Chapter 12
Laplace transform
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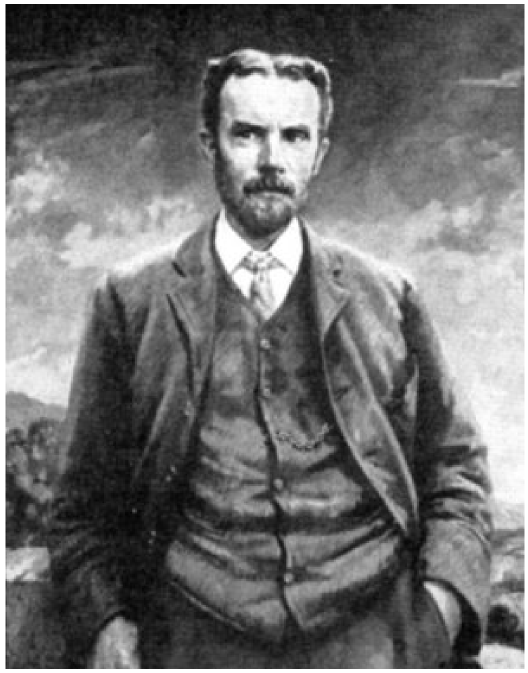
Pierre-Simon, marquis de Laplace (23 March 1749 – 5 March 1827) was a French mathematician and astronomer whose work was pivotal to the development of mathematical astronomy and statistics. He summarized and extended the work of his predecessors in his five volume *Mécanique Céleste* (Celestial Mechanics) (1799–1825). This work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems. In statistics, the so-called Bayesian interpretation of probability was mainly developed by Laplace. He formulated Laplace's equation, and pioneered the Laplace transform which appears in many branches of mathematical physics, a field that he took a leading role in forming. The Laplacian differential operator, widely used in applied mathematics, is also named after him.

http://en.wikipedia.org/wiki/Pierre-Simon_Laplace

Oliver Heaviside (18 May 1850 – 3 February 1925) was a self-taught English electrical engineer, mathematician, and physicist who adapted complex numbers to the study of electrical circuits, invented mathematical techniques to the solution of differential equations (later found to be equivalent to Laplace transforms), reformulated Maxwell's field equations in terms of electric and magnetic forces and energy flux, and independently co-formulated vector analysis. Although at odds with the scientific establishment for most of his life, Heaviside changed the face of mathematics and science for years to come



http://en.wikipedia.org/wiki/Oliver_Heaviside

Thomas John I'Anson Bromwich (1875 – 1929) was an English mathematician, and a Fellow of the Royal Society.

http://en.wikipedia.org/wiki/Thomas_John_l%27Anson_Bromwich

12.1. Definition

The Laplace transform is an integral transform perhaps second only to the Fourier transform in its utility in solving physical problems. The Laplace transform is particularly useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits.

The (unilateral) Laplace transform is defined by

$$F(s) = L[f(t)] = \int_0^\infty dt e^{-st} f(t),$$

with

$f(t)$: original function
$F(s)$: image function

((Link)) http://www.intmath.com/Laplace-transformation/Intro.php

Laplace transform of elementary functions.

(1) $f(t) = e^{at}$

$$L[e^{at}] = \int_0^\infty dte^{-st}e^{at} = \int_0^\infty dte^{-(s-a)t} = \frac{1}{s-a}$$ for Re(s)>Re(a).

(2) $f(t) = t^n$

$$L[t^n] = \int_0^\infty dte^{-st}t^n = \frac{t^n e^{-st}}{-s}\Big|_0^\infty + \frac{n}{s}\int_0^\infty dte^{-st}t^{n-1}.$$

For Re(s)>0,

$$\lim_{t\to\infty}\left|t^n e^{-st}\right| = \lim_{t\to\infty} t^n e^{-R(s)t} = 0.$$

Then we have

$$L[t^n] = \frac{n}{s}L[t^{n-1}] = \frac{n}{s}\frac{n-1}{s}L[t^{n-2}] = ... = \frac{n!}{s^{n+1}}.$$

(3) $f(t) = 1$

$$L[f(t)=1] = \int_0^\infty dte^{-st} = \frac{e^{-st}}{-s}\Big|_0^\infty = \frac{1}{s}.$$

12.2 Laplace transform of derivative

$$L[f'(t)] = \int_0^\infty dte^{-st}f'(t) = e^{-st}f(t)\Big|_0^\infty - (-s)\int_0^\infty dte^{-st}f(t).$$

$$I = \lim_{t\to\infty}\left|e^{-st}\right|f(t) = \lim_{t\to\infty} e^{-\mathrm{Re}(s)t}\left|f(t)\right|.$$

If $\left|f(t)\right| < Me^{at}$, then we have

$$I \le \lim_{t\to\infty} Me^{[a-\mathrm{Re}(s)]t} \to 0$$ for Re(s)>0.

