Cosmic microwave background (CMB)

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The blackbody radiation may be regarded as a gas consisting of photons. The photon gas is an ideal gas. Because of the angular momentum of the photons is integral (spin 1), this gas obeys Bose statistics. Here the physics of cosmic microwave background (CMB).

The **cosmic microwave background** (**CMB**) is the thermal radiation left over from the time of recombination in Big Bang cosmology. In older literature, the CMB is also variously known as cosmic microwave background radiation (CMBR) or "relic radiation". The CMB is a cosmic background radiation that is fundamental to observational cosmology because it is the oldest light in the universe, dating to the epoch of recombination. With a traditional optical telescope, the space between stars and galaxies (the *background*) is completely dark. However, a sufficiently sensitive radio telescope shows a faint background glow, almost isotropic, that is not associated with any star, galaxy, or other object. This glow is strongest in the microwave region of the radio spectrum. The accidental discovery of the CMB in 1964 by American radio astronomers Arno Penzias and Robert Wilson[1][2] was the culmination of work initiated in the 1940s, and earned the discoverers the 1978 Nobel Prize.
https://en.wikipedia.org/wiki/Cosmic_microwave_background

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1. Energy density of photon gas

The energy dispersion of photon is given by

$$\varepsilon = \hbar\omega = cp = c\hbar k = c\hbar\frac{2\pi}{\lambda}$$

where k is the wavenumber, c is the velocity of light, p is the momentum, and λ is the wavelength. We consider the mode ($|\boldsymbol{k}\rangle$) (simple harmonic oscillator) with the angular frequency $\omega_{\boldsymbol{k}}$. Using the canonical ensemble, the Canonical partition function is obtained as

$$Z_{C1} = \sum_{n=0}^{\infty} e^{-\beta n\hbar\omega_k} = \frac{1}{1-e^{-\beta\hbar\omega_k}}.$$

The average energy is

$$-\frac{\partial}{\partial\beta}\ln Z_{C1}=\frac{\partial}{\partial\beta}\ln(1-e^{-\beta\hbar\omega_{\boldsymbol{k}}})=\frac{\hbar\omega e^{-\beta\hbar\omega_{\boldsymbol{k}}}}{1-e^{-\beta\hbar\omega_{\boldsymbol{k}}}}=\frac{\hbar\omega_{\boldsymbol{k}}}{e^{\beta\hbar\omega_{\boldsymbol{k}}}-1}=\hbar\omega_{\boldsymbol{k}}\overline{n}_{\boldsymbol{k}}$$

where $\overline{n}_{\boldsymbol{k}}$ is the Planck distribution function

$$\overline{n}_{\boldsymbol{k}}=\frac{1}{e^{\beta\hbar\omega_{\boldsymbol{k}}}-1}.$$

which corresponds to the Bose-Einstein distribution function with zero chemical potential. When the system consists of many modes with $|\boldsymbol{k}\rangle$ (any $\boldsymbol{k}$), the total energy is given by

$$E_{tot}=\sum_{\boldsymbol{k}}\hbar\omega_{\boldsymbol{k}}\overline{n}_{\boldsymbol{k}}=\int\frac{V\omega^2}{\pi^2c^3}d\omega\,\overline{n}_{\omega}\,\hbar\omega=V\int u(\omega)d\omega$$

where we use the energy dispersion relation for photon,

$$\sum_{\boldsymbol{k}}\quad\rightarrow\qquad\frac{2V}{(2\pi)^3}4\pi k^2dk=\frac{2V}{(2\pi)^3}\frac{4\pi}{c^3}\omega^2d\omega=\frac{V\omega^2}{\pi^2c^3}d\omega$$

$$\overline{n}_{\omega}=\frac{1}{e^{\beta\hbar\omega}-1}.$$

Then the energy density is given by

$$\frac{E_{tot}}{V}=\overline{W}_T=u(\omega)=\int_0^{\infty}u(\omega)d\omega=\int_0^{\infty}u(\lambda)d\lambda$$

where

$$u(\omega)=\frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{\exp(\beta\hbar\omega)-1}=\frac{k_B{}^3T^3}{\pi^2\hbar^2c^3}\frac{x^3}{\exp(x)-1}$$

where, $x=\dfrac{\hbar\omega}{k_BT}=\beta\hbar\omega$. (Planck's law for the radiation energy density). It is clear that

$$\frac{u(\omega)}{\dfrac{k_B{}^3T^3}{\pi^2\hbar^2c^3}}=f(x)=\frac{x^3}{\exp(x)-1}$$

is dependent on a variable x given by

$$x = \frac{\hbar\omega}{k_B T}.$$

(scaling relation). The experimentally observed spectral distribution of the black body radiation is very well fitted by the formula discovered by Planck.

(1) Region of Wien ($x = \frac{\hbar\omega}{k_B T} >> 1$),

$$u_W(\omega) = \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} x^3 e^{-x}$$

(2) Region of Rayleigh-Jeans ($x = \frac{\hbar\omega}{k_B T} >> 1$),

$$u_{RJ}(\omega) = \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} \frac{x^3}{\exp(x) - 1} \approx \frac{k_B{}^3 T^3}{\pi^2 \hbar^2 c^3} x^2$$

