Radiation from electric dipole moment

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Maxwell's equations imply that all classical electromagnetic radiation is ultimately generated by accelerating electrical charges. Thomson[1] demonstrated a method for determining the electric field of a slowly moving charged particle that is abruptly decelerated to rest. His expression for the electric field due to the acceleration is correct to zero-th order in v/c, where c is the velocity of light. The Thomson's method of analyzing the kink in the electric field is discussed by Tessman and Finnell[2] for the charge moving at any velocity v less than c. The usual method of obtaining the field of an accelerating charge involves the use of the Lienard-Wiechert scalar and vector potential, and is fairly elaborate. Such a complicated mathematics sometimes obscures the physical interpretation that remains clear in the Thomson's simpler derivation.

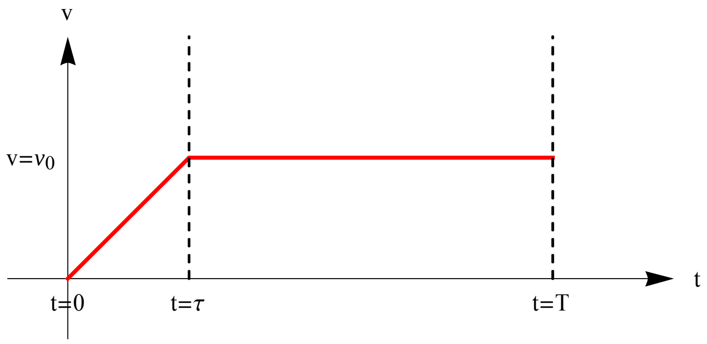
In this note, we discuss the radiation field due to an accelerating charge and the radiation field due to an oscillating electrical dipole moment. There have been so many references on these topics, including the textbooks for the introductory physics and standard textbooks for the electricity and magnetism, and optics, and lectures.[3-11] Nevertheless, we think that students studying the Introductory Physics Course may have some difficulty in understanding the origin of the radiation fields, partly because of the lack in visualization of the physical phenomena. In this note, we make many figures by using Mathematica. We think that such figures may be helpful for students to understand the radiation phenomena which are in general treated by complicated mathematics.

Contents

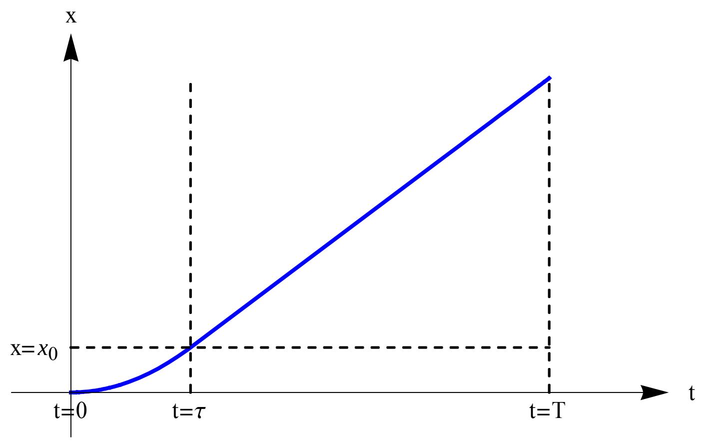
1. Radiation field from a charge with accelerated motion

We can gain some insight into the fields spreading out from an accelerated charge by the following argument. We consider a charge that is initially at rest, then is quickly accelerated for some short time interval τ, and then continues to move with a constant velocity v_0 (= $a\tau$) (which is small compared to the velocity of light c) for a long time. In

Fig.1(a) and Fig.1(b) we show the graph of velocity v versus time t, describing its motion and the graph of position versus x vs time t, respectively.



(a)



(b)

Fig.1 (a) Velocity-time diagram for a particle which is rest at $t = 0$. It then experiences a constant positive acceleration of magnitude $a = v_0/\tau$, which brought it to a velocity v_0 at time $t = \tau$, to $x = x_0 = v_0\tau/2$. After $t = \tau$, the charge continues to move at the constant velocity v_0. We assume v_0 is small compared to c. (b) Position-time diagram for $0 \le t \le T$.

In Fig.2 we show the situation at some time ($t = T$) after the charge undergoes an acceleration for $0 \le t \le \tau$. Prior to $t = 0$, the charge is always at rest at the origin O. The charge is then uniformly accelerated to until $t = \tau$, reaching a speed v_0. This velocity is kept constant after $t = \tau$. The disturbance of the electric field lines begins at $t = 0$ and ends at $t = \tau$. The wave front propagates outward from the origin (the initial position of the charge) and in a time T it reaches out to a distance cT. Beyond the sphere of the radius cT (called as the region I), the electric field lines in that region is uniform, straight, and centered on O, as if the charge were still at the origin O. The inner sphere travels outward from the position A at $t = \tau$; and by some later time T it reaches out to a distance c (T - τ). Within the inner sphere of radius (AQ = $c(T - \tau)$) (called the region II), the electrical field is the new radial field centered on the position B. For observers in the region I (the outside of the outer sphere), the electric field must be that of a charge located at the origin O. For observers in the region II (inside the inner sphere), it is that of a charge (Point B) at the position $x = x_0 = (v_0 t)/2 + v_0(T - t) = v_0(T - \tau/2)$.

The disturbance produced by the accelerated charge is confined to the space between the outer sphere and the inner spheres. The electric field lines in this transition region must connect the lines of the new field of the uniformly moving charge at the point B with the lines of the old field of the stationary charge at the point O. The electric field in this region of the kink has both a radial component and a tangential or transverse component. There now exists a transverse component of the electric field E_t, which propagates outward as a pulse.

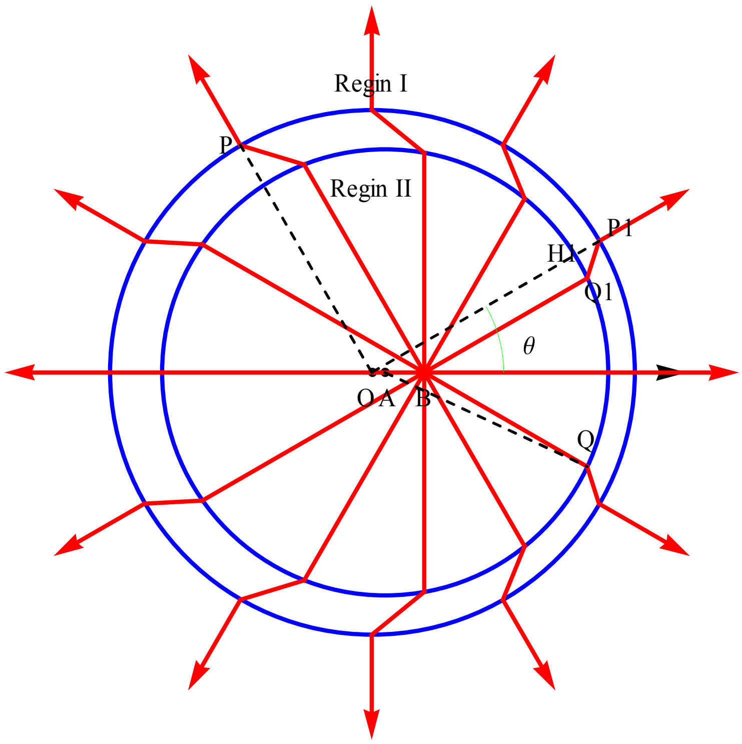


Fig.2 Electrical field lines of a charge that undergoes an acceleration for $0 \leq t \leq \tau$. Here B is the present position of the charge (at $t = T$, $T > \tau$), O is the initial position of the charge (the origin) at $t = 0$, and A is an intermediate position (AB = $v_0(T - \tau)$. Between O and A the charge suffered a constant

acceleration a (= $v_0\tau$). Between A and B the charge moves at the constant velocity v_0. The outer sphere (outer edge of the kink) has radius (OP = cT) and is centered at O. The inner sphere (inner edge of kink) has radius (AQ = $c(T - \tau)$) and is centered on A. $H_1P_1 = c\tau$, H_1Q_1 = OB $\sin\theta = v_0 T \sin\theta$, and OA = $(v_0\tau)/2$. For observers in the region I (the outside of the outer sphere), the electric field must be that of a charge located at the origin O. For observers in the region II (inside the inner sphere), it is that of a charge (Point B) at the position $x = x_0 = (v_0\tau)/2 + v_0(T - \tau) = v_0(T - \tau/2)$. Note that $\overrightarrow{OP_1}$ is parallel to $\overrightarrow{BQ_1}$. This figure is made using the Graphics of the Mathematica. See also Fig.4(a) and Fig.4(b) for the overviews of $\boldsymbol{E}$-lines.

