# Blackbody problem: Maxwell's equation Masatsugu Sei Suzuki <br> Department of Physics, SUNY at Binghamton 

(Date: September 03, 2018)
One of the most puzzling phenomena around 1900 was the spectral distribution of blackbody radiation. A blackbody is an ideal system that absorbs all the radiation incident on it. Max Planck proposed his theory that could explain the experimental data at all wavelengths. He assumed that the energy emitted and absorbed by the blackbody is not continuous, but is instead emitted or absorbed in quanta. The size of an energy quantum is proportional to the frequency of the radiation.

Max Planck (April 23, 1858 - October 4, 1947) was a German physicist. He is considered to be the founder of the quantum theory, and thus one of the most important physicists of the twentieth century. Planck was awarded the Nobel Prize in Physics in 1918.

http://en.wikipedia.org/wiki/Max_Planck

Wilhelm Carl Werner Otto Fritz Franz Wien (13 January 1864 - 30 August 1928) was a German physicist who, in 1893, used theories about heat and electromagnetism to deduce Wien's displacement law, which calculates the emission of a blackbody at any temperature from the emission at any one reference temperature. He also formulated an
expression for the black-body radiation which is correct in the photon-gas limit. His arguments were based on the notion of adiabatic invariance, and were instrumental for the formulation of quantum mechanics. Wien received the 1911 Nobel Prize for his work on heat radiation.

http://en.wikipedia.org/wiki/Wilhelm_Wien
John William Strutt, 3rd Baron Rayleigh, OM (12 November 1842 - 30 June 1919) was an English physicist who, with William Ramsay, discovered the element argon, an achievement for which he earned the Nobel Prize for Physics in 1904. He also discovered the phenomenon now called Rayleigh scattering, explaining why the sky is blue, and predicted the existence of the surface waves now known as Rayleigh waves. In 1910 Lord Rayleigh discovered that an electrical discharge in nitrogen gas produced "active nitrogen", an allotrope considered to be monatomic. The "whirling cloud of brilliant yellow light" produced by his apparatus reacted with quicksilver to produce explosive mercury nitride.

http://en.wikipedia.org/wiki/John_Strutt,_3rd_Baron_Rayleigh

## 1 Blackbody problem

We start with the Maxwell's equation

$$
\begin{aligned}
& \nabla \cdot \boldsymbol{E}=0 \\
& \nabla \cdot \boldsymbol{B}=0 \\
& \nabla \times \boldsymbol{E}=-\frac{\partial \boldsymbol{B}}{\partial t} \\
& \nabla \times \boldsymbol{B}=\frac{1}{c^{2}} \frac{\partial \boldsymbol{E}}{\partial t}
\end{aligned}
$$

We assume that

$$
\begin{aligned}
& \boldsymbol{E}=\operatorname{Re}\left[\widetilde{\boldsymbol{E}}_{0} e^{-i \omega t}\right] \\
& \boldsymbol{B}=\operatorname{Re}\left[\widetilde{\boldsymbol{B}}_{0} e^{-i \omega t}\right] \\
& \nabla \cdot \widetilde{\boldsymbol{E}}_{0}=0 \\
& \nabla \cdot \widetilde{\boldsymbol{B}}_{0}=0 \\
& \nabla \times \widetilde{\boldsymbol{E}}_{0}=i \omega \widetilde{\boldsymbol{B}}_{0}
\end{aligned}
$$

$$
\begin{aligned}
& \nabla \times \widetilde{\boldsymbol{B}}_{0}=-i \frac{\omega}{c^{2}} \widetilde{\boldsymbol{E}}_{0} \\
& \nabla \times(\nabla \times \boldsymbol{E})=\nabla(\nabla \cdot \boldsymbol{E})-\nabla^{2} \boldsymbol{E}=-\nabla^{2} \boldsymbol{E}=-\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B})=-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \boldsymbol{E}
\end{aligned}
$$

or

$$
\nabla^{2} \boldsymbol{E}=\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}} \boldsymbol{E}
$$

or

$$
\nabla^{2} \widetilde{\boldsymbol{E}}_{0}+k^{2} \widetilde{\boldsymbol{E}}_{0}=0
$$

with $\omega=c k$. Similarly, we have

$$
\begin{aligned}
& \nabla^{2} \boldsymbol{B}=\frac{1}{c^{2}} \frac{\partial \boldsymbol{B}}{\partial t} \\
& \nabla^{2} \widetilde{\boldsymbol{B}}_{0}+k^{2} \widetilde{\boldsymbol{B}}_{0}=0
\end{aligned}
$$

We now consider an electromagnetic wave in the closed cube with side $L$.


Fig. Boundary condition for the electric field (red) (tangential component continuous)) and the magnetic field (green) (normal component continuous).

From the boundary conditions we have

$$
\begin{aligned}
& E_{x}=E_{1}\binom{\sin \left(k_{1} x\right)}{\cos \left(k_{1} x\right)}\left(\begin{array}{l}
\sin \left(k_{2} y\right)
\end{array}\right)\left(\begin{array}{l}
\sin \left(k_{3} z\right) \\
E_{y}=E_{2}\left(\begin{array}{l}
\sin \left(k_{1} x\right)
\end{array}\right)\binom{\sin \left(k_{2} y\right)}{\cos \left(k_{2} y\right)}\binom{\sin \left(k_{3} z\right)}{)} \\
E_{z}=E_{3}\left(\begin{array}{l}
\sin \left(k_{1} x\right)
\end{array}\right)\left(\sin \left(k_{2} y\right)\right)\binom{\sin \left(k_{3} z\right)}{\cos \left(k_{3} y\right)}
\end{array}, ~\right.
\end{aligned}
$$

where

$$
\begin{aligned}
k_{1}=\frac{\pi}{L} n_{x}, \quad k_{2}=\frac{\pi}{L} n_{y}, \quad k_{3} & =\frac{\pi}{L} n_{z} \\
\left(n_{\mathrm{x}}, n_{\mathrm{y}}, n_{\mathrm{z}}\right. & =1,2,3, \ldots)
\end{aligned}
$$

Note that

$$
\begin{array}{ll}
E_{\mathrm{x}}=0 & \text { for } y=0 \text { and } y=L \text { planes and } z=0 \text { and } z=L \text { planes. } \\
E_{\mathrm{y}}=0 & \text { for } z=0 \text { and } z=L \text { planes and } x=0 \text { and } x=L \text { planes. } \\
E_{\mathrm{z}}=0 & \text { for } x=0 \text { and } x=L \text { planes and } y=0 \text { and } y=L \text { planes. }
\end{array}
$$

From the condition

$$
\nabla \cdot \widetilde{\boldsymbol{E}}=0
$$

we have

$$
\begin{aligned}
& E_{x}=E_{1} \cos \left(k_{1} x\right) \sin \left(k_{2} y\right) \sin \left(k_{3} z\right), \\
& E_{y}=E_{2} \sin \left(k_{1} x\right) \cos \left(k_{2} y\right) \sin \left(k_{3} z\right), \\
& E_{z}=E_{3} \sin \left(k_{1} x\right) \sin \left(k_{2} y\right) \cos \left(k_{3} y\right)
\end{aligned}
$$