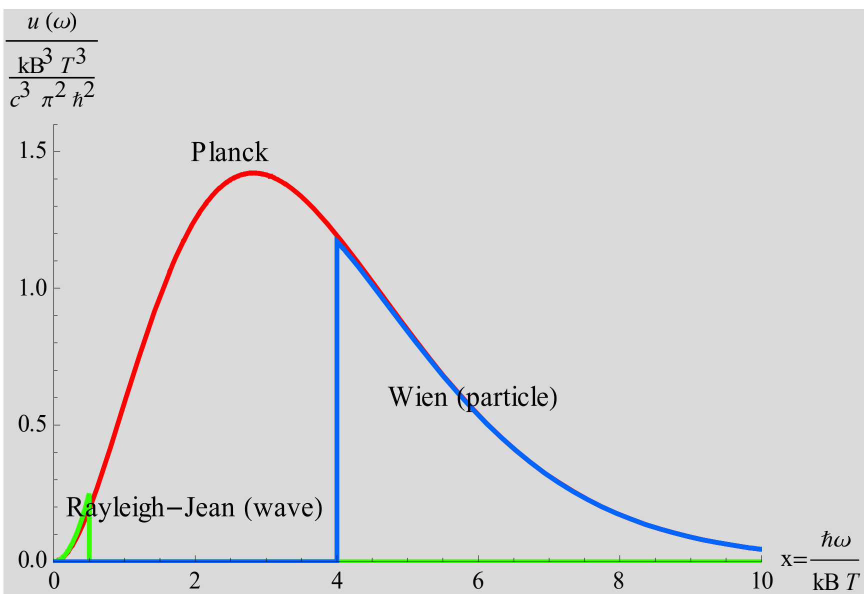


Fig. Scaling plot of $f(x)$ vs x for the Planck's law for the energy density of electromagnetic radiation at angular frequency ω and temperature T. Planck (red). Wien (blue, particle-like). Rayleigh-Jean (green, wave-like).

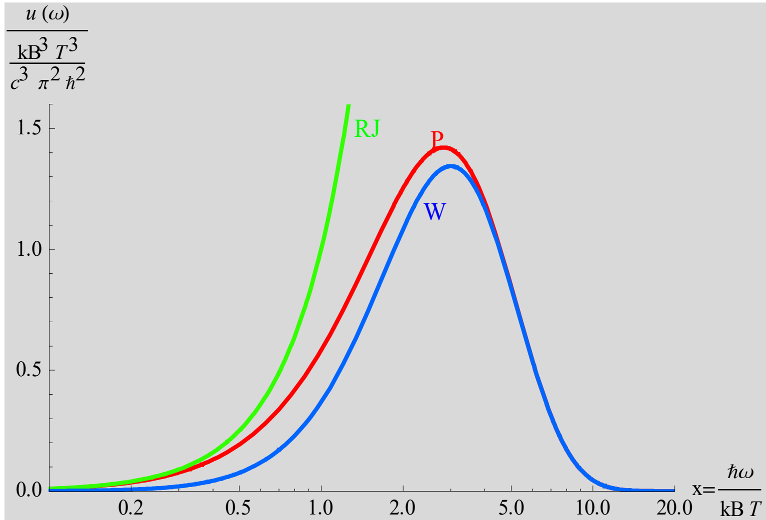


Fig. Scaling plot of Planck's law. Wien's law, and Rayleigh-Jean's law.

2. Deivation of $u(\lambda, T)$

$$\int_0^\infty u(\omega)d\omega = \int_0^\infty \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(\frac{\hbar\omega}{k_B T}) - 1} d\omega$$

Since $\omega = \frac{2\pi c}{\lambda}$, $\qquad d\omega = -2\pi c \frac{d\lambda}{\lambda^2}$

$$\int_0^\infty u(\omega)d\omega = \int_0^\infty \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(\frac{\hbar\omega}{k_B T}) - 1} d\omega$$

$$= \int_0^\infty \frac{\hbar\left(\frac{2\pi c}{\lambda}\right)^3}{\pi^2 c^3} \frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T}) - 1} 2\pi c \frac{d\lambda}{\lambda^2}$$

or

$$\int_0^\infty u(\omega)d\omega = \int_0^\infty u(\lambda)d\lambda = \int_0^\infty 16\pi^2\hbar c \frac{1}{\lambda^5} \frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T}) - 1} d\lambda$$

Then we have

$$u(\lambda) = \frac{16\pi^2 \hbar c}{\lambda^5} \frac{1}{\exp(\frac{2\pi \hbar c}{\lambda k_B T}) - 1}$$

where

$\hbar$ = 1.054571596 x 10^{-27} erg s, k_B = 1.380650324 x 10^{-16} erg/K
c = 2.99792458 x 10^{10} cm/s.
J = 10^7 erg

3. Wien's displacement law

$u(\lambda)$ has a maximum at

$$\frac{2\pi \hbar c}{\lambda k_B T} = 4.96511, \qquad \text{(dimensionless)}$$

Note that

$$\frac{\partial u(\lambda)}{\partial \lambda} = 0, \qquad e^{\frac{\alpha}{\lambda}}(\alpha - 5\lambda) + 5\lambda = 0$$

with

$$\alpha = \frac{2\pi \hbar c}{k_B T}$$

((**Mathematica**))

```
Clear["Global`*"]; f1 = (1/λ^5)/(Exp[α/λ] - 1);

eq1 = D[f1, λ] // Simplify
```

$$\frac{e^{\alpha/\lambda}(\alpha - 5\lambda) + 5\lambda}{(-1 + e^{\alpha/\lambda})^2 \lambda^7}$$

```
NSolve[Exp[x] (x - 5) + 5 == 0, x]
```

{{x → 0.}, {x → 4.96511}}

The above equation can be rewritten as

$$\lambda = \frac{0.28977}{T(K)} \qquad (\lambda \text{ in the units of cm})$$

or

$$\lambda = \frac{2.897768551}{T(K)} \times 10^6 . \qquad (\lambda \text{ in the units of nm})$$

T is the temperature in the units of K. λ is the wave-length in the unit of nm

T(K)	λ (nm)
1000	2897.77
1500	1931.85
2000	1448.89
2500	1159.11
3000	965.924
3500	827.935
4000	724.443
4500	643.949
5000	579.554
5500	526.867
6000	482.962
6500	445.811
7000	413.967
7500	386.369
8000	362.221
8500	340.914
9000	321.975
9500	305.029
10 000	289.777