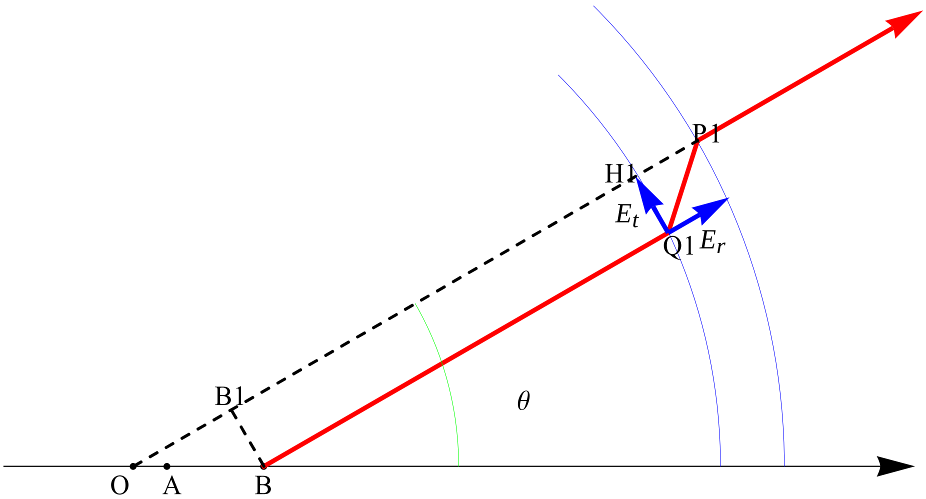


Fig.3 An inner-field line connects with an outer-field line. In the transition region of the kink, the electric field has both a radial component (E_r) and a transverse component (E_t). E_t decreases in proportion to $1/r$. E_r decreases in proportion to $1/r^2$. When the kink reaches a large distance from the source, the electric field in the transition region of the kink will be entirely transverse. Such an electric field at right angle to the direction of propagation is a feature of electromagnetic wave. The transverse kink propagates as a spherical wave. Note that $\overrightarrow{OP_1}$ is parallel to $\overrightarrow{BQ_1}$. BB_1=OB $\sin\theta$. $H_1P_1 = c\tau$. OA = $(v_0\tau)/2$. The center of the inner sphere is at A. H_1 and Q_1 are located on this inner sphere; $AH_1 = AQ_1 = c(T - \tau)$. The center of the outer sphere is at O. Note that OB = OA + AB = $(v_0\tau)/2 + v_0(T - \tau) = v_0(T - \tau/2) \cong v_0 T$. This figure is made using the Graphics of the Mathematica.

As shown in Fig.3 the electric field in the transition region, the spherical shell of thickness $c\tau$ has a radial component (E_r) and a tangential (or transverse) component (E_t). The transverse component is the radiation field of the accelerated charge. From Fig.3, we have the relation

$$\frac{E_t}{E_r} = \frac{H_1Q_1}{H_1P_1} \approx \frac{v_0 T \sin\theta}{c\tau}, \tag{1. 1}$$

where θ is the angle between the acceleration vector and the line from the charge to the observer. The Coulomb's law for the radial component of the *electric field* (electric force per unit charge) a distance from a charge is

$$E_r = \frac{q}{4\pi\varepsilon_0 r^2}. \tag{1. 2}$$

Since $T = r/c$ and $a = v_0/\tau$, we have

$$E_t = \frac{v_0 T \sin\theta}{c\tau} E_r = \frac{q\sin\theta}{4\pi\varepsilon_0 r^2}\frac{v_0}{c\tau}\frac{r}{c} = \frac{qa}{4\pi c^2\varepsilon_0}\frac{\sin\theta}{r}. \tag{1. 3}$$

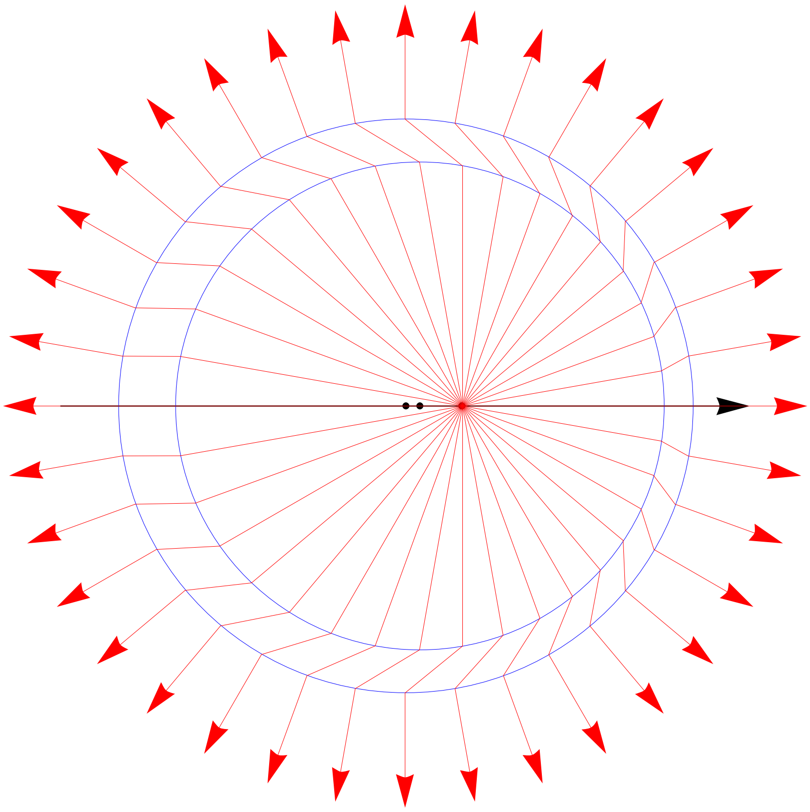


Fig.4(a) A kink in the ***E***-field lines (example-1). The notations in this figure are the same as those of Fig.2. This figure is made using the Graphics of the Mathematica.

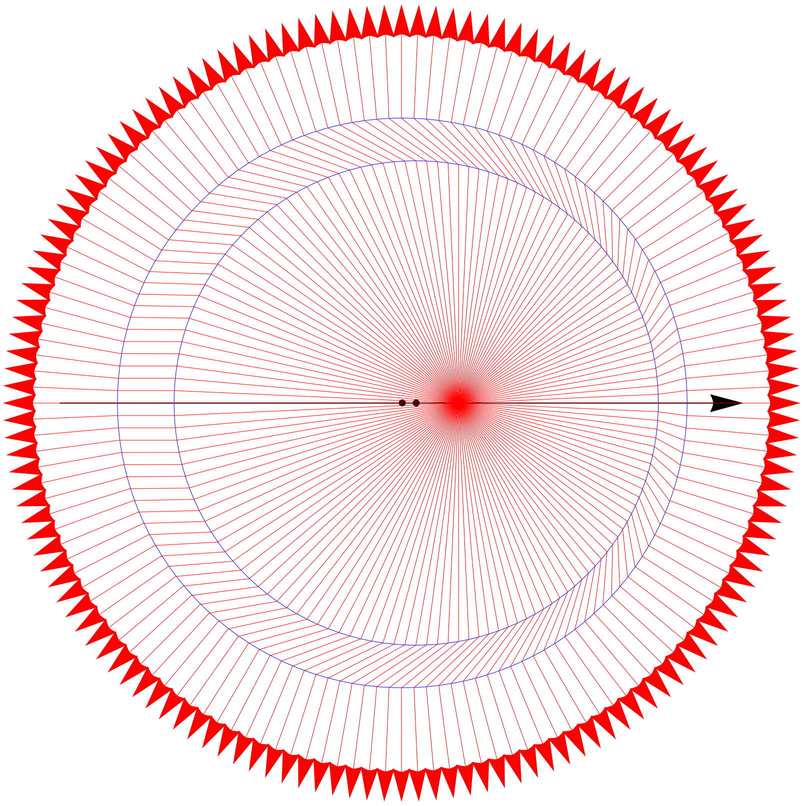


Fig.4(b) A kink in the ***E***-field lines (example-2). The notations in this figure are the same as those of Fig.2. This figure is made using the Graphics of the Mathematica.