From the condition

$$
\nabla \times \widetilde{\mathbf{E}}_{0}=i \omega \widetilde{\mathbf{B}}_{0}
$$

we have

$$
B_{x}=B_{1} \sin \left(k_{1} x\right) \cos \left(k_{2} y\right) \cos \left(k_{3} z\right),
$$

$$
\begin{aligned}
& B_{y}=B_{2} \cos \left(k_{1} x\right) \sin \left(k_{2} y\right) \cos \left(k_{3} z\right), \\
& B_{z}=B_{3} \cos \left(k_{1} x\right) \cos \left(k_{2} y\right) \sin \left(k_{3} z\right)
\end{aligned}
$$

where

$$
\begin{array}{ll}
B_{\mathrm{x}}=0 & \text { for } x=0 \text { and } x=L \text { planes } \\
B_{\mathrm{y}}=0 & \text { for } y=0 \text { and } y=L \text { planes. } \\
B_{\mathrm{z}}=0 & \text { for } z=0 \text { and } z=L \text { planes. }
\end{array}
$$

We note that

$$
\nabla \cdot \boldsymbol{E}=\left(E_{1} k_{1}+E_{2} k_{2}+E_{3} k_{3}\right) \sin \left(k_{1} x\right) \sin \left(k_{2} y\right) \sin \left(k_{3} z\right)=0
$$

This means that the vector $\left(E_{1}, E_{2}, E_{3}\right)$ is perpendicular to the wave vector $\boldsymbol{k}=\left(k_{1}, k_{2}, k_{3}\right)$. For each $\boldsymbol{k}$, there are two independent directions for $\left(E_{1}, E_{2}, E_{3}\right)$; polarization.


## 2. Density of states for the modes

Since $E_{1} k_{1}+E_{2} k_{2}+E_{3} k_{3}=0$, only one of $k_{1}, k_{2}, k_{3}$ can be zero at a time. Since if two or three are zero, $E_{1}=E_{2}=E_{3}=0$. There is no electromagnetic field in the cavity. Each set of integers ( $n_{\mathrm{x}}, n_{\mathrm{y}}, n_{\mathrm{z}}$ ) defines a mode of the radiation field and corresponds to two degrees of freedom of the field when two polarization directions are taken into account.


There are 2 states per $\left(\frac{\pi}{L}\right)^{3}$.

$$
\omega=c k=c \sqrt{k_{x}^{2}+k_{y}^{2}+k_{z}^{2}}
$$

or

$$
\frac{\omega^{2}}{c^{2}}=k_{x}^{2}+k_{y}^{2}+k_{z}^{2}
$$

The density of states ( $k$ to $k+\mathrm{d} k$ )

$$
\rho_{k} d k=\frac{1}{8} \frac{4 \pi k^{2} d k}{\left(\frac{\pi}{L}\right)^{3}} 2=\frac{V k^{2} d k}{\pi^{2}}
$$

where $V=L^{3}$.


Since $\omega=c k$,

$$
\rho_{\omega} d \omega=\frac{V\left(\frac{\omega}{c}\right)^{2} d \frac{\omega}{c}}{\pi^{2}}=\frac{V \omega^{2} d \omega}{\pi^{2} c^{3}}
$$

of modes having their frequencies between $\omega$ and $\omega+\mathrm{d} \omega$.

$$
\rho_{\omega}=\frac{V \omega^{2}}{\pi^{2} c^{3}}=V D(\omega) \quad(\text { density of modes })
$$

where $c$ is the velocity of light and

$$
D(\omega)=\frac{\omega^{2}}{\pi^{2} c^{3}}
$$

We have the following formula;

$$
\sum_{k} \rightarrow \int \rho_{k} d k
$$

or

$$
\sum_{k} \rightarrow \int \rho_{\omega} d \omega=V \int D(\omega) d \omega
$$

For single mode $|\boldsymbol{k}\rangle$, the energy is given by

$$
E_{n \cdot \mathbf{k}}=\left(n_{k}+\frac{1}{2}\right) \hbar \omega_{\mathbf{k}} .
$$

We use the Planck distribution. The total energy is given by

$$
E_{t o t}=\sum_{\mathbf{k}} n_{\mathbf{k}} \hbar \omega_{\mathbf{k}}=\int \frac{V \omega^{2}}{\pi^{2} c^{3}} d \omega n_{\omega} \hbar \omega=V \int u(\omega) d \omega
$$

or the energy density by

$$
\frac{E_{t o t}}{V}=\int_{0}^{\infty} u(\omega) d \omega=\int_{0}^{\infty} u(\lambda) d \lambda
$$

where

$$
u(\omega)=\bar{W}_{T}=\frac{\hbar \omega^{3}}{\pi^{2} c^{3}} \frac{1}{\exp \left(\frac{\hbar \omega}{k_{B} T}\right)-1}=\frac{\hbar \omega^{3}}{\pi^{2} c^{3}} \frac{1}{\exp (x)-1}=\frac{k_{B}^{3} T^{3}}{\pi^{2} \hbar^{2} c^{3}} \frac{x^{3}}{\exp (x)-1}
$$

(Planck's law for the radiation energy density). It is clear that

$$
\frac{u(\omega)}{\frac{k_{B}^{3} T^{3}}{\pi^{2} \hbar^{2} c^{3}}}=f(x)=\frac{x^{3}}{\exp (x)-1}
$$

is dependent on a variable $x$ given by

$$
x=\frac{\hbar \omega}{k_{B} T} .
$$

(scaling relation). The experimentally observed spectral distribution of the black body radiation is very well fitted by the formula discovered by Planck.
(1) Region of Wien $\left(x=\frac{\hbar \omega}{k_{B} T} \gg 1\right)$,

$$
u_{W}(\omega)=\frac{k_{B}^{3} T^{3}}{\pi^{2} \hbar^{2} c^{3}} x^{3} e^{-x}
$$

(2) Region of Rayleigh-Jeans $\left(x=\frac{\hbar \omega}{k_{B} T} \gg 1\right)$,

$$
u_{R J}(\omega)=\frac{k_{B}^{3} T^{3}}{\pi^{2} \hbar^{2} c^{3}} \frac{x^{3}}{\exp (x)-1} \approx \frac{k_{B}^{3} T^{3}}{\pi^{2} \hbar^{2} c^{3}} x^{2}
$$



Fig. Scaling plot of $f(x)$ vs $x$ for the Planck's law for the energy density of electromagnetic radiation at angular frequency $\omega$ and temperature $T$. Planck (red). Wien (blue, particle-like). Rayleigh-Jean (green, wave-like).


Fig. Scaling plot of Planck's law. Wien's law, and Rayleigh-Jean's law.