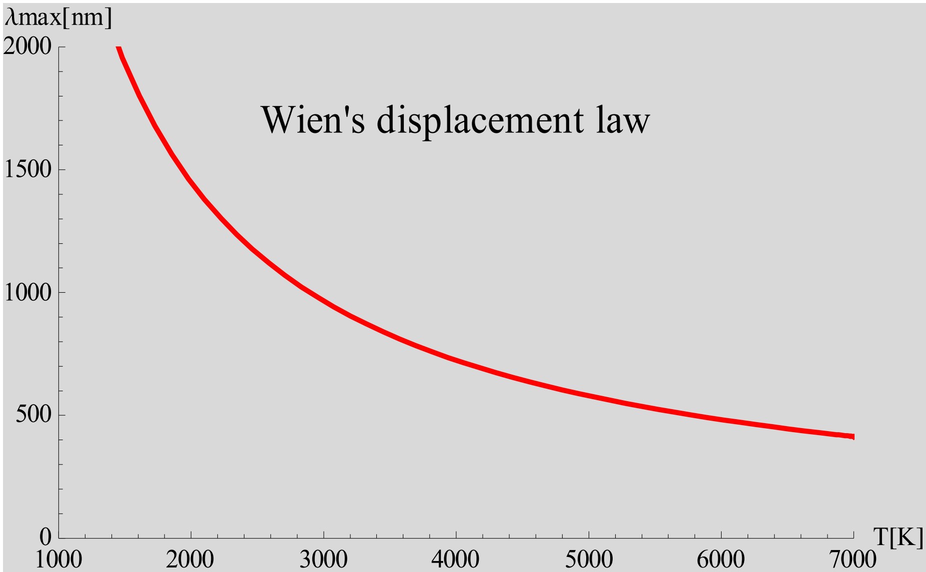


Fig. Wien's displacement law. The peak wavelength vs temperature T(K).

4. Rate of the energy flux density

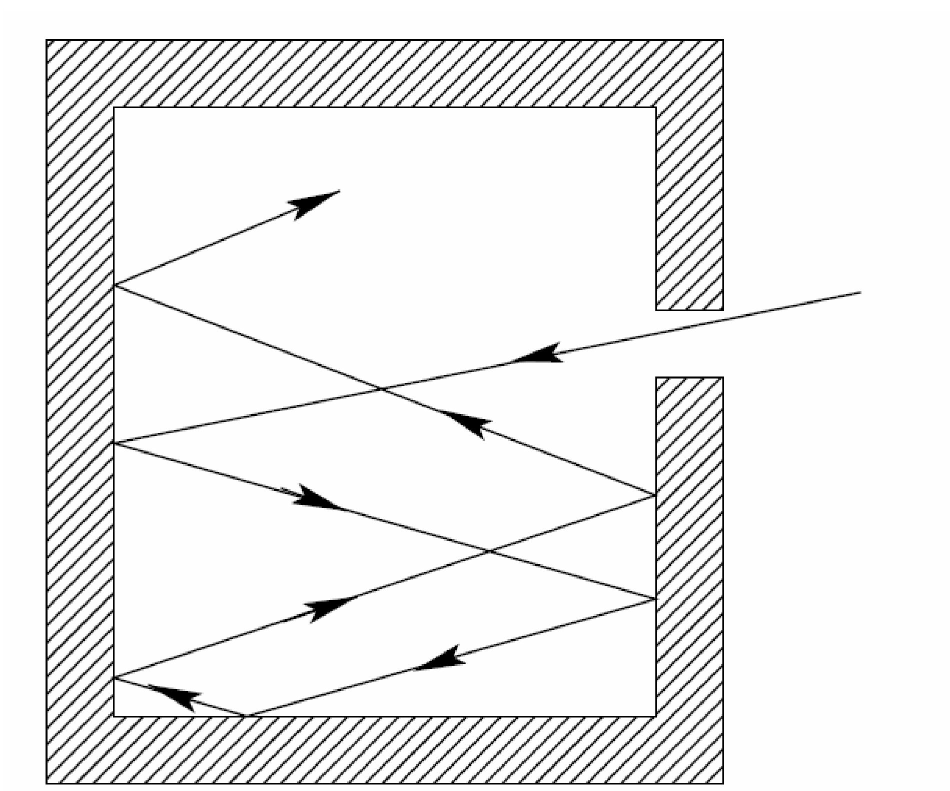


Fig. Experimental realization of a black body problem (from Bellac et al.)

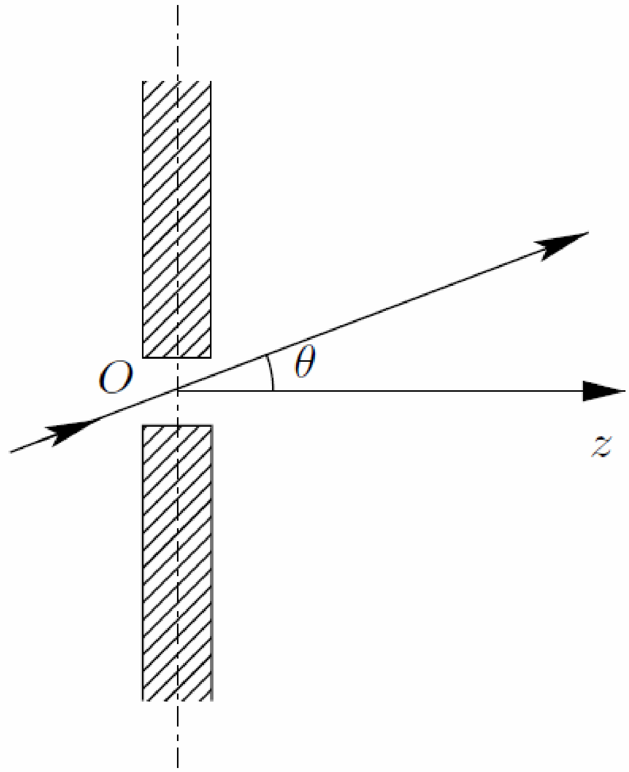


Fig. Convention for the axis in the calculation for black body radiation (from Bellac et al.).