Then

$$L[f'(t)] = sL[f(t)] - f(0) = sF(s) - f(0).$$

Similarly

$$\begin{aligned} L[f^{(2)}(t)] &= sL[f'(t)] - f'(0) \\ &= s[sL[f(t)] - f(0)] - f'(0), \\ &= s^2F(s) - sf(0) - f'(0) \end{aligned}$$

$$L[f^{(3)}(t)] = sL[f''(t)] - f''(0) = s[sL[f'(t)] - f'(0)] - f''(0),$$

or

$$L[f^{(3)}(t)] = s^2[sL[f(t)] - f(0)] - sf'(0)] - f''(0),$$

or

$$L[f^{(3)}(t)] = s^3L[f(t)] - s^2 f(0) - sf'(0)] - f''(0).$$

12.3. Laplace transform of the integral

$$L[\int_0^t dtf(t)] = \int_0^\infty dte^{-st}\left[\int_0^t dpf(p)\right] = [-\frac{1}{s}e^{-st}\int_0^t dpf(p)|_0^\infty + \frac{1}{s}\int_0^\infty dte^{-st} f(t).$$

We consider the first term

If $\left|f(t)\right| < Me^{at}$, then we have

$$\left|\int_0^t dpf(p)\right| \le M\int_0^t dte^{at} = M(\frac{e^{at}-1}{a}),$$

$$\left|\lim_{t\to\infty}\frac{1}{s}e^{-st}\int_0^t dpf(p)\right| < \lim_{t\to\infty}M\left|\frac{e^{-(\operatorname{Re}s-a)t} - e^{-(\operatorname{Re}s)t}}{as}\right| \to 0.$$

Thus for Re(s)>a

$$L[\int_0^t dtf(t)] = \frac{1}{s}F(s).$$

Next we consider the function

(1) Substitution

$$f(t) = t^n e^{at}.$$

$$L[t^n] = \frac{n!}{s^{n+1}}, \quad L[t^n e^{at}] = \int_0^\infty dt e^{-st} t^n e^{at} = \frac{n!}{(s-a)^{n+1}}.$$

More generally,

$$F(s-a) = \int_0^\infty dt e^{-(s-a)t} f(t) = L[e^{at} f(t)].$$

(2) Derivative

$$\frac{dF(s)}{ds} = -\int_0^\infty dt e^{-st} t f(t) = -L[tf(t)].$$

(3) Faltung (Convolution)

$$L[f_1(t)] = \int_0^\infty du e^{-su} f_1(u), \qquad L[f_2(t)] = \int_0^\infty dv e^{-sv} f_2(v),$$

$$L[f_1]L[f_2] = \int_0^\infty du \int_0^\infty dv e^{-s(u+v)} f_1(u) f_2(v),$$

$u + v = t$, $\quad u = \tau$ and $v = t - \tau$.

$[u \geq 0 \text{ and } v \geq 0] \rightarrow [0 \leq t \leq \infty \text{ and } 0 \leq \tau < t]$

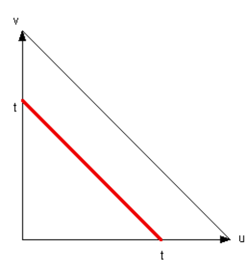


Fig.1

Jacobian:

$$dudv = \begin{vmatrix} \frac{\partial u}{\partial \tau} & \frac{\partial v}{\partial \tau} \\ \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \end{vmatrix} d\tau dt = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} d\tau dt = d\tau dt\,.$$

Then we have

$$L[f_1]L[f_2] = \int_0^\infty du \int_0^\infty dv e^{-s(u+v)} f_1(u) f_2(v) = \int_0^\infty e^{-st} dt \int_0^t d\tau f_1(\tau) f_2(t-\tau) = L[f_1 * f_2]\,,$$

with

$$f_1 * f_2 = \int_0^t d\tau f_1(\tau) f_2(t-\tau)$$

((Note)) There is a relation,

$$f_1 * f_2 = f_2 * f_1,$$

or

$$f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3)\,.$$

(4) Laplace transform of $g(t) = \int_0^t dx f(x)$

$$g'(t) = f(t)\,,$$

$$sG(s) - g(0) = F(s) \quad \text{and } g(0) = 0,$$

or

$$G(s) = \frac{F(s)}{s}\,.$$

12.4. Inverse Laplace transformation

The inverse Laplace transformation is defined as

$$f(t) = L^{-1}[F(s)] = f(t)\,.$$

If $\int_{-\infty}^{\infty} |f(t)| dt$ converges, then we have

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \int_{-\infty}^{\infty} d\tau f(\tau) e^{i\omega\tau} . \qquad \text{(Fourier Integral)}$$