## 3. Another approach for the density of states

We consider a cavity filled with electromagnetic radiation in thermal equilibrium with the fixed wall, with a periodic boundary condition. There are two independent transverse waves doe each mode (polarization). The wave function of photon is expressed by a plane wave,

$$
\psi(r)=A e^{i k \cdot r}
$$

with the energy eigen value

$$
\varepsilon_{\boldsymbol{k}}=\hbar c|\boldsymbol{k}|,
$$

The periodic boundary condition requires that $k_{i}$ satisfies

$$
k_{x}=\frac{2 \pi}{L} n_{x}, \quad k_{y}=\frac{2 \pi}{L} n_{y}, \quad k_{z}=\frac{2 \pi}{L} n_{z}
$$

with $n_{x}, n_{y} n_{z}=0, \pm 1, \pm 2, \ldots$. There are two states (including two polarization vectors) per

$$
\frac{2}{\left(\frac{2 \pi}{L}\right)^{3}}=\frac{2 L^{3}}{(2 \pi)_{3}}=\frac{V}{4 \pi^{3}}
$$

of the $\boldsymbol{k}$ space.


Fig. The energy dispersion for one dimensional photon. The state (denoted by red point) is quantized from the periodic boundary condition. There are two photon polarizations for each state.


Fig. Density of states in the 3D $\boldsymbol{k}$-space. There is one state per $(2 \pi / L)^{3}$. Each state is denoted by red points.

We consider the number of one-photon levels in the energy range from $\omega$ to $\omega+d \omega$; $D(\omega) d \omega D(\varepsilon) \mathrm{d} \varepsilon$

$$
D(\omega) d \omega=\frac{2 V}{(2 \pi)^{3}} 4 \pi k^{2} d k=\frac{V}{\pi^{2}} k^{2} d k,
$$

where $V=L^{3}$, and $D(\omega)$ is called a density of states. Since $\omega=c k$, we have

$$
d k=\frac{d \omega}{c} .
$$

Then we get the density of states

$$
D(\omega) d \omega=\frac{V}{\pi^{2}} \frac{\omega^{2}}{c^{3}} d \omega
$$

$$
D(\omega)=\frac{V}{\pi^{2}} \frac{\omega^{2}}{c^{3}}
$$

The total number of states below $\omega$

$$
N(\omega)=\int_{0}^{\omega} D(\omega) d \omega=\frac{V}{\pi^{2} c^{3}} \int_{0}^{\omega} \omega^{2} d \omega=\frac{V \omega^{3}}{3 \pi^{2} c^{3}} .
$$

The total number of photons in the black body radiation is

$$
\begin{aligned}
N_{t o t} & =\int_{0}^{\infty} D(\omega) \frac{d \omega}{e^{\beta \hbar \omega}-1} \\
& =\frac{V}{\pi^{2} c^{3}} \int_{0}^{\infty} \frac{\omega^{2} d \omega}{e^{\beta \hbar \omega}-1} \\
& =\frac{V k_{B}^{3} T^{3}}{\pi^{2} c^{3} \hbar^{3}} \int_{0}^{\infty} \frac{x^{2} d x}{e^{x}-1} \\
& =\frac{2 \varsigma(3)}{\pi^{2}} \frac{V k_{B}^{3} T^{3}}{c^{3} \hbar^{3}} \\
& =0.24359 \frac{V k_{B}^{3} T^{3}}{c^{3} \hbar^{3}}
\end{aligned}
$$

where

$$
\int_{0}^{\infty} \frac{x^{2} d x}{e^{x}-1}=2 \varsigma(3)=2.4041
$$

The number density of photons is

$$
n=\frac{N_{t o t}}{V}=0.24359 \frac{k_{B}^{3} T^{3}}{c^{3} \hbar^{3}}=2.02871 \times 10^{7} T^{3}\left(1 / \mathrm{m}^{3}\right)=20.2871 T^{3}\left(1 / \mathrm{cm}^{3}\right)
$$

When $T=2.73 \mathrm{~K}, \quad n=4.128 \times 10^{8} / \mathrm{m}^{3}$ for the entire observable universe (CMB).
When $T=300 \mathrm{~K}, \quad n=5.478 \times 10^{14} / \mathrm{m}^{3}$ (room temperature)
When $T=1500 \mathrm{~K} \quad n=6.847 \times 10^{16} / \mathrm{m}^{3}$.
When $T=6000 \mathrm{~K}, \quad n=4.382 \times 10^{18} / \mathrm{cm}^{3}$ (the temperature of sun's surface)

## 4. Deivation of $u(\lambda, T)$

The total energy:

$$
\begin{aligned}
U & =\langle E\rangle \\
& =\int d \omega D(\omega) \frac{\hbar \omega}{e^{\beta \hbar \omega}-1} \\
& =\frac{V}{\pi^{2} c^{3}} \int d \omega \frac{\hbar \omega^{3}}{e^{\beta \hbar \omega}-1}
\end{aligned}
$$

The energy density:

$$
\varepsilon=\frac{U}{V}=\frac{\hbar}{\pi^{2} c^{3}} \int_{0}^{\infty} d \omega \frac{\omega^{3}}{e^{\beta \hbar \omega}-1}
$$

or

$$
\varepsilon=\int_{0}^{\infty} u(\omega) d \omega=\frac{\hbar}{\pi^{2} c^{3}} \int_{0}^{\infty} \frac{\omega^{3}}{e^{\beta \hbar \omega}-1} d \omega
$$

With $\quad \beta=\frac{1}{k_{B} T}$. The energy density can be calculated as

$$
\varepsilon=\frac{k_{B}^{4} T^{4}}{\pi^{2} c^{3} \hbar^{3}} \int_{0}^{\infty} \frac{x^{3} d x}{e^{x}-1}=\frac{k_{B}^{4} T^{4}}{\pi^{2} c^{3} \hbar^{3}} \frac{\pi^{4}}{15}=\frac{k_{B}^{4} \pi^{2} T^{4}}{15 c^{3} \hbar^{3}}=\frac{4}{c} \sigma_{S B} T^{4}
$$

where $x=\beta \hbar \omega$. The Stefan-Boltzmann constant $\sigma_{S B}$ is

$$
\sigma_{S B}=\frac{\pi^{2} k_{B}{ }^{4}}{60 c^{2} \hbar^{3}}=0.5670400 \times 10^{-4} \mathrm{erg} /\left(\mathrm{s}-\mathrm{cm}^{2}-\mathrm{K}^{4}\right)=5.670400 \times 10^{-8} \mathrm{~W} \mathrm{~m}^{-2} \mathrm{~K}^{-4}
$$

and

$$
\int_{0}^{\infty} \frac{x^{3} d x}{e^{x}-1}=\frac{\pi^{4}}{15}
$$

We note that

$$
\int_{0}^{\infty} u(\omega) d \omega=\int_{0}^{\infty} u(\lambda) d \lambda
$$