It is assumed that the thermal equilibrium of the electromagnetic waves is not disturbed even when a small hole is bored through the wall of the box. The area of the hole is dS. The energy which passes in unit time through a solid angle dΩ, making an angle θ with the normal to dS is

$$J(\lambda,T,\theta)d\lambda d\Omega dS = cu(\lambda,T)d\lambda \cos\theta \frac{d\Omega}{4\pi} dS,$$

where c is the velocity of light. The right hand side is divided by 4π, because the energy density u comprises all waves propagating along different directions. The emitted energy unit time, per unit area is

$$\begin{aligned}\iint J(\lambda,T,\theta)d\lambda d\Omega &= \int cu(\lambda,T)d\lambda \int \cos\theta \frac{d\Omega}{4\pi} \\ &= \int \frac{cu(\lambda,T)}{4} d\lambda \\ &\equiv \frac{c}{4}\int u(\lambda,T)d\lambda \\ &= \frac{c}{4}\varepsilon\end{aligned}$$

where

$$\varepsilon = \int u(\lambda, T) d\lambda,$$

$$\begin{aligned}\int \cos\theta \frac{d\Omega}{4\pi} &= \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta \\ &= \frac{1}{4\pi} 2\pi \int_0^{\pi/2} \frac{1}{2}\sin(2\theta) d\theta \\ &= \frac{1}{4}\frac{1}{2}[-\cos(2\theta)]\Big|_0^{\pi/2} = \frac{1}{4}\end{aligned}$$

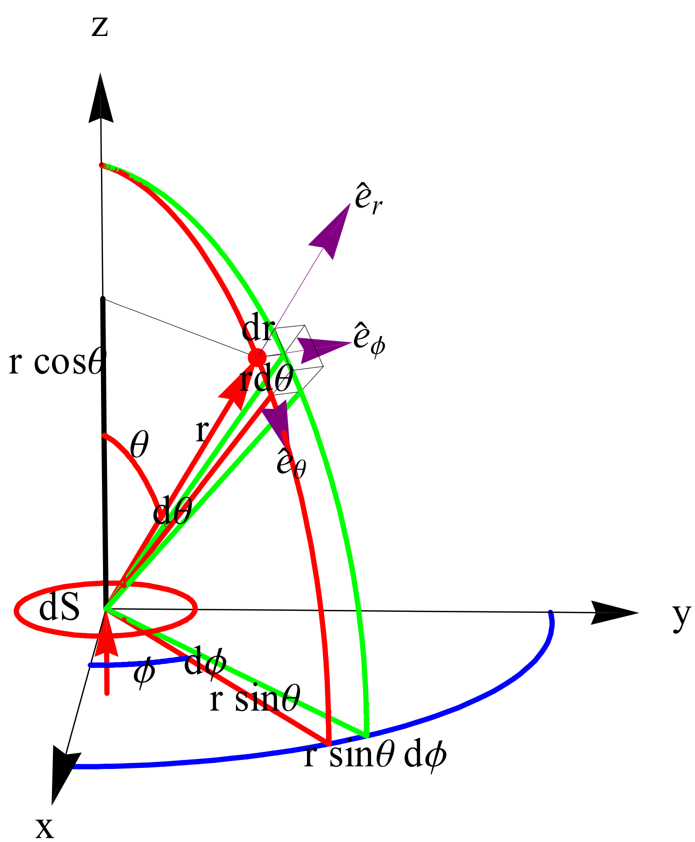


Fig. Radiation intensity is used to describe the variation of radiation energy with direction.

In other words, the geometrical factor is equal to 1/4. Then we have a measure for the intensity of radiation (the rate of energy flux density);

$$S(\lambda, T) = \frac{cu(\lambda, T)}{4} = \frac{4\pi^2 \hbar c^2}{\lambda^5} \frac{1}{\exp(\frac{2\pi\hbar c}{\lambda k_B T}) - 1}$$

where

$S(\lambda, T) d\lambda$ = power radiated per unit area in $(\lambda, \lambda + d\lambda)$

Unit

$$[\hbar c^2 \frac{1}{\lambda^5}] = \frac{erg.s}{cm^5} \frac{cm^2}{s^2} = \frac{erg}{cm^3} \frac{1}{s} = \frac{10^{-7} J}{(10^{-2} m)^3} \frac{1}{s} = 10^{-1} \frac{W}{m^3} = [\frac{W}{m^3}]$$

The energy flux density $S(\lambda, T)$ is defined as the rate of energy emission per unit area.

((Note)) The unit of the Poynting vector <*S*> is [W/m^2]. $\langle S \rangle$ is the energy flux (energy per unit area per unit time).

(1) Rayleigh-Jeans law (in the long-wavelength limit)

$$S_{RJ}(\lambda) = \frac{1}{4} cu_{RJ}(\lambda) = 4\pi^2 \hbar c \frac{1}{\lambda^5} \frac{1}{\frac{2\pi\hbar c}{\lambda k_B T}} = \frac{2\pi k_B T}{\lambda^4}$$

for

$$\frac{\lambda k_B T}{2\pi\hbar c} >> 1$$

(2) Wien's law (in short-wavelength limit)

$$S_W(\lambda) = \frac{1}{4} cu_W(\lambda) = \frac{\pi^2 \hbar c}{\lambda^5} \exp(-\frac{2\pi\hbar c}{\lambda k_B T})$$

$$\frac{\lambda k_B T}{2\pi\hbar c} << 1$$

We make a plot of $S(\lambda, T)$ as a function of the wavelength, where $S(\lambda, T)$ is in the units of W/m^3 and the wavelength is in the units of nm.

