#### Stefan-Boltzmann law from the book of Enrico Fermi Masatsugu Sei Suzuki Department of Physics, BUNY at Binghamton (Date: September 18, 2018)

#### ((What is the energy flux?))

The amount of energy emitted by a blackbody depends both on its temperature and on its surface area. This makes sense: A large burning log radiates much more heat than a burning match, even though the temperatures are the same. To consider the effects of temperature alone, it is convenient to look at the amount of energy emitted from each square meter of an object's surface in a second. This quantity is called the **energy flux** (F). Flux means "rate of flow," and thus F is a measure of how rapidly energy is flowing out of the object.

A body that absorbs *all* radiation incident on it is called an *ideal blackbody*. Here we discuss the Stefan-Boltzmann law.

#### 1. Introduction

The nineteenth-century Irish physicist David Tyndall performed the first careful measurements of the amount of radiation emitted by a blackbody. (He studied the light from a heated platinum wire, which behaves approximately like a blackbody.) By analyzing Tyndall's results, the Slovenian physicist Josef Stefan deduced in 1879 that an empirical relation between the power radiated by an ideal blackbody and the temperature T:

$$e = \sigma_{SB}T^4$$

where *e* is the power radiated per unit area (the energy flux), is the absolute temperature *T*, and is a constant called Stefan's constant. In 1884, this result was also derived on the basis of classical thermodynamics by his former pupil, Ludwig Boltzmann, by then a professor at Graz, and  $e = \sigma_{SB}T^4$  is now called the Stefan-Boltzmann law. Note that the power per unit area radiated by a blackbody depends only on the temperature and not on any other characteristic of the object, such as its color or the material of which it is composed.

# **2. Proof of the energy flux:** $e = \frac{1}{4}cu$

We consider the photon. The energy flux e is the power radiated per unit area (the energy flux), at the absolute temperature T. The total energy of photons passing through the surface element dA during the time  $\Delta t$  is obtained as follows. Note that u is the energy density of photon.



Fig. The photon energy passing in the time  $\Delta t$  through a solid angle  $d\Omega$ , making an angle  $\theta$  with the normal to dA. The volume of cylinder:  $(c \cos \theta \Delta t) dA$  during the time  $\Delta t$ .  $d\Omega = 2\pi \sin \theta \cdot c$  is the velocity of light.

Using the above figure, we have

$$\int_{0}^{\frac{\pi}{2}} u(c\Delta t) \cos \theta \Delta A \frac{2\pi \sin \theta d\theta}{4\pi} = \frac{1}{2} u(c\Delta t\Delta A) \int_{0}^{\frac{\pi}{2}} \cos \theta \sin \theta d\theta$$
$$= \frac{1}{4} u(c\Delta t\Delta A) \int_{0}^{\pi/2} \sin 2\theta d\theta$$
$$= \frac{1}{4} u(c\Delta t\Delta A)$$

where the factor  $\cos \theta$  is the *z* component of the energy flux, and  $\frac{2\pi \sin \theta d\theta}{4\pi}$  is the fraction of the solid angle. Then we have the energy flux of photon as

$$e = \frac{1}{4}cu$$
 [erg/(cm<sup>2</sup>s)] energy flux

per unit area per unit time.

#### ((Note))

### E. Fermi, Notes on Thermodynamics and Statistics (The University of Chicago, 1966).

From the book written by Fermi

Prove:  

$$e = \frac{c}{4}u$$
  
 $Se = \int_{0}^{\pi h} Sc \cos \theta \frac{2\pi \sin \theta}{4\pi} = S \frac{c}{4}u$   
solution rel. solid  
augle

# **3. Proof of the radiation pressure:** $P_{rad} = \frac{1}{3}u$

The radiation pressure by photon is calculated as follows. We consider The energy per the area  $\Delta A$  per unit time  $\Delta t$ , passing through the area  $\Delta A$ , which is perpendicular to the surface.



 $u = c\widetilde{p}$ (energy density) $\widetilde{p} = \frac{u}{c}$ (momentum density) $\varepsilon = cp$ (the energy dispersion relation for photon)

For the calculation of the pressure, we need to take into account of the reflection. The radiation pressure can be obtained as follows.

$$F_{rad}\Delta t = \int_{0}^{\pi/2} \frac{u}{c} (c\Delta t \cos\theta \Delta A) (2\cos\theta) \frac{2\pi \sin\theta d\theta}{4\pi}$$
$$= \frac{1}{2} u (\Delta t\Delta A) \int_{0}^{\pi/2} 2\sin\theta \cos^{2}\theta d\theta$$
$$= u (\Delta t\Delta A) \int_{0}^{\pi/2} \sin\theta \cos^{2}\theta d\theta$$
$$= \frac{1}{3} u (\Delta t\Delta A)$$

Here  $\frac{u}{c}$  is the momentum density,  $(c\Delta t \cos\theta)\Delta A$  is the volume, the factor  $2\cos\theta$  is due to the reflection of photons on the surface. The change of momentum upon reflection from the wall involves multiplication by the factor  $2\cos\theta$ , because the perpendicular component of the incident momentum changes sign upon reflection, whereas the tangential component remains unchanged.  $[2\pi\sin\theta d\theta]/4\pi$  is the fraction of the solid angle against  $4\pi$ .