Since $\omega=\frac{2 \pi c}{\lambda}, \quad d \omega=-2 \pi c \frac{d \lambda}{\lambda^{2}}$

$$
\begin{aligned}
\int_{0}^{\infty} u(\omega) d \omega & =\int_{0}^{\infty} \frac{\hbar \omega^{3}}{\pi^{2} c^{3}} \frac{1}{\exp \left(\frac{\hbar \omega}{k_{B} T}\right)-1} d \omega \\
& =\int_{0}^{\infty} \frac{\hbar\left(\frac{2 \pi c}{\lambda}\right)^{3}}{\pi^{2} c^{3}} \frac{1}{\exp \left(\frac{2 \pi \hbar c}{\lambda k_{B} T}\right)-1} 2 \pi c \frac{d \lambda}{\lambda^{2}}
\end{aligned}
$$

or

$$
\int_{0}^{\infty} u(\omega) d \omega=\int_{0}^{\infty} u(\lambda) d \lambda=\int_{0}^{\infty} 16 \pi^{2} \hbar c \frac{1}{\lambda^{5}} \frac{1}{\exp \left(\frac{2 \pi \hbar c}{\lambda k_{B} T}\right)-1} d \lambda
$$

Then we have

$$
u(\lambda)=\frac{16 \pi^{2} \hbar c}{\lambda^{5}} \frac{1}{\exp \left(\frac{2 \pi \hbar c}{\lambda k_{B} T}\right)-1}
$$

where

$$
\begin{aligned}
& \hbar=1.054571596 \times 10^{-27} \mathrm{erg} \mathrm{~s}, \quad k_{\mathrm{B}}=1.380650324 \times 10^{-16} \mathrm{erg} / \mathrm{K} \\
& c=2.99792458 \times 10^{10} \mathrm{~cm} / \mathrm{s} . \\
& \mathrm{J}=10^{7} \mathrm{erg}
\end{aligned}
$$

## 5. Wien's displacement law

$u(\lambda)$ has a maximum when

$$
1-e^{-x}=\frac{x}{5}
$$

where

$$
\begin{equation*}
x=\frac{2 \pi \hbar c}{\lambda k_{B} T}, \tag{dimensionless}
\end{equation*}
$$

The solution of the above equation is

$$
x=4.96511
$$

or

$$
\lambda=\frac{0.28977}{T(K)} \quad(\lambda \text { in the units of } \mathrm{cm})
$$

or

$$
\lambda=\frac{2.897768551}{T(K)} \times 10^{6} . \quad(\lambda \text { in the units of } \mathrm{nm})
$$

$T$ is the temperature in the units of $\mathrm{K} . \lambda$ is the wave-length in the unit of nm

| $T(\mathrm{~K})$ | $\lambda(\mathrm{nm})$ |
| :--- | :--- |
| 1000 | 2897.77 |
| 1500 | 1931.85 |
| 2000 | 1448.89 |
| 2500 | 1159.11 |
| 3000 | 965.924 |
| 3500 | 827.935 |
| 4000 | 724.443 |
| 4500 | 643.949 |
| 5000 | 579.554 |
| 5500 | 526.867 |
| 6000 | 482.962 |
| 6500 | 445.811 |
| 7000 | 413.967 |
| 7500 | 386.369 |
| 8000 | 362.221 |
| 8500 | 340.914 |
| 9000 | 321.975 |
| 9500 | 305.029 |
| 10000 | 289.777 |



Fig. Wien's displacement law. The peak wavelength vs temperature $T(\mathrm{~K})$.

## 6. Entropy of photon gas

The heat capacity of photon gas is

$$
C_{V}(T)=\left(\frac{\partial U}{\partial T}\right)_{V}=4 a T^{4}
$$

where

$$
U(T)=a T^{4}=\frac{4 \sigma_{S B} V}{c} T^{4} . \text { The entropy is obtained as }
$$

$$
S(T)=\int_{0}^{T} \frac{C_{V}\left(T^{\prime}\right)}{T^{\prime}} d T^{\prime}=4 a \int_{0}^{T} T^{\prime 2} d T^{\prime}=\frac{4 a}{3} T^{3}=\frac{16 \sigma_{S B} V}{3 c} T^{3}
$$

The total number of photons

$$
N_{t o t}=0.24359 \frac{V k_{B}^{3} T^{3}}{c^{3} \hbar^{3}}
$$

is given by the same formula as the entropy. The entropy is related to $U$ as

$$
S=\frac{4 U}{3 T} .
$$

In the adiabatic process

$$
S(T)=\frac{16 \sigma_{S B} V}{3 c} T^{3}=\text { constant }
$$

we have

$$
V T^{3}=\text { constant }
$$

## 7. Rate of the energy flux density



Fig. Experimental realization of a black body problem (from Bellac et al.)


Fig. Convention for the axis in the calculation for black body radiation (from Bellac et al.).


Fig. The photon energy passing in the time $\Delta t$ through a solid angle $\mathrm{d} \Omega$, making an angle $\theta$ with the normal to $\mathrm{d} A$. The volume of cylinder: $(c \cos \theta \Delta t) d A$ during the time $\Delta t . d \Omega=2 \pi \sin \theta . c$ is the velocity of light.

It is assumed that the thermal equilibrium of the electromagnetic waves is not disturbed even when a small hole is bored through the wall of the box. The area of the hole is $\mathrm{d} A$. The energy which passes in unit time through a solid angle $\mathrm{d} \Omega$, making an angle $\theta$ with the normal to $\mathrm{d} A$ is

$$
J(\lambda, T, \theta) d \lambda d \Omega d A=u(\lambda, T) d \lambda(c \cos \theta d A) \frac{d \Omega}{4 \pi}
$$

where $c$ is the velocity of light. The right hand side is divided by $4 \pi$, because the energy density $u$ comprises all waves propagating along different directions. The emitted energy per unit time, per unit area is

$$
\begin{aligned}
\iint J(\lambda, T, \theta) d \lambda d \Omega & =\int c u(\lambda, T) d \lambda \int \cos \theta \frac{d \Omega}{4 \pi}=\int \frac{c u(\lambda, T)}{4} d \lambda \\
& \equiv \frac{c}{4} \int u(\lambda, T) d \lambda \\
& =\frac{c}{4} \varepsilon
\end{aligned}
$$

where

$$
\begin{aligned}
& \varepsilon=\int u(\lambda, T) d \lambda \\
& \begin{aligned}
\int \cos \theta \frac{d \Omega}{4 \pi} & =\frac{1}{4 \pi} \int_{0}^{2 \pi} d \phi \int_{0}^{\pi / 2} \cos \theta \sin \theta d \theta \\
& =\frac{1}{4 \pi} 2 \pi \int_{0}^{\pi / 2} \frac{1}{2} \sin (2 \theta) d \theta \\
& =\left.\frac{1}{4} \frac{1}{2}[-\cos (2 \theta)]\right|_{0} ^{\pi / 2}=\frac{1}{4}
\end{aligned}
\end{aligned}
$$



Fig. Radiation intensity is used to describe the variation of radiation energy with direction.