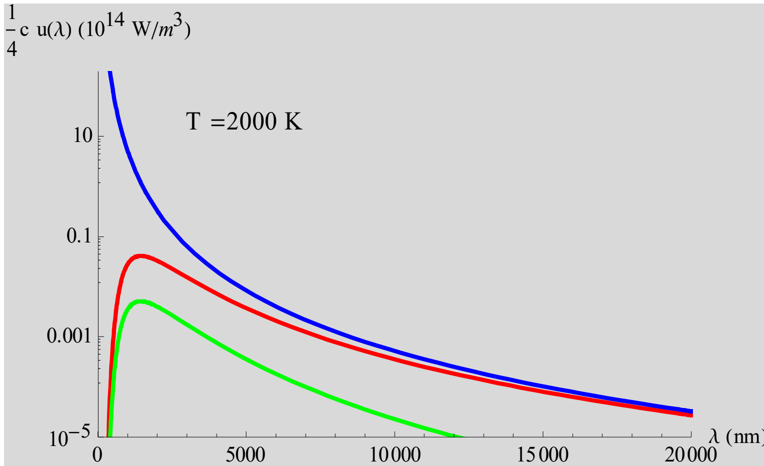
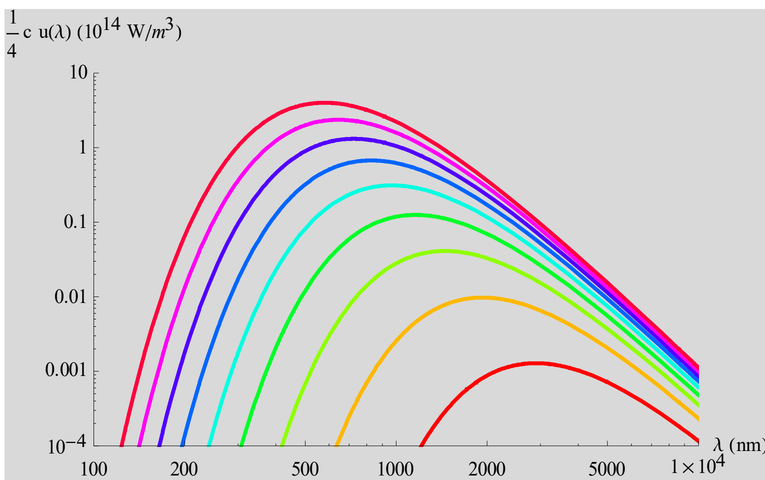


Fig. $cu(\lambda)/4$ (W/m^3) vs λ (nm). $T = 2 \times 10^3$ K. Red [Planck]. Green [Wien]. Blue [Rayleigh-Jean]. Wien's displacement law: The peak appears at $\lambda = 1448.89$ nm for $T = 2 \times 10^3$ K. This figure shows the misfit of Wien's law at long wavelength and the failure of the Rayleigh-Jean's law at short wavelength.



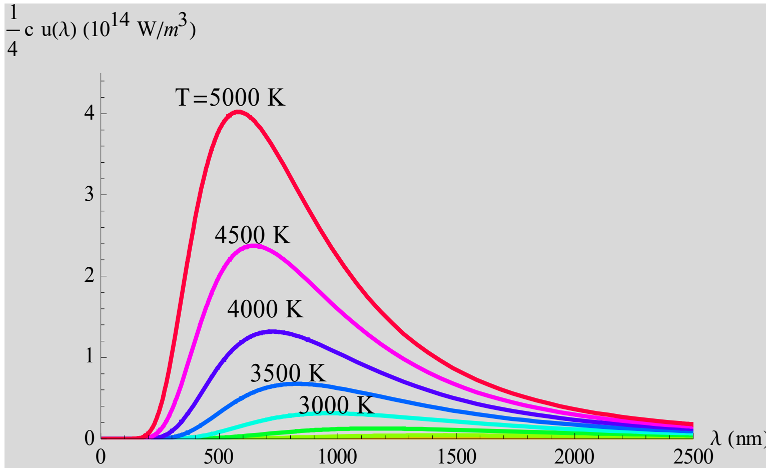


Fig. (a) and (b) $cu(\lambda)/4$ (W/m^3) vs λ (nm) for the Plank's law. T = 1000 K (red), 1500 K, 2000 K, 2500 K, 3000 K (blue), 3500 K, 4000 K (purple), 4500 K, and 5000 K. The peak shifts to the higher wavelength side as T decreases according to the Wien's displacement law.

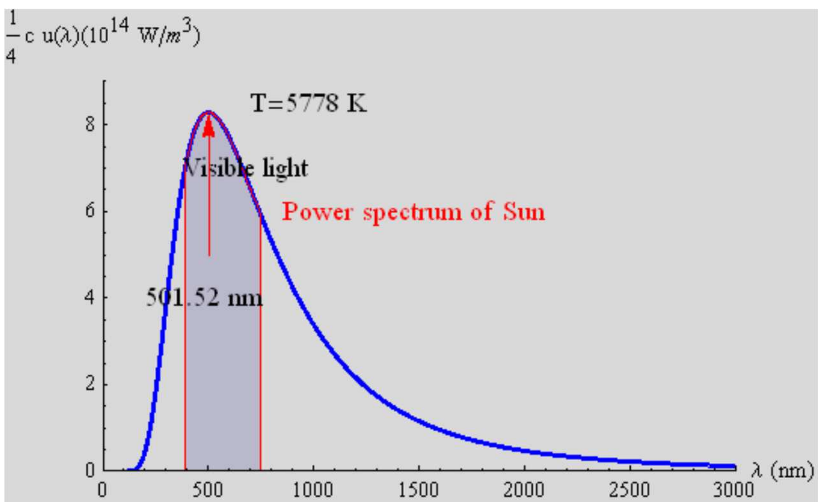


Fig. Power spectrum of sun. $cu(\lambda)/4$ (W/m^3) vs λ (nm). T = 5778 K. The peak wavelength is 501.52 nm according to the Wien's displacement law.