The radiation pressure is given by

$$P_{rad} = \frac{F_{rad}}{\Delta A} = \frac{1}{3}u = \frac{1}{3}\frac{U}{V}.$$

#### ((Note)) E. Fermi, Notes on Thermodynamics and Statistics (The University of Chicago, 1966).

From the book written by Fermi

Prove:  

$$p = \frac{u}{3}$$

$$p = \int u \times c \cos \theta \times \frac{\sin \theta d\theta}{2} \times 2\frac{1}{c} \cos \theta = \frac{u}{3}$$

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#### 4. Boltzmann's discussion

$$dU = TdS - PdV$$

$$\left(\frac{\partial U}{\partial V}\right)_T = T \left(\frac{\partial S}{\partial V}\right)_T - P$$

Here we use the Maxwell' relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

Thus we have

$$\left(\frac{\partial U}{\partial V}\right)_T = T \left(\frac{\partial P}{\partial T}\right)_V - P$$

or

$$T\left(\frac{\partial P}{\partial T}\right)_{V} = \left(\frac{\partial U}{\partial V}\right)_{T} + P$$

The pressure P is given by

$$P = \frac{1}{3}u(T) \,. \qquad U = u(T)V$$

Note that

$$T\left(\frac{\partial P}{\partial T}\right)_{V} = \frac{1}{3}Tu'(T)$$

and

$$\left(\frac{\partial U}{\partial V}\right)_T + P = u(T) + \frac{1}{3}u(T) = \frac{4}{3}u(T)$$

Then we have the differential equation for u(T),

$$\frac{1}{3}Tu'(T) = \frac{4}{3}u(T)$$

or

$$u'(T) = \frac{4}{T}u(T), \qquad \ln u(T) = 4\ln T + \text{const.}$$
$$u(T) = \alpha T^4$$

where

$$\alpha = \frac{4\sigma_{SB}}{c} = \frac{\pi^2 k_B^4}{15\hbar^3 c^3}$$

This is the calculation which Boltznlann carried out in 1884 and his name is justly associated with the Stefan-Boltzmann law.

# 5. Fermi's discussion on u(T)

$$U = uV$$
  

$$dQ = TdS$$
  

$$= dU + PdV$$
  

$$= d(uV) + PdV$$
  

$$= udV + Vdu + PdV$$
  

$$= udV + V\frac{du(T)}{dT}dT + \frac{1}{3}udV$$
  

$$= \frac{4}{3}udV + V\frac{du(T)}{dT}dT$$

The entropy:

$$dS = \frac{4}{3} \frac{u}{T} dV + \frac{V}{T} \frac{du(T)}{dT} dT$$
$$= \left(\frac{\partial S}{\partial V}\right)_T dV + \left(\frac{\partial S}{\partial T}\right)_V dT$$

For the exact differentia,

$$\frac{d}{dT}\left(\frac{4}{3}\frac{u(T)}{T}\right) = \frac{\partial}{\partial V}\left[\frac{V}{T}\frac{du(T)}{dT}\right] = \frac{1}{T}\frac{du(T)}{dT}$$

or

$$\frac{4}{3} \frac{Tu'(T) - u(t)}{T^2} = \frac{1}{T}u'(T)$$
$$u'(T) = 4\frac{u(T)}{T}$$

$$\frac{du}{u} = 4\frac{dT}{T}$$

leading to

$$u \propto T^4$$
 (Stefan-Boltzmann law)

# 6. Thermodynamics of radiation (Fermi)

Definition of absorption and emission of a body a, and e

Black body has a = 1

Prove: e/a is the same for all substances at the same temperature.

From Kelvin,

$$e_1 a_2 = e_2 a_1$$

or

or

$$\frac{e_1}{a_1} = \frac{e_2}{a_2} = e \qquad \text{of black body}$$

# 7. Stefan-Boltzmann law

$$u = \frac{U}{V}$$
$$= \int_{0}^{\infty} u_{\omega} d\omega$$
$$= \frac{\hbar}{\pi^{2} c^{3}} \int_{0}^{\infty} \frac{\omega^{3}}{\exp(\beta \hbar \omega) - 1} d\omega$$
$$= \frac{\hbar}{\pi^{2} c^{3}} \frac{1}{\beta^{4} \hbar^{3}} \int_{0}^{\infty} \frac{x^{3}}{\exp(x) - 1} dx$$
$$= \frac{\pi^{4}}{15} \frac{\hbar}{\pi^{2} c^{3}} \frac{1}{\beta^{4} \hbar^{3}}$$

or

$$u = \frac{U}{V} = \frac{\pi^2}{15} \frac{k_B^4 T^4}{c^3 \hbar^3}$$
 (Planck radiation law)

where

$$x = \beta \hbar \omega$$
$$\int_{0}^{\infty} \frac{x^{3}}{\exp(x) - 1} dx = \frac{\pi^{4}}{15}$$

Using this expression of u(T), we have

$$e = \frac{1}{4}cu = \frac{1}{4}c\frac{\pi^2 k_B^4}{15\hbar^3 c^3}T^4 = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}T^4 = \sigma_{SB}T^4$$
$$\sigma_{SB} = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$$
(Stefan-Boltzmann constant)

where 1/4 is called the geometrical factor.

The Stefan–Boltzmann constant (SI units)

 $\sigma_{SB} = 5.67036713 \times 10^{-8} \text{ W/(m^2K^4)}$ 

#### REFERENCES

- M.S. Longair, Theoretical Concepts in Physics An Alternative View of Theoretical Reasoning in Physics (Cambridge, 2003).
- E. Fermi, Notes on Thermodynamics and Statistics (The University of Chicago, 1966).
- M. Mansuripur<sup>+</sup> and P. Han: <u>arXiv:1708.09524</u> [physics.optics], Thermodynamics of Radiation Pressure and Photon Momentum.