In other words, the geometrical factor is equal to $1 / 4$. Then we have a measure for the intensity of radiation (the rate of energy flux density);

$$
S(\lambda, T)=\frac{c u(\lambda, T)}{4}=\frac{4 \pi^{2} \hbar c^{2}}{\lambda^{5}} \frac{1}{\exp \left(\frac{2 \pi \hbar c}{\lambda k_{B} T}\right)-1}
$$

where

$$
S(\lambda, T) d \lambda=\text { power radiated per unit area in }(\lambda, \lambda+\mathrm{d} \lambda)
$$

Unit

$$
\left[\hbar c^{2} \frac{1}{\lambda^{5}}\right]=\frac{\mathrm{erg} \cdot \mathrm{~s}}{\mathrm{~cm}^{5}} \frac{\mathrm{~cm}^{2}}{\mathrm{~s}^{2}}=\frac{\mathrm{erg}}{\mathrm{~cm}^{3}} \frac{1}{\mathrm{~s}}=\frac{10^{-7} \mathrm{~J}}{\left(10^{-2} \mathrm{~m}\right)^{3}} \frac{1}{\mathrm{~s}}=10^{-1} \frac{\mathrm{~W}}{\mathrm{~m}^{3}}=\left[\frac{\mathrm{W}}{\mathrm{~m}^{3}}\right]
$$

The energy flux density $S(\lambda, T)$ is defined as the rate of energy emission per unit area.
((Note)) The unit of the poynting vector $\langle S\rangle$ is [W/m $\left.\mathrm{m}^{2}\right] .\langle S\rangle$ is the energy flux (energy per unit area per unit time).
(1) Rayleigh-Jeans law (in the long-wavelength limit)

$$
S_{R J}(\lambda)=\frac{1}{4} c u_{R J}(\lambda)=4 \pi^{2} \hbar c \frac{1}{\lambda^{5}} \frac{1}{\frac{2 \pi \hbar c}{\lambda k_{B} T}}=\frac{2 \pi k_{B} T}{\lambda^{4}}
$$

for

$$
\frac{\lambda k_{B} T}{2 \pi \hbar c} \gg 1
$$

(2) Wien's law (in short-wavelength limit)

$$
\begin{aligned}
& S_{W}(\lambda)=\frac{1}{4} c u_{W}(\lambda)=\frac{\pi^{2} \hbar c}{\lambda^{5}} \exp \left(-\frac{2 \pi \hbar c}{\lambda k_{B} T}\right) \\
& \frac{\lambda k_{B} T}{2 \pi \hbar c} \ll 1
\end{aligned}
$$

We make a plot of $S(\lambda, T)$ as a function of the wavelength, where $S(\lambda, T)$ is in the units of $\mathrm{W} / \mathrm{m}^{3}$ and the wavelength is in the units of nm .


Fig. $\quad \mathrm{c} u(\lambda) / 4\left(\mathrm{~W} / \mathrm{m}^{3}\right)$ vs $\lambda(\mathrm{nm}) . T=2 \times 10^{3} \mathrm{~K}$. Red [Planck]. Green [Wien]. Blue [Rayleigh-Jean]. Wien's displacement law: The peak appears at $\lambda=1448.89 \mathrm{~nm}$
for $T=2 \times 10^{3} \mathrm{~K}$. This figure shows the misfit of Wien's law at long wavelength and the failure of the Rayleigh-Jean's law at short wavelangth.


Fig. (a) and (b) $\mathrm{c} u(\lambda) / 4\left(\mathrm{~W} / \mathrm{m}^{3}\right)$ vs $\lambda(\mathrm{nm})$ for the Plank's law. $T=1000 \mathrm{~K}$ (red), 1500 K, $2000 \mathrm{~K}, 2500 \mathrm{~K}, 3000 \mathrm{~K}$ (blue), $3500 \mathrm{~K}, 4000 \mathrm{~K}$ (purple), 4500 K , and 5000 K . The peak shifts to the higher wavelength side as $T$ decreases according to the Wien's displacement law.


Fig. Power spectrum of sun. $\mathrm{c} u(\lambda) / 4\left(\mathrm{~W} / \mathrm{m}^{3}\right)$ vs $\lambda(\mathrm{nm}) . T=5778 \mathrm{~K}$. The peak wavelength is 501.52 nm according to the Wien's displacement law.


Fig. Power spectrum of cosmic blackbody radiation at $T=2.726 \mathrm{~K}$. The peak wavelength is 1.063 mm (Wien's displacement law.

## 8. Stefan-Boltzmann radiation law for a black body (1879).

Joseph Stefan (24 March 1835-7 January 1893) was a physicist, mathematician and poet of Slovene mother tongue and Austrian citizenship.

http://en.wikipedia.org/wiki/Joseph_Stefan

Ludwig Eduard Boltzmann (February 20, 1844 - September 5, 1906) was an Austrian physicist famous for his founding contributions in the fields of statistical mechanics and statistical thermodynamics. He was one of the most important advocates for atomic theory at a time when that scientific model was still highly controversial.

http://en.wikipedia.org/wiki/Ludwig_Boltzmann
The total energy per unit volume is given by

$$
\varepsilon=\frac{E_{t o t}}{V}=\int u(\omega) d \omega=\int u(\lambda) d \lambda=\frac{\left(k_{B} T\right)^{4}}{\pi^{2} \hbar^{3} c^{3}} \int_{0}^{\infty} \frac{x^{3}}{\exp (x)-1}=\frac{\pi^{2}\left(k_{B} T\right)^{4}}{15 \hbar^{3} c^{3}}
$$

where

$$
\begin{aligned}
& \int_{0}^{\infty} \frac{x^{3}}{e^{x}-1} d x \\
& \frac{\pi^{4}}{15}
\end{aligned}
$$

A spherical enclosure is in equilibrium at the temperature $T$ with a radiation field that it contains. The power emitted through a hole of unit area in the wall of enclosure is

$$
P_{e}=\frac{1}{4} c \varepsilon=\frac{\pi^{2} k_{B}^{4}}{60 \hbar^{3} c^{2}} T^{4}=\sigma_{S B} T^{4}
$$

where $\sigma_{S B}$ is the Stefan-Boltzmann constant

$$
\sigma_{S B}=\frac{\pi^{2} k_{B}^{4}}{60 \hbar^{3} c^{2}}=0.5670400 \times 10^{-4} \mathrm{erg} /\left(\mathrm{s}-\mathrm{cm}^{2}-\mathrm{K}^{4}\right)=5.670400 \times 10^{-8} \mathrm{~W} \mathrm{~m}^{-2} \mathrm{~K}^{-4}
$$

and the geometrical factor is equal to $1 / 4$. The application of the Stefan-Boltzmann law is discussed in lecture notes of Phys. 131 (Chapter 18) (see URL at
http://bingweb.binghamton.edu/~suzuki/GeneralPhysLN.html