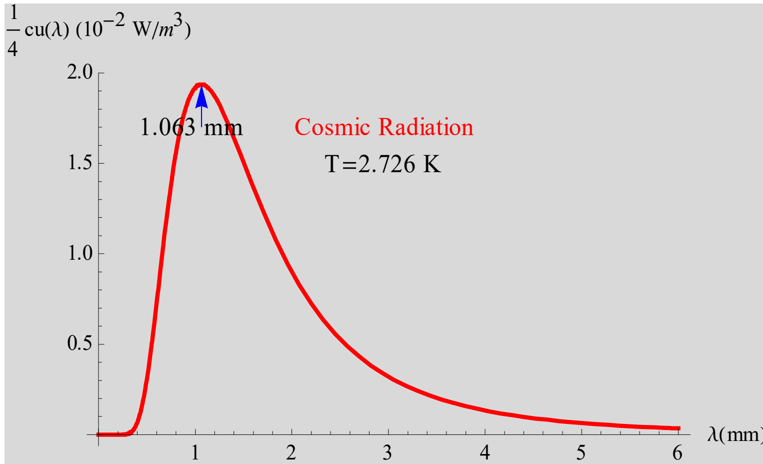
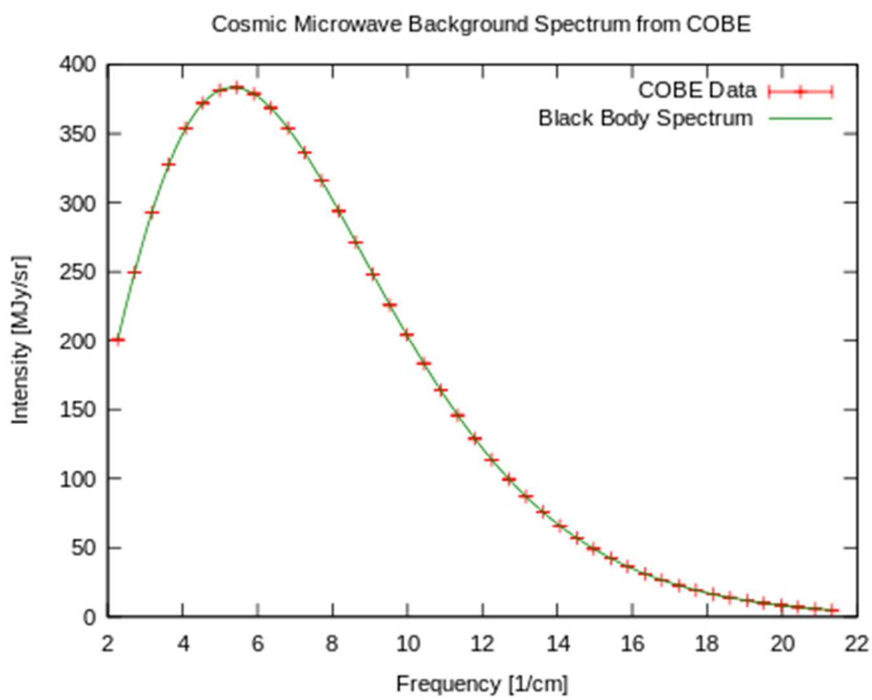


Fig. Power spectrum of cosmic blackbody radiation at T = 2.726 K. The peak wavelength is 1.063 mm (Wien's displacement law).



https://en.wikipedia.org/wiki/Cosmic_microwave_background

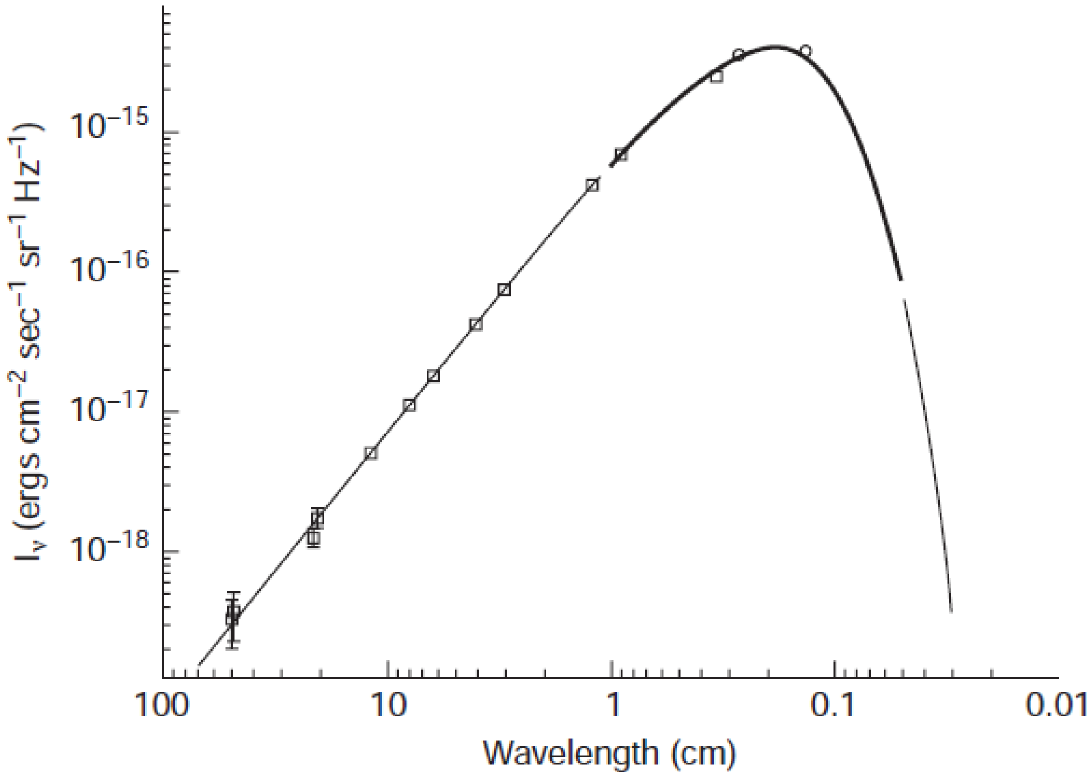


Fig. The spectrum of radiation that uniformly fills space, with greatest intensity at millimeter – microwave – wavelengths. [P.J. Peebles, L.A. Page, Jr., and R.B. Partridge, Finding the Big Bang (Cambridge, 2009)

5. Stefan-Boltzmann radiation law for a black body (1879).

Joseph Stefan (24 March 1835 – 7 January 1893) was a physicist, mathematician and poet of Slovene mother tongue and Austrian citizenship.

http://en.wikipedia.org/wiki/Joseph_Stefan
The total energy per unit volume is given by

$$\varepsilon = \frac{E_{tot}}{V} = \int u(\omega)d\omega = \int u(\lambda)d\lambda = \frac{(k_B T)^4}{\pi^2 \hbar^3 c^3}\int_0^\infty \frac{x^3}{\exp(x)-1} = \frac{\pi^2 (k_B T)^4}{15\hbar^3 c^3}$$

((**Mathematica**))

$$\int_0^\infty \frac{x^3}{e^x - 1}\, dx$$

$$\frac{\pi^4}{15}$$

A spherical enclosure is in equilibrium at the temperature T with a radiation field that it contains. The power emitted through a hole of unit area in the wall of enclosure is

$$P = \frac{1}{4}c\varepsilon = \frac{\pi^2 k_B{}^4}{60\hbar^3 c^2}T^4 = \sigma_{SB}T^4$$

where σ is the Stefan-Boltzmann constant

$$\sigma_{SB} = \frac{\pi^2 k_B{}^4}{60\hbar^3 c^2} = 0.5670400\times 10^{-4} \text{ erg/s-cm}^2\text{-K}^4 = 5.670400 \text{ x } 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$$

and the geometrical factor is equal to 1/4.

