beta} \sum_{k}\left[\frac{\beta \hbar \omega_{\boldsymbol{k}}}{2}-\ln \left(e^{\beta \hbar \omega_{k}}-1\right)\right] \\
& =\sum_{k}\left[-\ln \left(e^{\beta \hbar \omega_{k}}-1\right)+\beta \hbar \omega_{k} \frac{e^{\beta \hbar \omega_{k}}}{e^{\beta \hbar \omega_{k}}-1}\right]
\end{aligned}
$$

or

$$
\left.\frac{S}{k_{B}}=\sum_{\mathrm{k}}\left\{\frac{1}{2} \beta \hbar \omega_{\mathrm{k}} \operatorname{coth}\left(\frac{1}{2} \beta \hbar \omega_{\mathrm{k}}\right)\right]-\ln \left[2 \sinh \left(\frac{1}{2} \beta \hbar \omega_{\mathrm{k}}\right)\right]\right\}
$$

This distribution function can be used to discuss the black-body problem (Stefan-Boltzmann law).

$$
\begin{aligned}
& \omega_{k}=c|\boldsymbol{k}|=c k \\
& \sum_{k} \rightarrow \frac{2 V}{(2 \pi)^{3}} 4 \pi k^{2} d k=\frac{V}{\pi^{2} c^{3}} \omega^{2} d \omega
\end{aligned}
$$

The factor 2 comes from the two independent polarization. We drop the zero point energy. Then the total energy is rewritten as

$$
\begin{aligned}
U & =\sum_{k}\left\langle n_{k}\right\rangle \hbar \omega_{\boldsymbol{k}} \\
& =\frac{\hbar V}{\pi^{2} c^{3}} \int_{0}^{\infty} \frac{\omega^{3} d \omega}{e^{\beta \hbar \omega}-1}
\end{aligned}
$$

or

$$
u=\frac{U}{V}=\frac{\pi^{2}}{15 \hbar^{3} c^{3}}\left(k_{B} T\right)^{4}
$$

(Stefan-Boltzmann law of radiation).

Note that

$$
\int_{0}^{\infty} \frac{x^{3} d x}{e^{x}-1}=\frac{\pi^{4}}{15}
$$

The energy density:

$$
u=\frac{U}{V}=\frac{\hbar}{\pi^{2} c^{3}} \int_{0}^{\infty} \frac{\omega^{3} d \omega}{e^{\beta \hbar \omega}-1}=\int_{0}^{\infty} u_{\omega} d \omega
$$

So the spectral density is

$$
u_{\omega}=\frac{\hbar}{\pi^{2} c^{3}} \frac{\omega^{3}}{e^{\beta h \omega}-1}
$$

Note that

$$
\int_{0}^{\infty} u(\omega) d \omega=\int_{0}^{\infty} \frac{\hbar \omega^{3}}{\pi^{2} c^{3}} \frac{1}{\exp \left(\frac{\hbar \omega}{k_{B} T}\right)-1} d \omega
$$

Since $\omega=\frac{2 \pi c}{\lambda}, \quad d \omega=-2 \pi c \frac{d \lambda}{\lambda^{2}}$

$$
\begin{aligned}
\int_{0}^{\infty} u(\omega) d \omega & =\int_{0}^{\infty} \frac{\hbar \omega^{3}}{\pi^{2} c^{3}} \frac{1}{\exp \left(\frac{\hbar \omega}{k_{B} T}\right)-1} d \omega \\
& =\int_{0}^{\infty} \frac{\hbar\left(\frac{2 \pi c}{\lambda}\right)^{3}}{\pi^{2} c^{3}} \frac{1}{\exp \left(\frac{2 \pi \hbar c}{\lambda k_{B} T}\right)-1} 2 \pi c \frac{d \lambda}{\lambda^{2}}
\end{aligned}
$$

or

$$
\int_{0}^{\infty} u(\omega) d \omega=\int_{0}^{\infty} u(\lambda) d \lambda=\int_{0}^{\infty} 16 \pi^{2} \hbar c \frac{1}{\lambda^{5}} \frac{1}{\exp \left(\frac{2 \pi \hbar c}{\lambda k_{B} T}\right)-1} d \lambda
$$

Then we have

$$
u(\lambda)=\frac{16 \pi^{2} \hbar c}{\lambda^{5}} \frac{1}{\exp \left(\frac{2 \pi \hbar c}{\lambda k_{B} T}\right)-1}
$$