## 9. Duality of wave and particle

Region of Rayleigh-Jeans: wave-like nature
Region of Wien:
particle-like nature

The mean energy contained in a volume $\Delta V$ in the frequency range between $\omega$ and $\omega+\Delta \omega$, is given by

$$
\bar{E}(\omega)=\langle E(\omega)\rangle=\Delta V \bar{W}_{T}(\omega) \Delta \omega=\Delta V D(\omega) \hbar \omega \Delta \omega \bar{n}=\Delta V D(\omega) \hbar \omega \Delta \omega \frac{1}{e^{\beta \hbar \omega}-1}
$$

where

$$
\bar{n}=\frac{1}{e^{\beta \hbar \omega}-1},
$$

and

$$
\begin{aligned}
& D(\omega)=\frac{\omega^{2}}{\pi^{2} c^{3}}, \quad \bar{n}=\frac{1}{e^{\beta \hbar \omega}-1} \\
& \bar{W}_{T}(\omega)=\frac{\omega^{2}}{\pi^{2} c^{3}} \frac{\hbar \omega}{e^{\beta \hbar \omega}-1}=D(\omega) \hbar \omega \bar{n}=D(\omega) \hbar \omega \frac{1}{e^{\beta \hbar \omega}-1}
\end{aligned}
$$

The mean-square of the fluctuation in energy is obtained as

$$
[\Delta E(\omega)]^{2}=\left\langle[E(\omega)]^{2}\right\rangle-\langle E(\omega)\rangle^{2}=k_{B} T^{2} \frac{\partial}{\partial T}\langle E(\omega)\rangle
$$

from the general theory of thermodynamics,
or

$$
\begin{aligned}
{[\Delta E(\omega)]^{2} } & =\Delta V \Delta \omega D(\omega) \hbar \omega k_{B} T^{2} \frac{\partial}{\partial T} \frac{1}{e^{\beta \hbar \omega}-1} \\
& =\Delta V \Delta \omega D(\omega) \hbar^{2} \omega^{2}\left[\frac{1}{e^{\beta \hbar \omega}-1}+\frac{1}{\left(e^{\beta \hbar \omega}-1\right)^{2}}\right]
\end{aligned}
$$

or

$$
\begin{aligned}
{[\Delta E(\omega)]^{2} } & =\Delta V \Delta \omega D(\omega) \hbar^{2} \omega^{2}\left[\bar{n}+\bar{n}^{2}\right] \\
& =\Delta V \Delta \omega D(\omega) \hbar^{2} \omega^{2}(\Delta n)^{2}
\end{aligned}
$$

where

$$
\bar{n}+\bar{n}^{2}=(\Delta n)^{2}
$$

(See the Appendix for the detail). Note that
$(\Delta n)^{2}=<(n-\bar{n})^{2}>=\overline{n^{2}}-\bar{n}^{2}$ (from the definition).

## (i) Rayleigh-Jean (wave-like)

$$
\begin{aligned}
& \text { For } \frac{\hbar \omega}{k_{B} T}=\beta \hbar \omega \ll 1, \quad \bar{n}^{2} \gg \bar{n} \\
& (\Delta n)^{2} \approx \bar{n}^{2}, \quad \text { or } \quad(\Delta n) \approx \bar{n} \quad \text { (wave-like, Rayleigh-Jeans) } \\
& {[\Delta E(\omega)]^{2} \approx \Delta V \Delta \omega D(\omega) \hbar^{2} \omega^{2} \bar{n}^{2}}
\end{aligned}
$$

Then we have

$$
\frac{[\Delta E(\omega)]^{2}}{\langle E(\omega)\rangle^{2}}=\frac{\Delta V \Delta \omega D(\omega) \hbar^{2} \omega^{2} \bar{n}^{2}}{(\Delta V \Delta \omega D(\omega) \hbar \omega \bar{n})^{2}}=\frac{1}{\Delta V \Delta \omega D(\omega)}=\frac{1}{\Delta V \Delta \omega} \frac{c^{3}}{\omega^{2}}
$$

or

$$
[\Delta E(\omega)]^{2}=\frac{1}{\Delta V \Delta \omega} \frac{c^{3}}{\omega^{2}}\langle E(\omega)\rangle^{2}
$$

(ii) Wien (particle-like)

$$
\begin{aligned}
& \text { For } \frac{\hbar \omega}{k_{B} T}=\beta \hbar \omega \gg 1, \quad \bar{n}^{2}<\bar{n} \\
& (\Delta n)^{2} \approx \bar{n} \quad(\text { particle-like, corpuscle, Wien) } \\
& {[\Delta E(\omega)]^{2} \approx \Delta V \Delta \omega D(\omega) \hbar^{2} \omega^{2} \bar{n}=\hbar \omega(\Delta V \Delta \omega D(\omega) \hbar \omega \bar{n})=\hbar \omega\langle E(\omega)\rangle}
\end{aligned}
$$

or

$$
\frac{[\Delta E(\omega)]^{2}}{\langle E(\omega)\rangle}=\hbar \omega, \quad \text { or } \quad[\Delta E(\omega)]^{2}=\hbar \omega\langle E(\omega)\rangle
$$

(iii) Planck

$$
[\Delta E(\omega)]^{2}=\hbar \omega\langle E(\omega)\rangle+\frac{1}{\Delta V \Delta \omega} \frac{c^{3}}{\omega^{2}}\langle E(\omega)\rangle^{2}
$$

## 10. Einstein A and B coefficient

$$
\bar{W}_{T}(\omega)=\frac{\omega^{2}}{\pi^{2} c^{3}} \frac{\hbar \omega}{e^{\beta \hbar \omega}-1}
$$

Planck's law for the radiative energy density (Black body)
Suppose that a gas of $N$ identical atoms is placed in the interior of the cavity:

$$
\hbar \omega=E_{2}-E_{1} .
$$

Two atomic levels are not degenerate.
$N_{1}, N_{2}$ : level population


$$
\bar{W}(\omega)=\overline{W_{T}}(\omega)+\overline{W_{E}}(\omega)
$$

$\bar{W}(\omega)$ : cycle-average energy density of radiation at $\omega$ $\overline{W_{T}}(\omega)$ : thermal part
$\overline{W_{E}}(\omega)$ : contribution from some external source of electromagnetic radiation


$$
\left\{\begin{array}{l}
\frac{d N_{1}}{d t}=A_{21} N_{2}-N_{1} B_{12} \bar{W}(\omega)+N_{2} B_{21} \bar{W}(\omega) \\
\frac{d N_{2}}{d t}=-A_{21} N_{2}+N_{1} B_{12} \bar{W}(\omega)-N_{2} B_{21} \bar{W}(\omega)
\end{array}\right.
$$