7. Thermodynamics

Here we discuss the thermodynamics of photon gas.

(a) The photon number density

$$N = \sum_{\boldsymbol{k}} \bar{n}_k = \frac{2V}{(2\pi)^3} \int 4\pi k^2 dk\, \bar{n}_{\boldsymbol{k}} = \frac{V}{\pi^2} \int \frac{k^2 dk}{e^{\beta\hbar\omega} - 1}$$

or

$$n = \frac{N}{V} = \frac{1}{\pi^2 c^3} \int \frac{\omega^2 d\omega}{e^{\beta\hbar\omega} - 1} = \frac{(k_B T)^3}{\pi^2 c^3 \hbar^3} \int_0^\infty \frac{x^2 dx}{e^x - 1} = 2.40411 \frac{k_B{}^3}{\pi^2 c^3 \hbar^3} T^3$$

where

$$\int_0^\infty \frac{x^2 dx}{e^x - 1} = 2.40411$$

When $T = 2.73$ K, we have

$n = 412.767/\text{cm}^3$.

(b) Energy density:

$$\frac{E}{V} = \langle \varepsilon \rangle = \int u(\omega) d\omega = \frac{(k_B T)^4}{\pi^2 \hbar^3 c^3} \int_0^\infty \frac{x^3}{\exp(x) - 1} = \frac{\pi^2 (k_B T)^4}{15 \hbar^3 c^3}$$

or

$$\frac{E}{V} = \frac{4}{c} \sigma_{SB} T^4,$$

where

$$\sigma_{SB} = \frac{\pi^2 k_B{}^4}{60 \hbar^3 c^2}$$

(c) Pressure P;

$$PV = \frac{1}{3}E$$

The pressure is given by

$$P = \frac{1}{3}\frac{E}{V} = \frac{4}{3c}\sigma_{SB}T^4$$

(d) Helmholtz free energy F;

$$\begin{aligned} F &= -k_B T \ln Z_C \\ &= k_B T \sum_{\boldsymbol{k}} \ln(1 - e^{-\beta\hbar\omega_{\boldsymbol{k}}}) \\ &= k_B T \frac{2V}{(2\pi)^3} \int 4\pi k^2 dk \ln(1 - e^{-\beta\hbar\omega_{\boldsymbol{k}}}) \end{aligned}$$

The integration by part leads to

$$\frac{F}{V} = -\frac{{k_B}^4}{3\pi^2\hbar^3 c^3} T^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} = -\frac{1}{45}\frac{\pi^2 {k_B}^4}{\hbar^3 c^3} T^4 = -\frac{4}{3c}\sigma_{SB}T^4$$

where

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$

(e) Entropy S;

$$\begin{aligned} \frac{S}{V} &= \frac{1}{T}\left(\frac{E}{V} - \frac{F}{V}\right) \\ &= \frac{1}{T}\left(\frac{4}{c}\sigma_{SB}T^4 + \frac{4}{3c}\sigma_{SB}T^4\right) \\ &= \frac{16}{3}\frac{\sigma_{SB}}{c}T^3 \end{aligned}$$

From the relation; $dF = TdS - PdV$ we can get the entropy S as

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4V}{3c}\sigma_{SB}\frac{\partial}{\partial T}T^4 = \frac{16}{3c}\sigma_{SB}T^3 V$$

(f) The heat capacity:

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V = \frac{16}{c}\sigma_{SB}T^3V\,.$$

(g) The pressure P

The pressure P can be obtained as

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{4}{3c}\sigma_{SB}T^4$$

8. Chemical potential (Landau-Lifshitz)

The mechanism by which equilibrium can be established, consists in the absorption and emission of photons by matter. This results in an important specific property of the photon gas: the number of photons N in it is variable, and not a given constant as in an ordinary gas. Thus N itself must be determined from the conditions of thermal equilibrium. From the condition that the free energy of the gas should be a minimum (for given T and V), we obtain as one of the necessary conditions $\partial F/\partial N)_{T,V} = 0$. Since

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$$

this gives $\mu = 0$, i.e., the chemical potential of the photon gas is zero.

L.D. Landau and E.M. Lifshitz, Statistical Physics (Pergamon Press 1976).