Since

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-\tau)} = \delta(t-\tau) ,$$

then we have

$$f(t) = \int_{-\infty}^{\infty} d\tau f(\tau) \delta(t-\tau) .$$

Now we consider

$$g(t) = e^{-\gamma t} f(t) \Theta(t) .$$

Then we have

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \int_{-\infty}^{\infty} d\tau g(\tau) e^{i\omega\tau} ,$$

or

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{\gamma t} e^{i\omega t} \int_{0}^{\infty} d\tau f(\tau) e^{-\gamma\tau} e^{-i\omega\tau} ,$$

with the choice of variable

$s = \gamma + i\omega$,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{(\gamma + i\omega)t} F(\gamma + i\omega) .$$

Since ds = i$d\omega$

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} ds e^{st} F(s) .$$

(Bromwich integral , or Bromwich -Wagner integral)

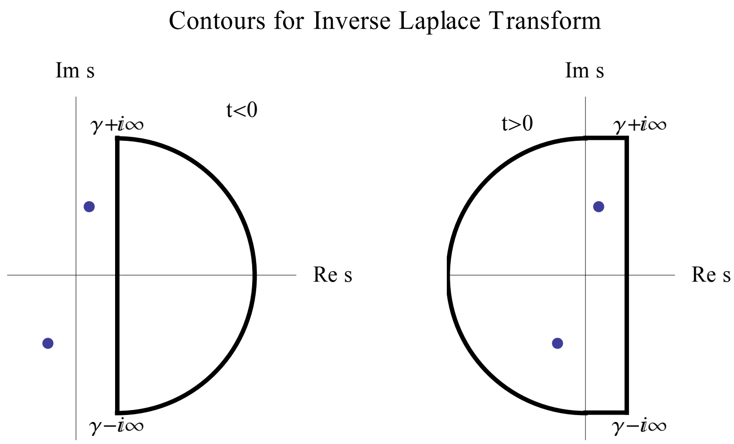


Fig.2

((**Bromwich integral)**) from Wikipedia

An integral formula for the inverse Laplace transform, called the **Bromwich integral**, the **Fourier-Mellin integral**, and **Mellin's inverse formula**, is given by the line integral:

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds$$

where the integration is done along the vertical line $x = \gamma$ in the complex plane such that γ is greater than the real part of all singularities of $F(s)$. This ensures that the contour path is in the region of convergence. If all singularities are in the left half-plane, then γ can be set to zero and the above inverse integral formula above becomes identical to the inverse Fourier transform.

In practice, computing the complex integral can be done by using the Cauchy residue theorem. It is named after Hjalmar Mellin (Finland 1854 – 1933), Joseph Fourier, and Thomas John I'Anson Bromwich (1875-1929).

12.5 Example

We now consider the example of the Bromwich integral of $F(s) = \dfrac{1}{s+a}$.