The transverse electric field E_t instantaneously reflects the applied acceleration. We note that E_t is proportional to $1/r$, in contrast to E_r ($\propto 1/r^2$). For large r, only E_t contribute significantly to the radiation field. The magnetic field B is perpendicular to $\boldsymbol{r}$ and $\mathbf{E}_t$. How much power is radiated in each direction? In vacuum, the poynting vector is given by

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{c\mu_0}E_\perp{}^2\hat{r} = \frac{1}{c\mu_0}\left(\frac{q\dot{v}}{4\pi c^2\varepsilon_0}\frac{\sin\theta}{r}\right)^2\hat{r} = \frac{\mu_0 q^2\dot{v}^2}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{r} . \qquad (1.4)$$

The total radiated power is obtained by integrating the Poynting vector over a closed surface,

$$
\begin{aligned}
P_{rad} &= \int \mathbf{S}\cdot d\mathbf{A} = \int_0^{\pi} \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c}\frac{\sin^2\theta}{r^2}(2\pi r^2 \sin\theta)d\theta \\
&= \frac{\mu_0 q^2 \dot{v}^2}{16\pi^2 c}2\pi\int_0^{\pi} \sin^3\theta d\theta \\
&= \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3} = \frac{1}{4\pi\varepsilon_0}\frac{2\dot{p}^2}{3c^3} \quad . \qquad (1.5)
\end{aligned}
$$

where p is the electric dipole moment and $a = \dot{v}$ is the acceleration. This is *Larmor's power formula* for an accelerating charge. It states that any charged particle radiates when it is accelerated and that the total power is proportional to a^2. Figure 4 shows the angular dependence of the intensity for the radiation from an accelerated point charge. The direction of the acceleration is the positive x axis.

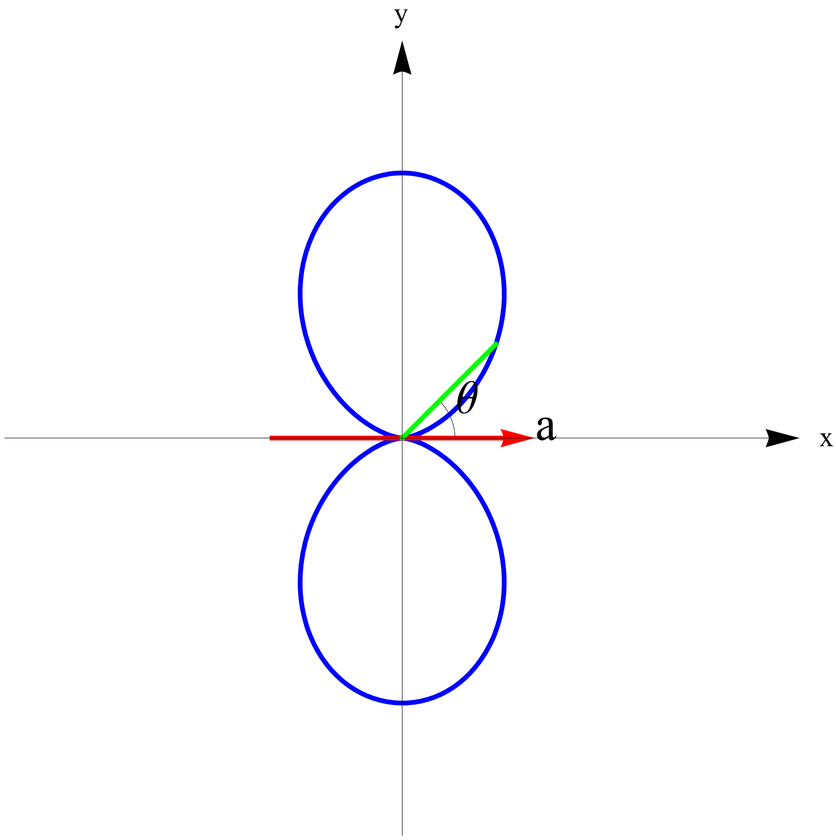


Fig.5 Angular dependence of the intensity pattern for the radiation from an accelerated point charge. The direction of the acceleration is the positive x axis. The radiation is maximum perpendicular to the acceleration vector and

the total power radiated is proportional to q^2a^2. This figure is made using the PolarPlot and Graphics of the mathematica.

2. Radiation from oscillating electric dipole moment

2.1 Formulation

An oscillating electric dipole (an antenna) is used to generate electromagnetic radiation. It is a pair of electric charges that vary sinusoidally with time such that at any instant the two charges equal magnitude but opposite sign. One charge could be equal to $q(t) = Q\cos(\omega t)$ and the other to $-q(t)$, where w is the angular frequency. One technique that works well for radio frequencies is to connect two straight conductors to the terminal of an ac source (see Fig.6).

((Note)) Typical values of f (frequency) and λ (wavelength)

$$c = \lambda f \qquad (2.\ 1)$$

where c is the velocity of light, c = 2.99792458 x 10^8 m/s. When f = 10 GHz, λ = 3 cm. When f = 75 MHz, λ = 4 m.

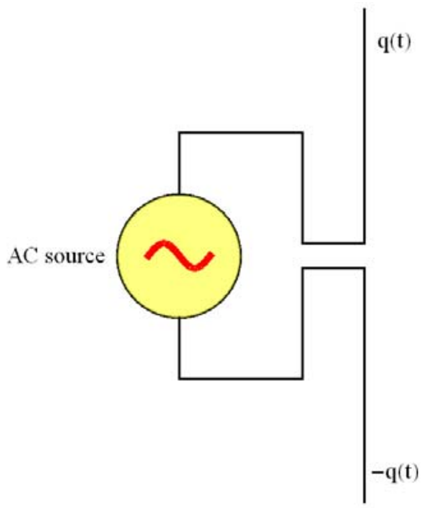
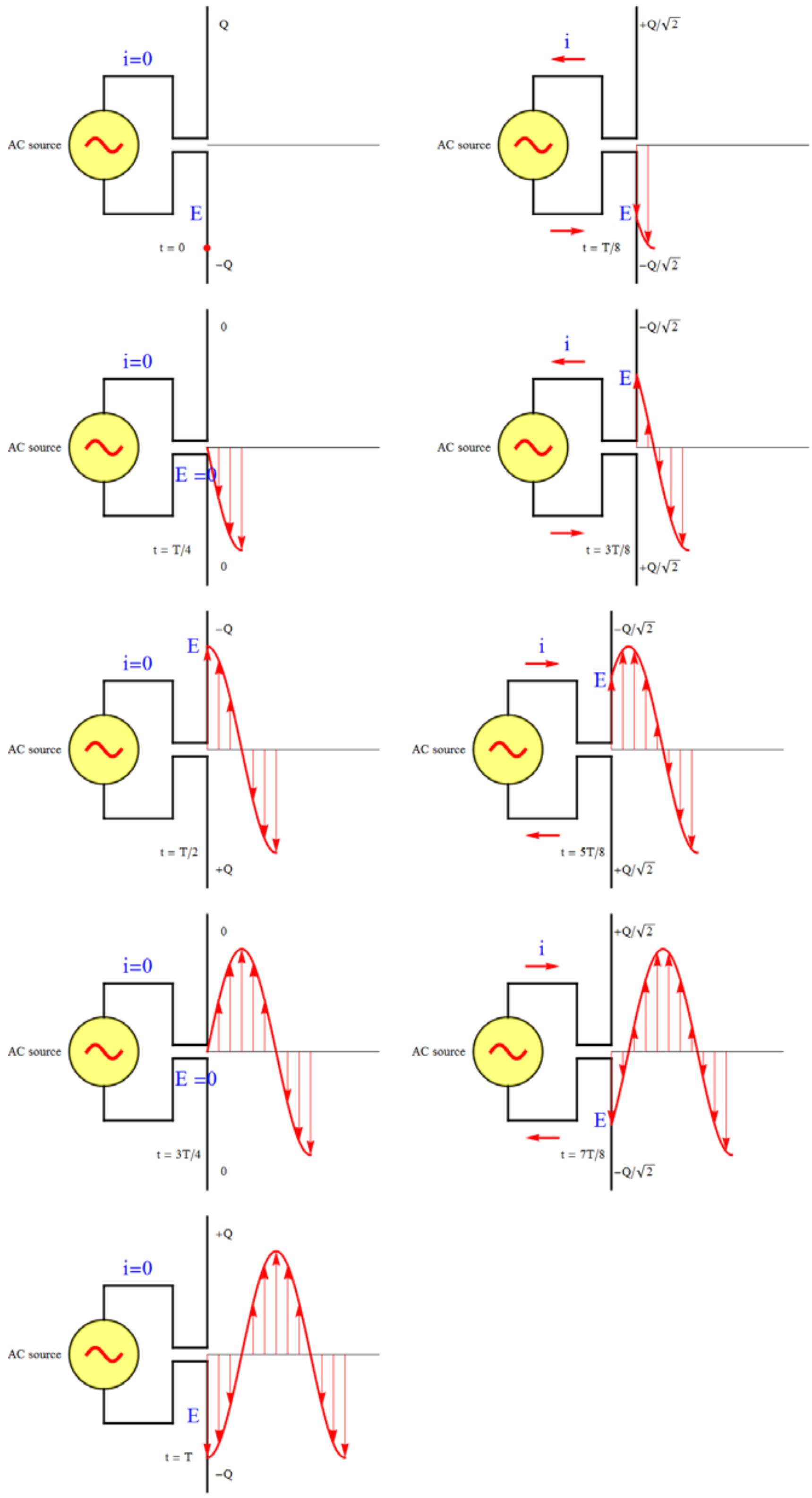


