Chemical potential of a pair of electron and positron and photon

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1. Discussion by Bellac et al.

When the number of particles can change, which is allowed in relativistic quantum mechanics where particles can be destroyed and created, the chemical potential of the different particle species is in general constrained by conservation laws, e.g. conservation of electric charge. However, the simplest case occurs when particle production/destruction is not constrained by any conservation law, which means, as we shall demonstrate, that the chemical potential is zero. This is the case, for example, for a gas of photons in thermal equilibrium: the photon chemical potential is zero. Since the number of photons is not fixed *a priori*, the number of particles N should be viewed as an internal variable to be determined by minimizing the free energy F, which is also a function of the controlled parameters V and T,

$$\frac{\partial}{\partial N} F(T,V;N)|_{T,V} = 0$$

In other words we have $\mu = 0$. We now examine a case with a conservation law, for example a system of electrons and positrons in thermal equilibrium. If these are the only charged particles whose number can change, then charge conservation means that the number of electrons, $N-$, minus the number of positrons, $N+$, is constant:

$$N_- - N_+ = N_0$$

The free energy F is a function of the controlled parameters V, T and N_0 as well as the internal variable N_- (or N_+). It may also be considered as a function of the controlled parameters T and V and the two internal variables N_+ and N_- which are related by a conservation law

$$F(T,V,N_0;N) = F(T,V;N_-,N_+ = N_- - N_0)$$

At equilibrium, the value of N_- (and therefore N_+) is given by the minimization of the free energy

$$\frac{\partial F}{\partial N_-}|_{T,V,N_0} = \frac{\partial F}{\partial N_-}|_{T,V,N_+} + \frac{\partial F}{\partial N_+}|_{T,V,N_-} \frac{\partial N_+}{\partial N_-} = \mu_- + \mu_+ = 0$$

The chemical potentials of the electrons and positrons therefore obey

$$\mu_+ + \mu_- = 0$$

This result can be interpreted using the results on chemical reactions. In fact, a thermal bath of electrons (*e*–) and positrons (*e*+) also contains photons (*γ*). We may thus interpret the thermal and chemical equilibrium as the result of the following reaction

$$e^+ + e^- \to \gamma$$

Since the photonic chemical potential is zero, it is concluded that $\mu_+ + \mu_- = 0$.

2. A pair production (Huang): the grand canonical ensemble

The grand canonical ensemble includes systems with different particle numbers, with a mean value *N* determined by the chemical potential. This makes sense only if *N* is a conserved quantity, for otherwise the chemical potential would be zero, as in the case of photon. We consider a reaction

$$e^+ + e^- \to \gamma$$

The reaction establishes an average value for the conserved quantum number $N_- - N_+$. The grand partition function is given by

$$Z_G = \sum_{N_-}\sum_{N_+} \lambda^{N_- - N_+} Z_{N_-} Z_{N_+}$$

where

$$\lambda = e^{\beta\mu},$$

$$F_{N_-} = -k_B T \ln Z_{N_-} \qquad \text{or} \qquad Z_{N_-} = e^{-\beta F_{N_-}}$$

$$F_{N_+} = -k_B T \ln Z_{N_+} \qquad \text{or} \qquad Z_{N_+} = e^{-\beta F_{N_+}}$$

Then we have

$$\lambda^{N_- - N_+} Z_{N_-} Z_{N_+} = \exp[\beta\mu(N_- - N_+) - \beta(F_{N_-} + F_{N_+})]$$

We put

$$f = \mu(N_- - N_+) - (F_{N_-} + F_{N_+})$$

We find the condition that

$$\frac{\partial f}{\partial N_-} = \mu - \left(\frac{\partial F_{N_-}}{\partial N_-}\right)_{T,V} = 0 \qquad \text{or} \qquad \mu - \mu_- = 0\,.$$

$$\frac{\partial f}{\partial N_+} = -\mu - \left(\frac{\partial F_{N_+}}{\partial N_+}\right)_{T,V} = 0 \qquad \text{or} \qquad \mu + \mu_+ = 0\,.$$

where

$$\left(\frac{\partial F_{N_-}}{\partial N_-}\right)_{T,V} = \mu_-, \text{ and } \left(\frac{\partial F_{N_+}}{\partial N_+}\right)_{T,V} = \mu_+\,.$$

Thus we have

$$\mu_- + \mu_+ = 0\,.$$

3. Disussion by Landau and Lifshiz

Pair production (and the reverse process, annihilation) can be regarded thermodynamically as a chemical reaction ($e^+ + e^- \to \gamma$), where the symbols e^+ and e^- denote a positron and an electron, and γ denotes one or more photons. The chemical potential of the photon gas is zero. The condition of equilibrium for pair production is therefore

$$\mu^+ + \mu^- = 0$$

where μ^+ and μ^- are the chemical potentials of positron and electron gases

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REFERENCES

M.Le Bellac, F. Mortessagne, and G.G. Batrouni, Equilibrium and Non-equilibrium Statistical Thermodynamics (Cambridge, 2004). ISBN-10 0-521-82143-6 (Hardback).

L.D. Landau and E.M. Lifshitz, Statistical Physics (Pergamon Press 1976).

K. Huang, Introduction to Statistical Physics, second edition (CRC Press, 2010). ISBN: 978-1-4200-7902-9.