## Wien's law:

$$
\frac{d}{d \lambda} \frac{1}{\lambda^{5}}\left[\frac{1}{\exp \left(\frac{2 \pi \hbar c}{\lambda k_{B} T}\right)-1}\right]=0
$$

$u(\lambda)$ has a maximum when

$$
\begin{equation*}
1-e^{-x}=\frac{x}{5} \tag{1}
\end{equation*}
$$

where

$$
x=\frac{2 \pi \hbar c}{\lambda k_{B} T}, \quad \quad \text { (dimensionless) }
$$

The solution of Eq.(1) is given by

$$
x=4.96511
$$

or

$$
\lambda_{\max }=\frac{0.28977}{T(K)}
$$

( $\lambda$ in the units of cm )
or

$$
\lambda_{\max }=\frac{2.897768551}{T(K)} \times 10^{6} . \quad(\lambda \text { in the units of } \mathrm{nm})
$$

When $T=6000 \mathrm{~K}$ (sun), $\lambda_{\text {max }}=483 \mathrm{~nm}$.

## 6. Photon number density

The photon number density is evaluated from

$$
N=\sum_{k} \bar{n}_{k}=\frac{2 V}{(2 \pi)^{3}} \int 4 \pi k^{2} d k \bar{n}_{k}=\frac{V}{\pi^{2}} \int \frac{k^{2} d k}{e^{\beta \hbar \omega}-1}
$$

as

$$
n=\frac{N}{V}=\frac{1}{\pi^{2} c^{3}} \int \frac{\omega^{2} d \omega}{e^{\beta \hbar \omega}-1}=\frac{\left(k_{B} T\right)^{3}}{\pi^{2} c^{3} \hbar^{3}} \int_{0}^{\infty} \frac{x^{2} d x}{e^{x}-1}=2.40411 \frac{k_{B}{ }^{3}}{\pi^{2} c^{3} \hbar^{3}} T^{3}
$$

where

$$
\int_{0}^{\infty} \frac{x^{2} d x}{e^{x}-1}=2 \zeta(3)=2.40411
$$

At $T=2.73 \mathrm{~K}, \quad n=412.7 / \mathrm{cm}^{3}$

At $T=3000 \mathrm{~K}, \quad n=5.4775 \times 10^{11} / \mathrm{cm}^{3}$

Then we have the average energy per photon

$$
\frac{U}{N}=\frac{\frac{\pi^{2}\left(k_{B} T\right)^{4}}{15 \hbar^{3} c^{3}}}{2.40411 \frac{k_{B}^{3}}{\pi^{2} c^{3} \hbar^{3}} T^{3}}=\frac{\pi^{4}}{15 \times 2.40411} k_{B} T=2.70119 k_{B} T
$$

or

$$
\frac{U}{N}=2.70119 k_{B} T
$$

For $T=2.73 \mathrm{~K}(\mathrm{CMB})$

$$
\frac{U}{N}=2.70119 k_{B} T=1.01813 \times 10^{-22} \mathrm{~J} . \quad \text { (average energy per photon) }
$$

For $T=6000 \mathrm{~K}$ (Sun)

$$
\frac{U}{N}=2.70119 k_{B} T=2.23764 \times 10^{-19} \mathrm{~J} . \quad \text { (average energy per photon) }
$$

7. Summary

$$
\begin{aligned}
& U=\frac{4 \sigma_{B}}{c} V T^{4} \\
& F=-\frac{4 \sigma_{B}}{3 c} V T^{4} \\
& S=\frac{16 \sigma_{B}}{3 c} V T^{3}
\end{aligned}
$$

$$
P=\frac{4 \sigma_{B}}{3 c} T^{4}
$$

$$
P V=\frac{U}{3}
$$

$$
C=\frac{16 \sigma_{B}}{c} V T^{3}
$$

$$
\left.\sigma_{B}=\frac{\pi^{2} k_{B}}{60 \hbar^{3} c^{2}} \quad \quad \text { (Stefan-Boltzmann constant }\right)
$$

## REFERENCES

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S.J. Blundell and K.M. Blundell, Concepts in Thermal Physics (Oxford, 2006).