Fig.6 An oscillating electric dipole antenna. Each terminal of an ac source is connected to a straight conductor; the two conductors together comprise the antenna. As the voltage across the source oscillates, the charges on the two conductors also oscillate. The charges are always equal in magnitude and

apposite in sign. This figure is made using the Graphics and Plot (sine curve of the ac source)) of the Mathematica



i=0
AC source
E
t = 0
Q
−Q
i
AC source
E
t = T/8
$+Q/\sqrt{2}$
$-Q/\sqrt{2}$
i=0
AC source
E = 0
t = T/4
0
0
i
AC source
E
t = 3T/8
$-Q/\sqrt{2}$
$+Q/\sqrt{2}$
i=0
AC source
E
t = T/2
−Q
+Q
i
AC source
E
t = 5T/8
$-Q/\sqrt{2}$
$+Q/\sqrt{2}$
i=0
AC source
E = 0
t = 3T/4
0
0
i
AC source
E
t = 7T/8
$+Q/\sqrt{2}$
$-Q/\sqrt{2}$
i=0
AC source
E
t = T
+Q
−Q

Fig.7 One cycle in the production of an electromagnetic wave by an oscillating electric dipole antenna. $T = 2\pi/\omega$. The red curve and arrows depict the ***E*** field at points on the *x* axis (where $\theta = \pi/2$); the magnetic field is not shown. At $t = 0$, the electric field is directed from the upper part of the dipole to the lower part. At $t = T/2$, the electric field is directed from the lower part of the dipole to the upper part. For $0<t<T/2$ the current flows from the upper part of the dipole to the lower part, while for $T/2<t<T$, the current flows from the lower part of the dipole to the upper part. The time dependence of $q(t)$ at the upper part of the dipole and the current flowing the wire, $i(t)$ is shown in Fig.10. This figure is made using the Graphics and Plot (programs) of the Mathematica.

The geometry of the antenna determines the geometrical properties of radiated electric and magnetic fields. We assume a dipole antenna, which can be considered simply as straight conductors. Charges surge back and forth in these two conductors at the angular frequency ω, driven by the oscillator.

The antenna can be regarded as an oscillating electric dipole, in which one branch carries an instantaneous charge $q(t)$ and the other branch carries $-q(t)$. The charge $q(t)$ varies sinusoidally with time and changes sign every half cycle. The charges certainly accelerated as they move back and force in antenna, and as a result the antenna is a source of electric dipole radiation. At any point in space there are electric and magnetic fields that vary sinusoidally with time.

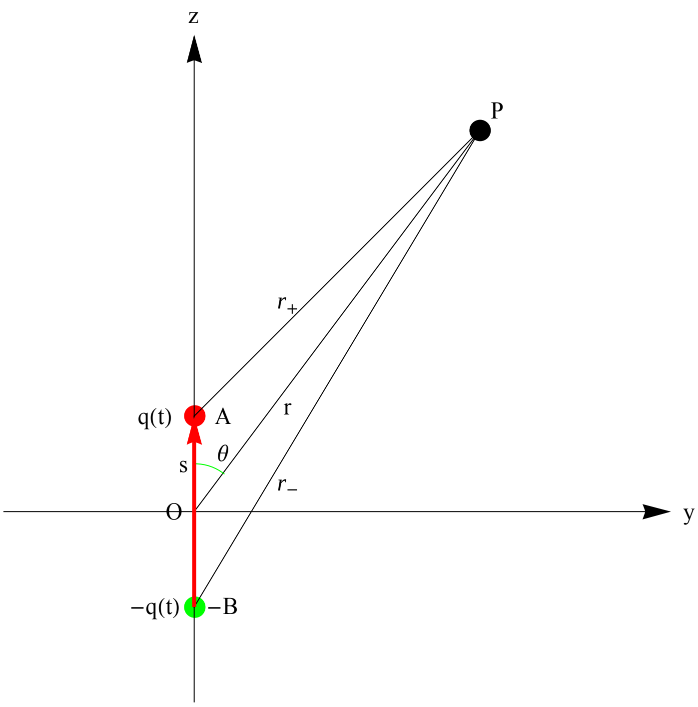


Fig.8 The electric dipole moment with a charge $q(t)$ at the point A (0, 0, s/2) and a charge $-q(t)$ at the point B (0, 0, s/2). s ($\cong$ 0) is the distance between the points A and B

Imagine that two tiny metal spheres separated by a distance s and connected by a fine wire. The system as a whole is electrically neutral. We assume that there are a charge $q(t)$ at the point A (0, 0, s/2) and a charge -$q(t)$ at the point B (0, 0, -s/2), where

$$q(t) = Q\cos(\omega t), \tag{2. 2}$$

and ω is the angular frequency. The electric dipole is defined by

$$\mathbf{p}(t) = q(t)\mathbf{s} = p_0 \cos(\omega t)\mathbf{u}_z, \tag{2. 3}$$

where

$$p_0 = Qs. \tag{2. 4}$$

Suppose further that we somehow contrive to drive the charge back and forth through the wire, from one end to the other, at an angular frequency ω.

The retarded potential at P is given by

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q\cos\omega(t-\frac{r_+}{c})}{r_+} - \frac{Q\cos\omega(t-\frac{r_-}{c})}{r_-}\right], \qquad (2.\ 5)$$

where

$$r+ = \sqrt{r^2 + \frac{s^2}{4} - rs\cos\theta}$$
$$r- = \sqrt{r^2 + \frac{s^2}{4} + rs\cos\theta}. \qquad (2.\ 6)$$

Expanding V as a function of s to the first order of s (using the Mathematica, Series) we have

$$V = \frac{p_0}{4\pi\varepsilon_0}\frac{\cos\theta}{r}[-k\sin(\omega t - kr) + \frac{1}{r}\cos(\omega t - kr)], \qquad (2.\ 7)$$

where k is the wave number and is given by

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c}, \qquad (2.\ 8)$$

(the dispersion relation). In the static limit ($\omega \to 0$, and $k \to 0$,),

$$V = \frac{p_0}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}.$$

This is not, however, the term that concerns us now. We are interested in the fields that survive at large distances from the source, in the so-called radiation zone;

$$V = \frac{p_0}{4\pi\varepsilon_0}\frac{\cos\theta}{r}[-k\sin(\omega t - kr) + \frac{1}{r}\cos(\omega t - kr)] \quad , \qquad (2.\ 9)$$

or

$$V = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\frac{\cos\theta}{r}\sin(\omega t - kr)] \quad . \qquad (2.10)$$

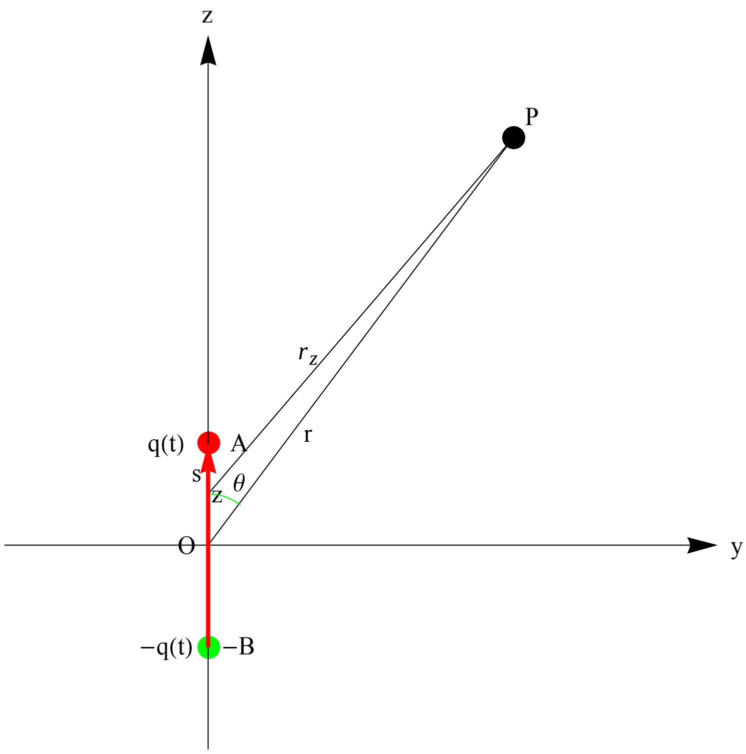


Fig.9 The current (= d$q(t)$/dt) flowing along the wire between A and B. The vetopr potential ***A*** is due to the distribution of current on the wire. The position P is located at (x, y, z).