Case of thermal equilibrium

$$
\frac{d N_{1}}{d t}=\frac{d N_{2}}{d t}=0
$$

or

$$
N_{2} A_{21}-N_{1} B_{12} \bar{W}(\omega)+N_{2} B_{21} \bar{W}(\omega)=0
$$

For thermal equilibrium with no external radiation introduced into the cavity

$$
\begin{aligned}
& \bar{W}(\omega)=\overline{W_{T}}(\omega) \\
& \overline{W_{T}}(\omega)=\frac{A_{21}}{\left(\frac{N_{1}}{N_{2}} B_{12}-B_{21}\right)}
\end{aligned}
$$

The level populations $N_{1}$ and $N_{2}$ are related in thermal equilibrium by Boltzman's law

$$
\frac{N_{1}}{N_{2}}=\frac{e^{-\beta E_{1}}}{e^{-\beta E_{2}}}=\exp (\beta \hbar \omega),\left(\beta=1 / k_{\mathrm{B}} T\right)
$$

Then

$$
\bar{W}_{T}(\omega)=\frac{A_{21}}{B_{12} e^{\beta \hbar \omega}-B_{21}}
$$

which is compared with the Planck's law

$$
\begin{aligned}
& \bar{W}_{T}(\omega)=\frac{\left(\frac{\hbar \omega^{3}}{\pi^{2} c^{3}}\right)}{e^{\beta \hbar \omega}-1} \\
& \Rightarrow \quad\left\{\begin{array}{l}
B_{12}=B_{21} \\
\frac{A_{21}}{B_{12}}=\frac{\hbar \omega^{3}}{\pi^{2} c^{3}}
\end{array}\right. \\
& \bar{W}_{T}(\omega)=\frac{A_{21}-}{B_{12}}, \text { where } \bar{n}=\frac{1}{e^{\beta \hbar \omega}-1}
\end{aligned}
$$

or

$$
\frac{A_{21}}{B_{21} \bar{W}_{T}(\omega)}=e^{\beta \hbar \omega}-1
$$

$(($ Example $)) ~ \hbar \omega=k_{B} T$
For $T=300 \mathrm{~K}, v_{\mathrm{T}}=6 \times 10^{12} \mathrm{~Hz}=6 \mathrm{THz}$
For $\hbar \omega<k_{\mathrm{B}} T, A_{21}<B_{21} \bar{W}_{T}(\omega) \quad\left(v \ll v_{\mathrm{T}}\right)$
For $\hbar \omega » k_{\mathrm{B}} T, A_{21}>B_{21} \bar{W}_{T}(\omega) \quad\left(v » v_{\mathrm{T}}\right)$

For optical experiments that use electromagnetic radiation in the near-infrared, we have visible, ultraviolet region of the spectrum ( $v>5 \mathrm{THz}$ ).

We have
(i) $\quad A_{21} \gg B_{21} \bar{W}_{T}(\omega)$
$A_{21}$ : spontaneous emission rate
$B_{21}$ : rate of thermally stimulated emission
(ii) $\bar{W}(\omega)=\overline{W_{T}}(\omega)+\overline{W_{E}}(\omega) \cong \overline{W_{E}}(\omega)$

Therefore the radioactive process of interest involve the absorption and stimulated emission associated with the external source.

((Note))
Calculation of $\frac{A_{21}}{B_{12} \bar{W}_{T}(\omega)}=\frac{1}{n}=e^{\hbar \omega / k_{B} T}-1$ at $T=300 \mathrm{~K}$ as a typical example. This factor is larger than 1 when $v=4.333 \mathrm{THz}$.

## 11. Example: Blundell-Blundell Problem 23.7

(23.7) Show that the total number $N$ of photons in black body radiation contained in a volume $V$ is

$$
\begin{equation*}
N=\int_{0}^{\infty} \frac{g(\omega) \mathrm{d} \omega}{\mathrm{e}^{\hbar \omega / k_{\mathrm{B}} T}-1}=\frac{2 \zeta(3)}{\pi^{2}}\left(\frac{k_{\mathrm{B}} T}{\hbar c}\right)^{3} V \tag{23.74}
\end{equation*}
$$

where $\zeta(3)=1.20206$ is a Riemann-zeta function (see Appendix C.4). Hence show that the average energy per photon is

$$
\begin{equation*}
\frac{U}{N}=\frac{\pi^{4}}{30 \zeta(3)} k_{\mathrm{B}} T=2.701 k_{\mathrm{B}} T \tag{23.75}
\end{equation*}
$$

and that the average entropy per photon is

$$
\begin{equation*}
\frac{S}{N}=\frac{2 \pi^{4}}{45 \zeta(3)} k_{\mathrm{B}}=3.602 k_{\mathrm{B}} \tag{23.76}
\end{equation*}
$$

The result for the internal energy of a photon gas is therefore $U=2.701 N k_{\mathrm{B}} T$, whereas for a classical ideal gas one obtains $U=\frac{3}{2} N k_{\mathrm{B}} T$. Why should the two results be different? Compare the expression for the entropy of a photon gas with that for an ideal gas (the Sackur-Tetrode equation); what is the physical reason for the difference?
((Solution))

$$
\frac{N}{V}=\frac{k_{B}^{3} T^{3}}{\pi^{2} \hbar^{3} c^{3}} \int_{0}^{\infty} \frac{x^{2} d x}{\exp (x)-1}=2 \varsigma(3) \frac{k_{B}{ }^{3} T^{3}}{\pi^{2} \hbar^{3} c^{3}}
$$

where

$$
\begin{aligned}
& \int_{0}^{\infty} \frac{x^{2} d x}{e^{x}-1}=\Gamma(3) \varsigma(3)=2!\varsigma(3)=2.40411 \\
& \varsigma(3)=1.20206
\end{aligned}
$$

The total energy is given by

$$
\frac{U}{V}=\frac{k_{B}^{4} T^{4}}{\pi^{2} \hbar^{3} c^{3}} \int_{0}^{\infty} \frac{x^{3} d x}{\exp (x)-1}=\frac{k_{B}^{4} T^{4}}{\pi^{2} \hbar^{3} c^{3}} \varsigma(4) \Gamma(4)=3!\varsigma(4) \frac{k_{B}^{4} T^{4}}{\pi^{2} \hbar^{3} c^{3}}
$$

where

$$
\varsigma(4)=\frac{\pi^{4}}{90}=1.0823 .
$$

Then we have the ratio $\frac{U}{N}$ as

$$
\frac{U}{N}=\frac{6 \varsigma(4) \frac{k_{B}{ }^{4} T^{4}}{\pi^{2} \hbar^{3} c^{3}}}{2 \varsigma(3) \frac{k_{B}^{3} T^{3}}{\pi^{2} \hbar^{3} c^{3}}}=\frac{3 \varsigma(4)}{\varsigma(3)} k_{B} T=2.70111 k_{B} T
$$

The entropy is defined by $S=\frac{4 U}{3 T}$

$$
\frac{S}{N}=\frac{4}{3 T} \frac{U}{N}=\frac{4}{3 T} 2.70111 k_{B} T=3.60148 k_{B}
$$

The ideal gas is in a limit where there are far more possible states than particles, permitting quite a low average energy ( $1.5 k_{\mathrm{B}} T$ per particle) but quite high entropy.