## APPENDIX

## Canonical ensemble for the $f$ - simple harmonics

The one-oscillator partition function is given by

$$
Z_{C 1}=\sum_{n=0}^{\infty} e^{-\beta \hbar \omega\left(n+\frac{1}{2}\right)}=e^{-\frac{\beta \hbar \omega}{2}} \sum_{n=0}^{\infty} e^{-\beta n \hbar \omega}=e^{-\frac{\beta \hbar \omega}{2}} \frac{1}{1-e^{-\beta \hbar \omega}}
$$

The $f$-oscillator partition function is given by

$$
\left(Z_{C f}\right)^{f}=e^{-\frac{\beta \hbar \hbar \omega}{2}}\left(\frac{1}{1-e^{-\beta \hbar \omega}}\right)^{f}
$$

when $x=e^{-\beta \hbar \omega}$. The partition function for the $f$-harmonic oscillators can be rewritten as

$$
\begin{aligned}
Z_{C f} & =e^{-\frac{\beta \hbar \hbar \omega}{2}}\left(\frac{1}{1-x}\right)^{f} \\
& =e^{-\frac{\beta \hbar \hbar \omega}{2}}\left(1-e^{-\beta \hbar \omega}\right)^{-f} \\
& =e^{-\frac{\beta f \hbar \omega}{2}} \sum_{m=0}^{\infty} \frac{(f-1+m)!}{(f-1)!m!} e^{-m \beta \hbar \omega} \\
& =\sum_{m=0}^{\infty} \frac{(f-1+m)!}{(f-1)!m!} e^{-\frac{\beta \hbar \hbar \omega}{2}} e^{-m \beta \hbar \omega}
\end{aligned}
$$

where

$$
\left(\frac{1}{1-x}\right)^{f}=\sum_{m=0}^{\infty} \frac{(f-1+m)!}{(f-1)!m!} x^{m} \quad \text { (see the proof using Mathematica below). }
$$

The multiplicity (degeneracy) for the $\left|E_{m}\right\rangle$ state is

$$
g(f, m)=\frac{(f-1+m)!}{(f-1)!m!}
$$

and the energy for the $\left|E_{m}\right\rangle$ state is given by

$$
E_{m}=m \hbar \omega+\frac{1}{2} f \hbar \omega=\left(m+\frac{f}{2}\right) \hbar \omega .
$$

where $\frac{1}{2} f \hbar \omega$ is the zero point energy.

## ((Mathematica))

Clear["Global`*"]; $g\left[f_{-}, m_{-}\right]:=\frac{(f+m-1)!}{(f-1)!m!} ;$

$$
\begin{aligned}
& \text { eq1 }=\text { Series }\left[\left(\frac{1}{1-x}\right)^{15},\{x, 0,20\}\right] \\
& 1+15 x+120 x^{2}+680 x^{3}+3060 x^{4}+11628 x^{5}+ \\
& 38760 x^{6}+116280 x^{7}+319770 x^{8}+817190 x^{9}+ \\
& 1961256 x^{10}+4457400 x^{11}+9657700 x^{12}+ \\
& 20058300 x^{13}+40116600 x^{14}+77558760 x^{15}+ \\
& 145422675 x^{16}+265182525 x^{17}+471435600 x^{18}+ \\
& 818809200 x^{19}+1391975640 x^{20}+0[x]^{21}
\end{aligned}
$$

A1[ $n_{-}$] := Coefficient[eq1, $\left.\mathbf{x}, n\right]$;
list1 = Table $[\{n, g[15, n], A 1[n]\},\{n, 0,20\}]$;
Prepend [Table[\{n, $g[15, n], A 1[n]\},\{n, 0,20\}]$, \{"n", " g[f=15,n]", "coefficient( $\left.\left.\mathrm{x}^{\mathrm{n}}\right) \mathrm{l}\right\}$ ] //

| $n$ |  |  |
| :--- | :--- | :--- |
| 0 | 1 | $[f=15, n]$ |
| 1 | 15 | coefficient $\left(\mathrm{x}^{\mathrm{n}}\right)$ |
| 2 | 120 | 15 |
| 3 | 680 | 120 |
| 4 | 3060 | 680 |
| 5 | 11628 | 3060 |
| 6 | 38760 | 31628 |
| 7 | 116280 | 116280 |
| 8 | 319770 | 319770 |
| 9 | 817190 | 817190 |
| 10 | 1961256 | 1961256 |
| 11 | 4457400 | 4457400 |
| 12 | 9657700 | 9657700 |
| 13 | 20058300 | 20058300 |
| 14 | 40116600 | 40116600 |
| 15 | 77558760 | 77558760 |
| 16 | 145422675 | 145422675 |
| 17 | 265182525 | 265182525 |
| 18 | 471435600 | 471435600 |
| 19 | 818809200 | 818809200 |
| 20 | 1391975640 | 1391975640 |