The vector potential ***A*** is determined by the current flowing in the wire;

$$\mathbf{i}(t) = \frac{dq(t)}{dt}\mathbf{u}_z = -Q\omega\sin(\omega t)\mathbf{u}_z \quad . \qquad (2.11)$$

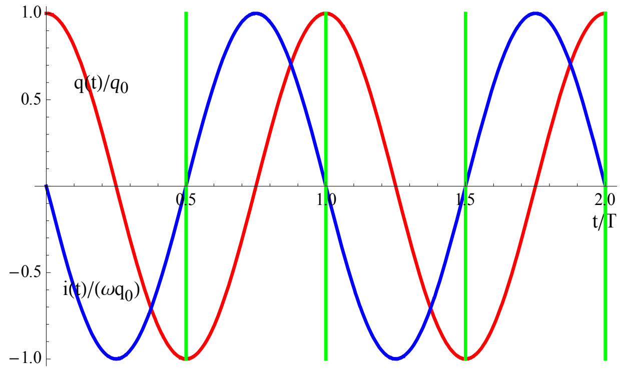


Fig.10 Charge $q(t)$ at the point A (0, 0, s/2) and current flowing a wire between the points A and B (0, 0, -s/2). When $q(t)$ decreases from Q to $-Q$ for $0<t<T/2$ the current defined by $i(t) = dq(t)/dt$ flows from A to B through the wire. When $q(t)$ increases from $-Q$ to Q for $T/2<t<T$, the current flows from the point B to the point A through the wire. See also Fig.7.

We may take the current i to be the same at all points along the length. Then we have the vector potential

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{-s/2}^{s/2} \frac{-Q\omega \sin\omega(t-\frac{r_z}{c})}{r_z} dz \, , \qquad (2.12)$$

where

$$r_z = \sqrt{r^2 + z^2 - 2rz\cos\theta} \, . \qquad (2.13)$$

The integrand of the vector potential is expanded as a series of z around $z = 0$ (Mathematica);

$$-\frac{\mu_0 Q\omega u_z}{4\pi r^2}[r\sin\omega(t-\frac{r}{c}) + \cos\theta\{r\frac{\omega}{c}\cos\omega(t-\frac{r}{c}) + \sin\omega(t-\frac{r}{c})\}]z$$

Noting that the second term in the parenthesis is an odd function of z, we have

$$\mathbf{A} = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin(\omega t - kr)\hat{z} \quad . \tag{2.14}$$

The electric field ***E*** can be evaluated as

$$\mathbf{E} = -\nabla V - \frac{\partial}{\partial t}\mathbf{A} \quad , \tag{2.15}$$

or

$$\mathbf{E} = \frac{p_0\omega}{4c\pi\varepsilon_0}\frac{\cos\theta}{r^2}\sin(kr-\omega t)\mathbf{u}_r + \frac{p_0\omega}{4c\pi\varepsilon_0}\frac{\sin\theta}{r^2}[\sin(kr-\omega t) - kr\cos(kr-\omega t)]\mathbf{u}_\theta \quad . \tag{2.16}$$

E consists of $(1/r)$ terms and $(1/r^2)$ term. When the $(1/r^2)$ term is neglected, we have the approximation for ***E***, as $\boldsymbol{E}_1$,

$$\mathbf{E}_1 \approx -\frac{p_0\omega}{4c\pi\varepsilon_0}\sin\theta\frac{k}{r}\cos(kr-\omega t)\mathbf{u}_\theta \quad . \tag{2.17}$$

The magnetic field ***B*** can be evaluate as

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{p_0\omega}{4c^2\pi\varepsilon_0}\sin\theta[-\frac{k}{r}\cos(kr-\omega t) + \frac{1}{r^2}\sin(kr-\omega t)]\mathbf{u}_\phi \quad . \tag{2.18}$$

B consists of $(1/r)$ terms and $(1/r^2)$ term. When the $(1/r^2)$ term is neglected, we have the approximation for ***B***, as $\boldsymbol{B}_1$,

$$\mathbf{B}_1 \approx -\frac{p_0\omega}{4c^2\pi\varepsilon_0}\sin\theta\frac{k}{r}\cos(kr-\omega t)]\mathbf{u}_\phi \quad . \tag{2.19}$$

We find that the ratio E_1/B_1 is evaluated as

$$\frac{E_1}{B_1} = c \quad , \tag{2.20}$$

which is the same as that for the plane waves. At large distances the fields look locally to be plane. The only difference is that the amplitudes do not remain constant in the direction of propagation but fall off slowly as $1/r$, because the wavefornts are spherical rather than truly planar. The electric and magnetic field are in phase. The fields are mutually perpendicular. The pointing vector ***S*** (energy radiated) is obtained as

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}_1 \times \mathbf{B}_1 = \frac{p_0{}^2\omega^4}{16c^3\pi^2 r^2 \varepsilon_0}\sin^2\theta\cos^2(kr - \omega t)\mathbf{u}_r \quad . \tag{2.21}$$

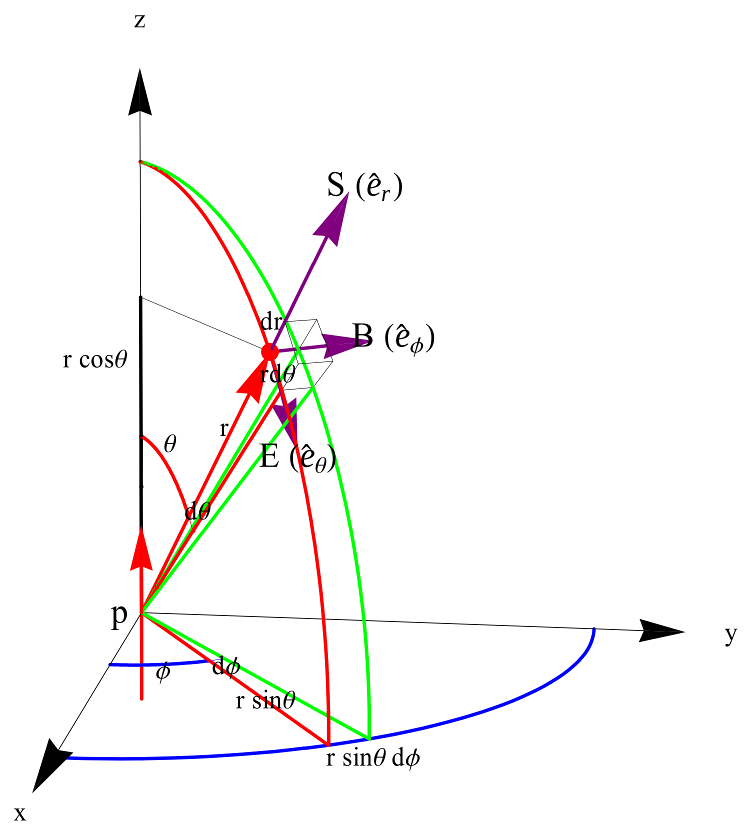


Fig.11 ***p***, ***E***, and ***B*** in the spherical co-ordinate system. ***p***, ***E***, and ***r*** are in the same plane. ***E*** and ***r*** are perpendicular to each other.

(1) The vectors ***p***, ***E*** and ***r*** are in the same plane.
(2) ***E*** is always perpendicular to ***r***.

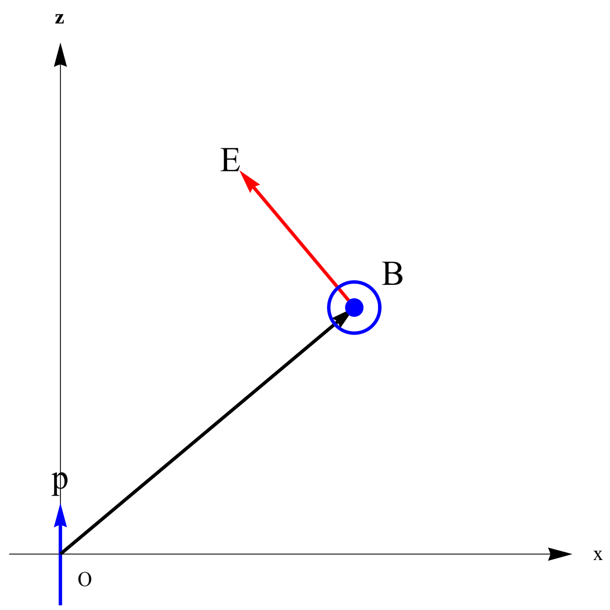


Fig.12 An oscillating electric dipole oriented along the *z* axis. The electric field ***E***, the electric dipole moment ***p*** , and the position vector ***r*** lie in the same plane. The electric field ***E*** is perpendicular to r and the magnetic field ***B***.