## REFERENCES

L.D. Landau and E.M. Lifshitz, Statistical Physics, $3^{\text {rd }}$ edition Part 1 (Pergamon Press, 1993).
C. Kittel and H. Kroemer, Thermal Physics, 2nd edition (W.H. Freeman and Company, New York, 1980).
F. Reif, Fundamentals of statistical and thermal physics (McGraw-Hill Publishing Company, New York, 1965).
E.G. Steward, Quantum Mechanics; Its Early Development and the Road to Entanglement
R. Loudon, The Quantum Theory of Light, 2nd edition (Clarendon Press, Oxford, 1983).
W. Heisenberg, The Physical Principles of the Quantum Theory (Dover Publications, Inc.,1949)
S. Tomonaga, Quantum Mechanics (Misuzu Syobou, Tokyo, 1962).
M.Le Bellac, F. Mortessagne, and G.G. Batrouni, Equilibrium and Non-equilibrium Statistical Thermodynamics (Cambridge, 2004).

## APPENDIX

## A-1 Planck's law

In thermal equilibrium at temperature $T$, the probability $P_{\mathrm{n}}$ that the mode oscillator is thermally excited to the $n$-th excited state is given by the usual Boltzmann factor

$$
P_{n}=\frac{\exp \left(-\frac{E_{n}}{k_{B} T}\right)}{\sum_{n} \exp \left(-\frac{E_{n}}{k_{B} T}\right)} .
$$

The zero-point energy cancels when the quantized energy expression is substituted and, with the shorthand notation

$$
U=\exp \left(-\frac{\hbar \omega}{k_{B} T}\right)
$$

the thermal probability becomes

$$
P_{n}=\frac{U}{\sum_{n=0}^{\infty} U^{n}}=\frac{1}{1-U}
$$

where $0<U<1$. We define that

$$
<n^{m}>=\sum_{n=0}^{\infty} n^{m} P_{n} .
$$

Then we have

$$
\begin{aligned}
& \bar{n}=\langle n\rangle=\sum_{n=0}^{\infty} n^{m} P_{n}=\frac{U}{1-U}=\frac{1}{\exp \left(\frac{\hbar \omega}{k_{B} T}\right)-1} \\
& \left\langle n^{2}>=\frac{U+U^{2}}{(1-U)^{2}}\right. \\
& \left\langle n^{3}>=\frac{U\left(1+4 U+U^{2}\right)}{(1-U)^{3}}\right.
\end{aligned}
$$

$$
\begin{aligned}
(\Delta n)^{2} & =\left\langle n^{2}\right\rangle-\langle n\rangle^{2} \\
& =\frac{U}{(1-U)^{2}}
\end{aligned}
$$

The fluctuation in the number is characterized by the root-mean square deviation $\Delta n$ of the distribution.

$$
\begin{aligned}
(\Delta n)^{2} & =<(n-\bar{n})^{2}> \\
& =<n^{2}>-\bar{n}^{2} \\
& =\frac{U}{(1-U)^{2}}
\end{aligned}
$$

Since

$$
\bar{n}^{2}+\bar{n}=\frac{U^{2}}{(1-U)^{2}}+\frac{U}{1-U}=\frac{U}{(1-U)^{2}}
$$

we get the relation

$$
(\Delta n)^{2}=\bar{n}^{2}+\bar{n} .
$$

## ((Mathematica))

Fluctuation in photon number (Planck distribution)

$$
\begin{aligned}
& \mathrm{P}\left[n_{-}\right]=(1-\mathrm{U}) \mathrm{U}^{\mathrm{n}} ; \\
& \mathrm{K}\left[m_{-}\right]:=\sum_{\mathrm{n}=0}^{\infty}\left(\mathrm{n}^{m} \mathrm{P}[\mathrm{n}]\right) / / \text { Simplify }[\#, 0<\mathrm{U}<1] \& ; \\
& \mathrm{K}[1] / / \text { Simplify } \\
& -\frac{\mathrm{U}}{-1+\mathrm{U}} \\
& \mathrm{~K}[2] / / \text { Simplify } \\
& \frac{\mathrm{U}(1+\mathrm{U})}{(-1+U)^{2}}
\end{aligned}
$$

K[3] // Simplify

$$
-\frac{U\left(1+4 U+U^{2}\right)}{(-1+U)^{3}}
$$

$$
\mathrm{K}[2]-\mathrm{K}[1]^{2} / / \text { Simplify }
$$

$$
\frac{U}{(-1+U)^{2}}
$$

$$
K[1]+K[1]^{2} / / \text { Simplify }
$$

$$
\frac{U}{(-1+U)^{2}}
$$

## A-2 Energy fluctuation

$$
\begin{aligned}
(\Delta E)^{2} & =\left\langle(E-\langle E\rangle)^{2}\right\rangle \\
& =\left\langle E^{2}\right\rangle-\langle E\rangle^{2} \\
& =k_{B} T^{2} \frac{\partial}{\partial T}\langle E\rangle \\
\langle E\rangle= & \sum_{n} n \hbar \omega P_{n}=\hbar \omega\langle n\rangle
\end{aligned}
$$

$$
\begin{aligned}
\left\langle E^{2}\right\rangle & =\sum_{n}(n \hbar \omega)^{2} P_{n} \\
& =\hbar^{2} \omega^{2}\left\langle n^{2}\right\rangle \\
& =\hbar^{2} \omega^{2}\left\langle n^{2}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
(\Delta E)^{2} & =\hbar^{2} \omega^{2}\left[\left\langle n^{2}\right\rangle-\langle n\rangle^{2}\right] \\
& =\hbar^{2} \omega^{2}(\Delta n)^{2} \\
& =\hbar^{2} \omega^{2}\left[\langle n\rangle^{2}+\langle n\rangle\right] \\
k_{B} T^{2} \frac{\partial}{\partial T}\langle E\rangle & =k_{B} T^{2} \hbar \omega \frac{\partial}{\partial T}\langle n\rangle \\
& =k_{B} T^{2} \hbar \omega \frac{\partial \beta}{\partial T} \frac{\partial}{\partial \beta}\left(\frac{1}{e^{\beta \hbar \omega}-1}\right) \\
& =k_{B} T^{2} \hbar \omega\left(-\frac{1}{k_{B} T^{2}}\right) \frac{-\hbar \omega e^{\beta \hbar \omega}}{\left(e^{\beta \hbar \omega}-1\right)^{2}} \\
& =(\hbar \omega)^{2} \frac{e^{\beta \hbar \omega}}{\left(e^{\beta \hbar \omega}-1\right)^{2}}
\end{aligned}
$$