9. Example (Huamg 17.1)

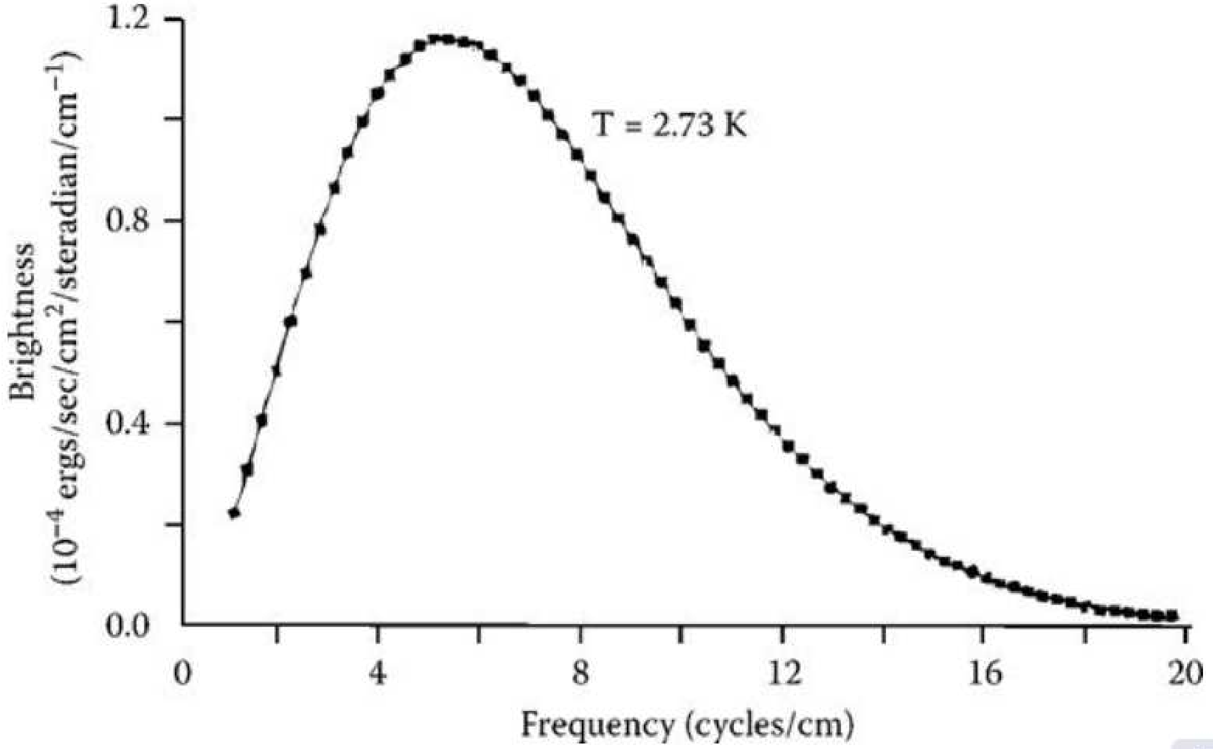
Huang 17.6

The background cosmic radiation has a Planck distribution temperature with temperature 2.73 K, as shown in Fig.17.1.

(a) What is the phonon density in the universe?

(b) What is the entropy per photon?

(c) Suppose the universe expands adiabatically. What would the temperature be when the volume of the universe doubles?



((**Solution**))

Here we discuss the thermodynamics of photon gas.

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or

$$n = \frac{N}{V} = \frac{1}{\pi^2 c^3}\int \frac{\omega^2 d\omega}{e^{\beta\hbar\omega}-1} = \frac{(k_B T)^3}{\pi^2 c^3 \hbar^3}\int_0^\infty \frac{x^2 dx}{e^x - 1} = \frac{2.40411}{\pi^2}\left(\frac{k_B T}{c\hbar}\right)^3$$

where

$$\int_0^\infty \frac{x^2 dx}{e^x - 1} = 2.40411$$

When $T = 2.73$ K, we have

$n = 412.767\,/\text{cm}^3.$

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$$\frac{U}{V} = \langle \varepsilon \rangle = \int u(\omega) d\omega = \frac{(k_B T)^4}{\pi^2 \hbar^3 c^3} \int_0^\infty \frac{x^3}{\exp(x) - 1} = \frac{\pi^2 (k_B T)^4}{15 \hbar^3 c^3}$$

or

$$\frac{U}{V} = \frac{4}{c} \sigma_{SB} T^4,$$

where

$$\sigma_{SB} = \frac{\pi^2 k_B{}^4}{60 \hbar^3 c^2} \qquad \text{(Stefan-Boltzmann constant)}$$

(c) Pressure P;

$$PV = \frac{1}{3} U$$

The pressure is given by

$$P = \frac{1}{3} \frac{E}{V} = \frac{4}{3c} \sigma_{SB} T^4$$

(d) Helmholtz free energy F;

$$\begin{aligned} F &= -k_B T \ln Z_C \\ &= k_B T \sum_{\boldsymbol{k}} \ln(1 - e^{-\beta \hbar \omega_k}) \\ &= k_B T \frac{2V}{(2\pi)^3} \int 4\pi k^2 dk \ln(1 - e^{-\beta \hbar \omega_k}) \end{aligned}$$

The integration by part leads to

$$\frac{F}{V} = -\frac{{k_B}^4}{3\pi^2\hbar^3 c^3} T^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} = -\frac{1}{45}\frac{\pi^2 {k_B}^4}{\hbar^3 c^3} T^4 = -\frac{4}{3c}\sigma_{SB} T^4$$

where

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$

(e) Entropy S;

$$\begin{aligned} \frac{S}{V} &= \frac{1}{T}(\frac{U}{V} - \frac{F}{V}) \\ &= \frac{1}{T}(\frac{4}{c}\sigma_{SB} T^4 + \frac{4}{3c}\sigma_{SB} T^4) \\ &= \frac{16}{3}\frac{\sigma_{SB}}{c} T^3 \end{aligned}$$

From the relation; $dF = TdS - PdV$ we can get the entropy S as

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4V}{3c}\sigma_{SB}\frac{\partial}{\partial T} T^4 = \frac{16}{3c}\sigma_{SB} T^3 V$$

or

$$\frac{S}{V} = k_B \frac{4\pi^2}{45}\left(\frac{k_B T}{\hbar c}\right)^3$$

(f) The heat capacity:

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V = \frac{16}{c}\sigma_{SB} T^3 V \, .$$

(g) The pressure P

The pressure P can be obtained as

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{4}{3c}\sigma_{SB} T^4$$

10. Example (Kittel 4-1)

4-1. Number of thermal photons. Show that the number of photons in equilibrium at temperature T in a cavity of volume V is

$$n = \frac{N}{V} = 2.4040411 \frac{1}{\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 .$$

The entropy S is given by

$$\frac{S}{V} = k_B \frac{4\pi^2}{45} \left(\frac{k_B T}{\hbar c} \right)^3 ,$$

whence

$$\frac{S}{N} = k_B \frac{4\pi^4}{45 \times 2.40411} = 3.60158 k_B$$

It is believed that the total number of photons in the universe is 10^8 larger than the total number of nucleons (protons, neutrons). Because both entropies are of the order of the respective number of particles, the photons provide the dominant contribution to the entropy of the universe, although the particles dominate the total energy. We believe that the entropy of the photons is essentially constant, so that the entropy of the universe is approximately constant with time.

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