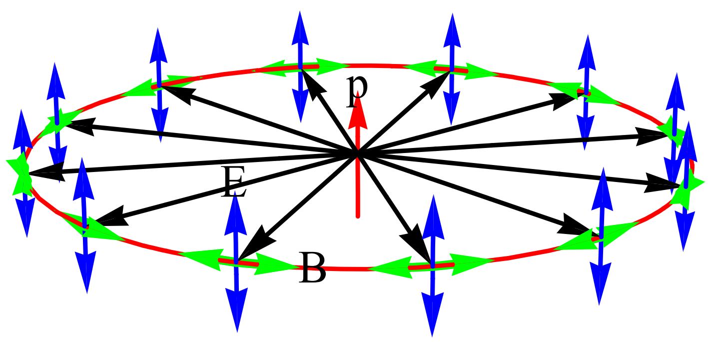


Fig.13 The instantaneous electric field on a sphere centered at a localized linearly oscillating charge. The electric field is along the z axis. The magnetic field B is tangential to the circle. p is the electrical dipole moment. E and B are the electric field and the magnetic field, respectively. This figure is made using the ParametricPlot3D and the Graphics3D of the Mathematica.

The magnitude of the time-averaged Poynting vector is obtained by averaging (in time) over a complete cycle ($T = 2\pi/\omega$)

$$P = \langle S \rangle = \frac{1}{T}\int_0^T S_r dt = \frac{{p_0}^2\omega^4}{32c^3\pi^2\varepsilon_0 r^2}\sin^2\theta \quad . \tag{2.22}$$

P is the total power radiated and is given by

$$P = \int_0^\pi \langle S \rangle (2\pi r^2 \sin\theta) d\theta = \frac{{p_0}^2\mu_0\omega^4}{12c\pi} \quad . \tag{2.23}$$

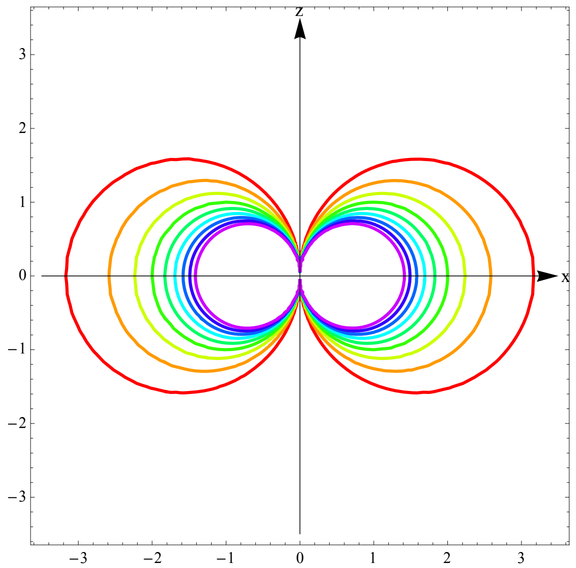


Fig.14(a) Dipole radiation pattern $P = A \sin^2\theta/r^2$. $A = 1$. The distance r is changed as a parameter. This figure is made using the PolarPlot of the Mathematica.

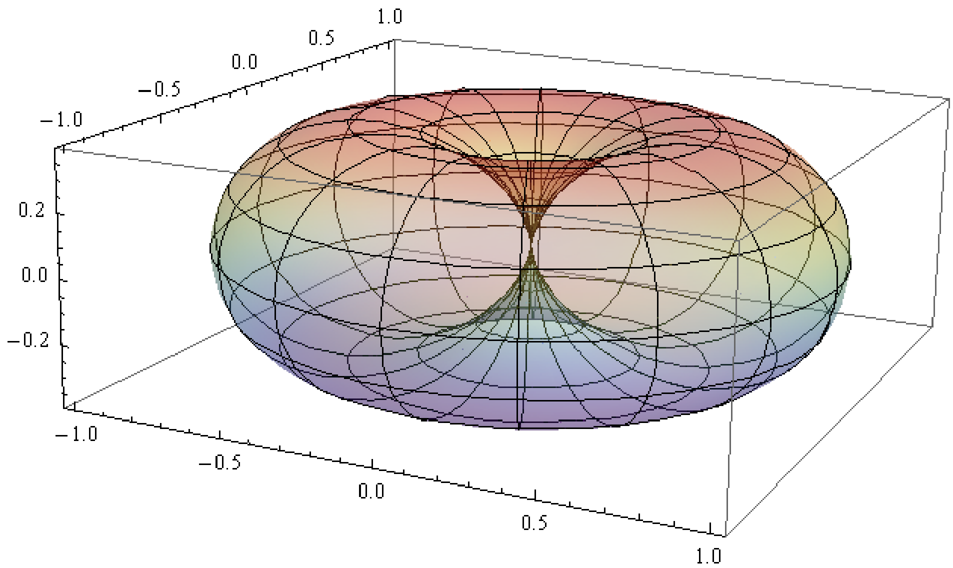


Fig.14(b) Dipole radiation pattern $\sin^2\theta$. This figure is made using the SphericalPlot3D of the Mathematica.

2.2 Simulation using the Mathematica

We now make a plot of the directions of the electric field

$$\mathbf{E} = \frac{p_0\omega}{4c\pi\varepsilon_0}\frac{\cos\theta}{r^2}\sin(kr-\omega t)\mathbf{u}_r + \frac{p_0\omega}{4c\pi\varepsilon_0}\frac{\sin\theta}{r^2}[\sin(kr-\omega t) - kr\cos(kr-\omega t)]\mathbf{u}_\theta, \tag{2.24}$$

and the magnetic field

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{p_0\omega}{4c^2\pi\varepsilon_0 r^2}\sin\theta[-kr\cos(kr-\omega t) + \sin(kr-\omega t)]\mathbf{u}_\phi. \tag{2.25}$$

Here we use the relations between the unit vectors in the spherical coordinates ($\boldsymbol{u}_r$, $\boldsymbol{u}_\theta$, $\boldsymbol{u}_\phi$) and the those in the Cartesian coordinates ($\boldsymbol{u}_x$, $\boldsymbol{u}_y$, $\boldsymbol{u}_z$)

$$\mathbf{u}_r = (\sin\theta\cos\phi)\mathbf{u}_x + (\sin\theta\sin\phi)\mathbf{u}_y + (\cos\theta)\mathbf{u}_z$$
$$\mathbf{u}_\theta = (\cos\theta\cos\phi)\mathbf{u}_x + (\cos\theta\sin\phi)\mathbf{u}_y - (\sin\theta)\mathbf{u}_z$$
$$\mathbf{u}_\phi = (-\sin\phi)\mathbf{u}_x + (\cos\phi)\mathbf{u}_y \quad . \tag{2.26}$$

Then we have the x, y, and z components (E_x, E_y, and E_z) of the electric field as

$$E_x = -\frac{p_0\omega}{8c\pi\varepsilon_0 r^2}\cos\phi\sin(2\theta)[kr\cos(kr-\omega t) - 2\sin(kr-\omega t)] \quad , \tag{2.27 a}$$

$$E_y = -\frac{p_0\omega}{8c\pi\varepsilon_0 r^2}\sin\phi\sin(2\theta)[kr\cos(kr-\omega t) - 2\sin(kr-\omega t)] \quad , \tag{2.27 b}$$

$$E_z = \frac{p_0\omega}{4c\pi\varepsilon_0 r^2}[\sin^2\theta(kr)\cos(kr-\omega t) + \cos(2\theta)\sin(kr-\omega t)] \quad , \tag{2.27c}$$

and the x, y, and z components (B_x, B_y, and B_z) of the magnetic field as

$$B_x = -\frac{p_0\omega}{4c^2\pi\varepsilon_0 r^2}\sin\theta\sin\phi[-kr\cos(kr-\omega t) + \sin(kr-\omega t)] \quad , \tag{2.28a}$$

$$B_y = \frac{p_0\omega}{4c^2\pi\varepsilon_0 r^2}\sin\theta\cos\phi[-kr\cos(kr-\omega t) + \sin(kr-\omega t)] \quad , \tag{2.28b}$$

$$B_z = 0. \tag{2.28c}$$

For simplicity, we use

$$k = \frac{2\pi}{\lambda}$$
$$\omega = ck = \frac{2\pi}{\lambda}c$$
$$\omega t = 2\pi\alpha$$

Then, we get

$$E_x = -A\cos\phi\sin(2\theta)\frac{1}{\frac{r^2}{\lambda^2}}[\pi\frac{r}{\lambda}\cos(2\pi(\frac{r}{\lambda}-\alpha)) - \sin(2\pi(\frac{r}{\lambda}-\alpha))] \quad , \tag{2.29a}$$

$$E_y = -A\sin\phi\sin(2\theta)\frac{1}{\frac{r^2}{\lambda^2}}[\pi\frac{r}{\lambda}\cos(2\pi(\frac{r}{\lambda}-\alpha)) - \sin(2\pi(\frac{r}{\lambda}-\alpha))] \quad , \tag{2.29b}$$

$$E_z = A\frac{1}{\frac{r^2}{\lambda^2}}[\sin^2\theta(2\pi\frac{r}{\lambda})\cos(2\pi(\frac{r}{\lambda}-\alpha)) + \cos(2\theta)\sin(2\pi(\frac{r}{\lambda}-\alpha))] \quad , \tag{2.29c}$$

and

$$B_x = -\frac{A}{c}\sin\theta\sin\frac{1}{\frac{r^2}{\lambda^2}}\phi[-2\pi\frac{r}{\lambda}\cos(2\pi(\frac{r}{\lambda}-\alpha)) + \sin(2\pi(\frac{r}{\lambda}-\alpha))] \quad , \tag{2.30a}$$

$$B_y = \frac{A}{c}\sin\theta\cos\phi\frac{1}{\frac{r^2}{\lambda^2}}[-2\pi\frac{r}{\lambda}\cos(2\pi(\frac{r}{\lambda}-\alpha)) + \sin(2\pi(\frac{r}{\lambda}-\alpha))] \tag{2.30b}$$

where

$$A = \frac{p_0\omega^3}{16c^3\pi^3\varepsilon_0} \quad . \tag{2.31}$$

Using the relations (spherical coordinates)

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \quad , \tag{2.32}$$

or

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arccos[\frac{z}{\sqrt{x^2 + y^2 + z^2}}]$$
$$\phi = \arctan[\frac{y}{x}] \quad , \qquad (2.33)$$

we get the expressions of ***E*** and ***B*** in the Cartesian coordinate.

$$E_x = -A\frac{1}{\frac{x^2 + y^2 + z^2}{\lambda^2}}\cos[\arctan(\frac{y}{x})]\sin[2\arccos(\frac{z}{\sqrt{x^2 + y^2 + z^2}})]$$
$$\times[\pi\frac{\sqrt{x^2 + y^2 + z^2}}{\lambda}\cos(2\pi(\alpha - \frac{\sqrt{x^2 + y^2 + z^2}}{\lambda})) + \sin(2\pi(\alpha - \frac{\sqrt{x^2 + y^2 + z^2}}{\lambda})] \quad , \quad (2.34a)$$

$$E_y = -A\frac{1}{\frac{x^2 + y^2 + z^2}{\lambda^2}}\sin[\arctan(\frac{y}{x})]\sin[2\arccos(\frac{z}{\sqrt{x^2 + y^2 + z^2}})]$$
$$\times[\pi\frac{\sqrt{x^2 + y^2 + z^2}}{\lambda}\cos(2\pi(\alpha - \frac{\sqrt{x^2 + y^2 + z^2}}{\lambda}) + \sin(2\pi(\alpha - \frac{\sqrt{x^2 + y^2 + z^2}}{\lambda})] \quad , \quad (2.34b)$$

$$E_z = -A\frac{1}{\frac{(x^2 + y^2 + z^2)^2}{\lambda^4}}[2\pi\frac{(x^2 + y^2)}{\lambda^2}\frac{\sqrt{x^2 + y^2 + z^2}}{\lambda}\cos(2\pi(\alpha - \frac{\sqrt{x^2 + y^2 + z^2}}{\lambda}))$$
$$-\frac{(x^2 + y^2 + z^2)}{\lambda^2}\cos(2\arccos[\frac{z}{\sqrt{x^2 + y^2 + z^2}})\sin(2\pi(\alpha - \frac{\sqrt{x^2 + y^2 + z^2}}{\lambda}))] \quad ,$$

$$(2.34c)$$

and

$$B_x = -\frac{A}{c}\frac{1}{\frac{(x^2+y^2+z^2)}{\lambda^2}}\frac{y}{\sqrt{x^2+y^2+z^2}}[-2\pi\frac{\sqrt{x^2+y^2+z^2}}{\lambda}\cos(2\pi(\frac{\sqrt{x^2+y^2+z^2}}{\lambda}-\alpha))$$
$$+\sin(2\pi(\frac{\sqrt{x^2+y^2+z^2}}{\lambda}-\alpha))]$$
,

(2.35a)

$$B_y = \frac{A}{c}\frac{1}{\frac{x^2+y^2+z^2}{\lambda^2}}\frac{x}{\sqrt{x^2+y^2+z^2}}[-2\pi\frac{\sqrt{x^2+y^2+z^2}}{\lambda}\cos(2\pi(\frac{\sqrt{x^2+y^2+z^2}}{\lambda}-\alpha))$$
$$+\sin(2\pi(\frac{\sqrt{x^2+y^2+z^2}}{\lambda}-\alpha))]$$
.

(2.35b)

(1) StreamPlot of (E_x, E_z) in the z-x plane

The SteamPlot plots streamlines that show the local direction of the vector field at each point. The length of the arrow does not correspond to the magnitude of the vector. When $y = 0$, $E_y = 0$, we make a plot of the direction of (E_x, E_z) by using the Mathematica (StreamPlot). We assume that $A = 1$ ($\lambda = 1$ as normalization factor). The value α is changed as a parameter. Here we use $\alpha = 3$.

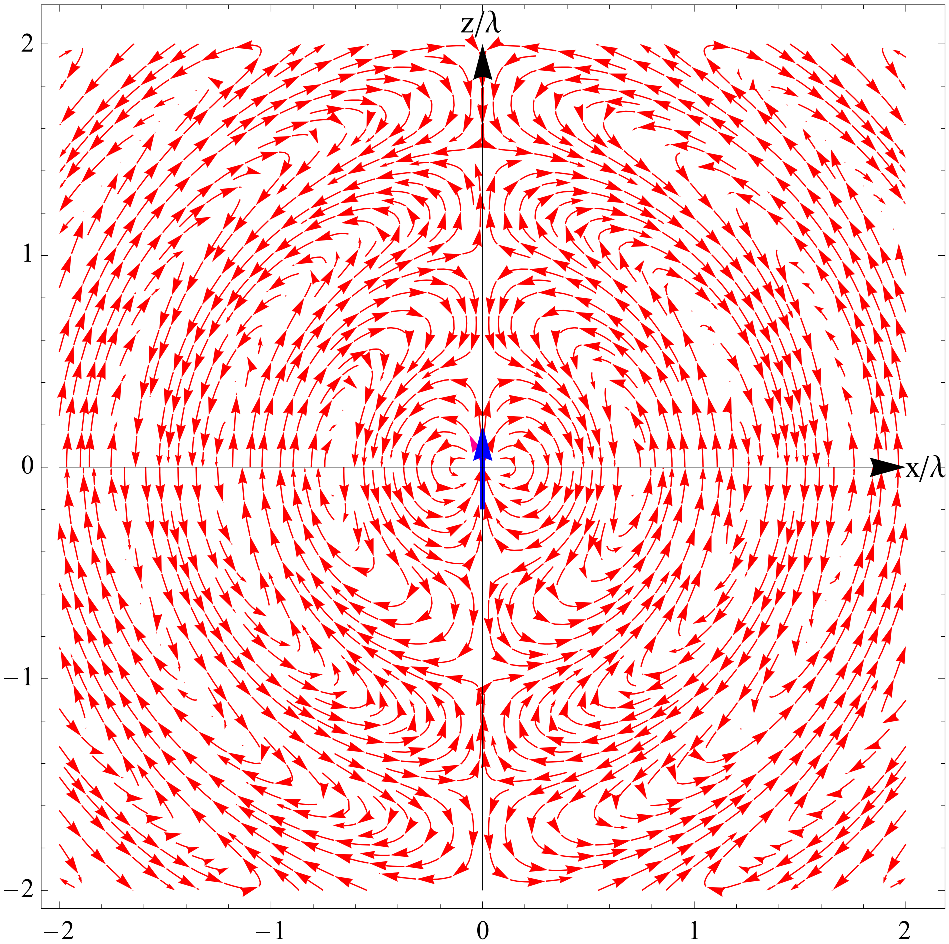


Fig.15 Direction of the electric field lines in the (x, z) plane. The magnitude of each arrow is the same, although the magnitude of $\boldsymbol{E}$ strongly depends on the position in the (x, z) plane. Note that $E_z = 0$ on the z axis. The direction of the oscillating electric dipole (located at the origin) is the z axis. $A = 1$. $\alpha = 3$. $\lambda = 1$ as a normalization factor. This figure is made using the StreamPlot of the Mathematica.

(2) ContourPlot of $E_x{}^2 + E_z{}^2 = \beta$

The ContourPlot (Mathematica) can be used to make a plot of the contour lines where the magnitude of $E_x{}^2 + E_z{}^2$ is kept constant ($= \beta$). β is changed as a parameter. This figure is made by using the ContourPlot of the Mathematica.

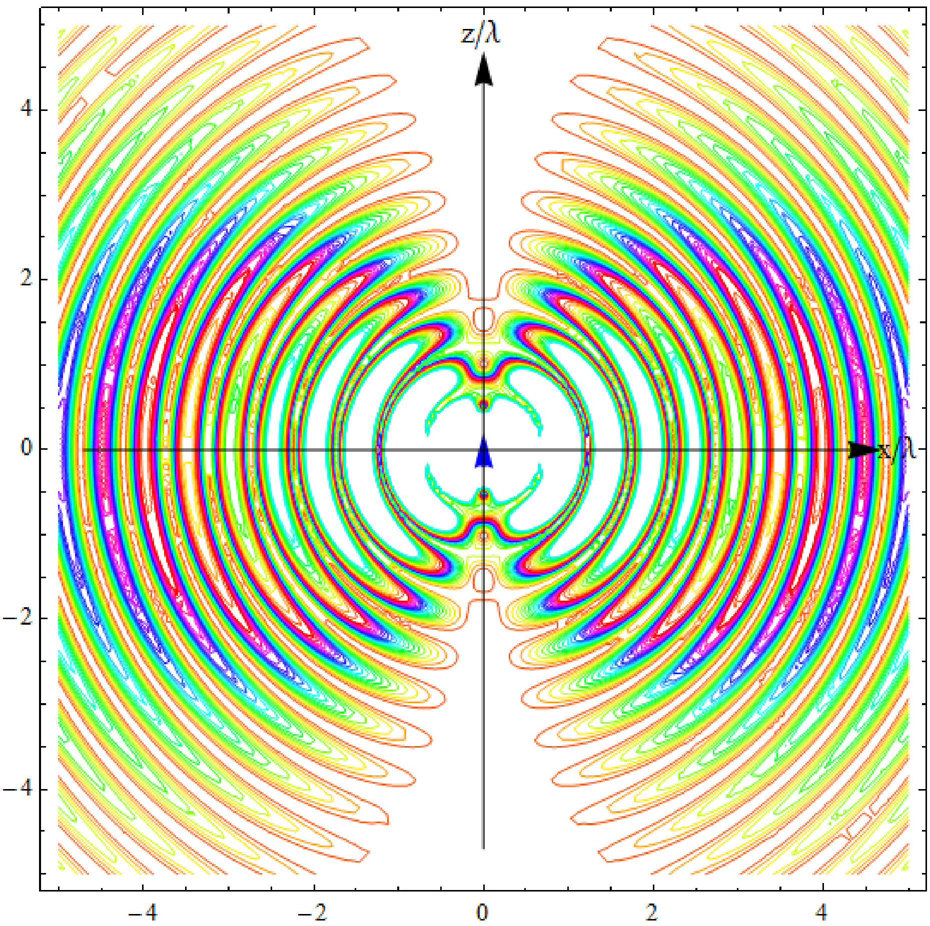


Fig.16 Contour plot of $E_x^{\ 2} + E_z^{\ 2} = \beta$ in the (x/λ, z/λ) plane, where y = 0 and β is changed as a parameter. The direction of the oscillating electric dipole (located at the origin) is the z axis. This figure is made usimg the ContourPlot of the Mathematica. During one period, the loop of $\boldsymbol{E}$ shown closest to the source moves out and expands to become the loop shown farthest from the source.

(3) Propagation of the electric field lines

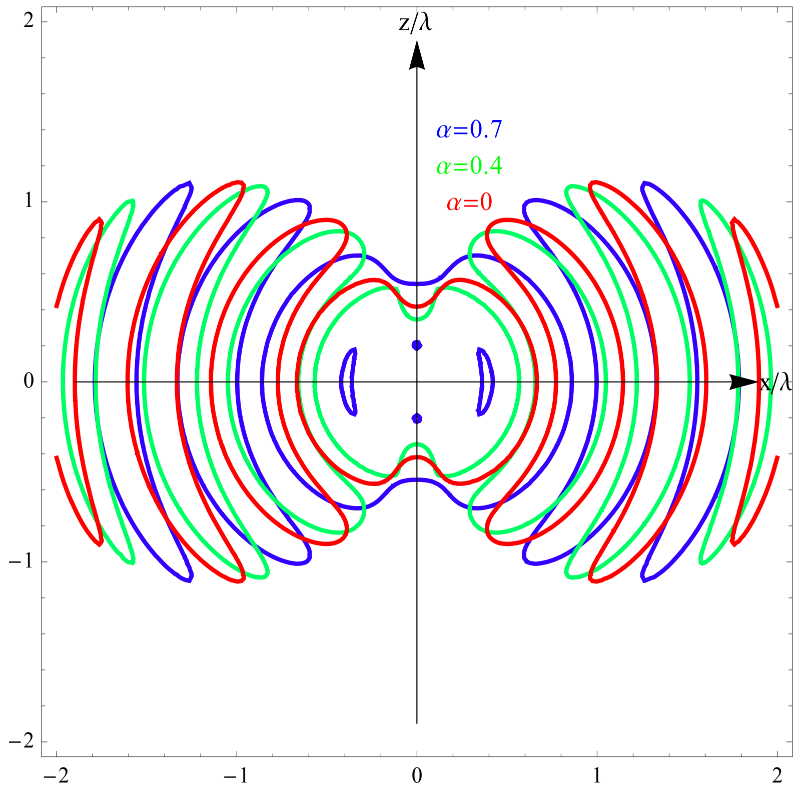


Fig.17 Propagation of the electric field lines, whose time dependence is periodic with a period of T (= $2\pi/\omega$). It takes time for $\boldsymbol{E}$ and $\boldsymbol{B}$ fields to spread outward from oscillating charges on two conductors (the antenna) connected to an ac source, to distant points. A = 1. λ = 1. y = 0. β = 8. The parameter a is changed as a parameter; $\alpha = \omega t$ = 0 (red), 0.4 (green), and 0.7 (purple). This figure is made using the ContourPlot of the Mathematica.

(4) Plot of B_ϕ in the (x, y) plane ($\theta = \pi/2$)

$$B_\phi = \frac{A}{c}\frac{1}{\frac{r^2}{\lambda^2}}\sin\theta[-2\pi\frac{r}{\lambda}\cos(2\pi(\alpha-\frac{r}{\lambda}))-\sin(2\pi(\alpha-\frac{r}{\lambda}))]\mathbf{u}_\phi \qquad (2.36)$$

We notice that B_ϕ is scaling function of r/λ. We now make a plot of B_ϕ for $\theta = \pi/2$ in the (x, y) plane, where the electric dipole p is directed along the positive z direction. In this case, $r = \sqrt{x^2+y^2}$. The electric field is directed along the z axis. We assume that A/c = 1 and α = 3.

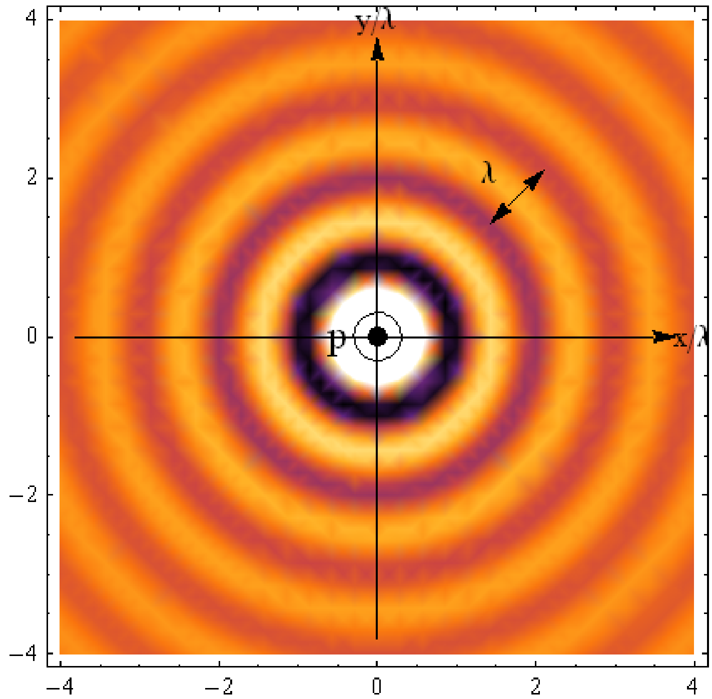


Fig.18 The magnitude of B_ϕ in the (x, y) plane with $z = 0$, forming many rings. The distance between adjacent rings is λ. This figure is made using the DensityPlot of the Mathematica. $A/c = 1$. $\lambda = 1$. $\alpha = 3$. Note that $B_z = 0$.

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