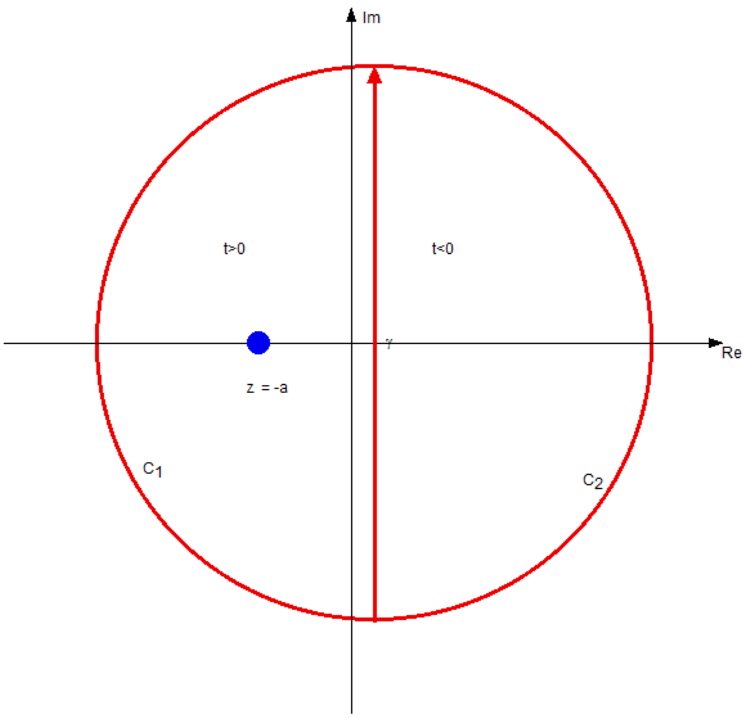


Fig.3

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{1}{s+a}ds$$

Using the Jordan’s lemma, this integral can be rewritten as

$$f(t) = \frac{1}{2\pi i}\oint e^{zt}\frac{1}{z+a}dz$$

along the contour C_1 (counter-clock wise) for t>0 and the contour C_2 (clock wise) for t<0.

There is a single pole at z = -a (<0). Using the residue the torem, we get

$$f(t) = \mathrm{Re}\, s(z=-a) = e^{-at} \qquad \text{for } t>0$$

and

$$f(t) = 0 \qquad \text{for } t<0.$$

12.6 LaplaceTransform and InverseLaplaceTransform (Mathematica)

Using the mathematica, one can calculate the Laplace transfrom and inverse Laplace transform.

(a) `LaplaceTransform[`$f(t)$`, `t`, `s`]`
To represent the Laplace transform of the function f(t) with respect to the variable t and the kernel.

(b) `InverseLaplaceTransform[`$f(s)$`, `s`, `t`]`
to represents the inverse Laplace transform of the function *f*(s) with respect to the variable s and the kernel

12.6.1 Example-1: Laplace transformation

```
Clear["Global`*"]
```

```
LaplaceTransform[f'[t], t, s]
```

```
-f[0] + s LaplaceTransform[f[t], t, s]
```

```
LaplaceTransform[f''[t], t, s]
```

$-s\,f[0] + s^2\,\text{LaplaceTransform}[f[t], t, s] - f'[0]$

```
LaplaceTransform[f'''[t], t, s]
```

$-s^2\,f[0] + s^3\,\text{LaplaceTransform}[f[t], t, s] - s\,f'[0] - f''[0]$

$\text{LaplaceTransform}\left[\int_0^t g[t1]\, dt1, t, s\right]$

$\dfrac{\text{LaplaceTransform}[g[t], t, s]}{s}$

12.6.2 Example-2: Laplace transformation

```
LaplaceTransform[DiracDelta[t], t, s]
```

1

```
InverseLaplaceTransform[1, t, s]
```

DiracDelta[s]

```
LaplaceTransform[t^n, t, s]
```

s^{-1-n} Gamma[1 + n]

```
LaplaceTransform[Exp[k t], t, s]
```

$$\frac{1}{-k+s}$$

```
LaplaceTransform[t Exp[k t], t, s]
```

$$\frac{1}{(k-s)^2}$$

```
LaplaceTransform[Cosh[k t], t, s]
```

$$\frac{s}{-k^2+s^2}$$

```
LaplaceTransform[Sinh[k t], t, s]
```

$$\frac{k}{-k^2+s^2}$$

```
LaplaceTransform[Cos[k t], t, s]
```

$$\frac{s}{k^2+s^2}$$

```
Simplify[LaplaceTransform[Sin[k t], t, s], k > 0]
```

$$\frac{k}{k^2+s^2}$$

```
LaplaceTransform[-Log[t] - EulerGamma, t, s] // Simplify
```

$$\frac{\text{Log}[s]}{s}$$

```
LaplaceTransform[2 - 3 Exp[-10 t] + Exp[-30 t], t, s] // Simplify //
 Factor
```

$$\frac{600}{s\ (10+s)\ (30+s)}$$

12.6.3 Example-3: inverse Laplace transformation

```
Clear["Global`*"]
```

```
f1 = LaplaceTransform[Exp[-a t^2], t, s]
```

$$\frac{e^{\frac{s^2}{4a}}\sqrt{\pi}\,\mathrm{Erfc}\left[\frac{s}{2\sqrt{a}}\right]}{2\sqrt{a}}$$

$$\frac{e^{\frac{s^2}{4a}}\sqrt{\pi}\,\mathrm{Erfc}\left[\frac{s}{2\sqrt{a}}\right]}{2\sqrt{a}}$$

```
Simplify[InverseLaplaceTransform[f1, s, t], a > 0]
```

$$e^{-a t^2}$$

((Note))

$$\mathrm{Erfc}[x] = 1 - \mathrm{Erf}[x],$$

$$\mathrm{Erf}[x] = \frac{1}{\sqrt{\pi}}\int_0^{\pi}\mathrm{Exp}\left[-u^2\right]du$$

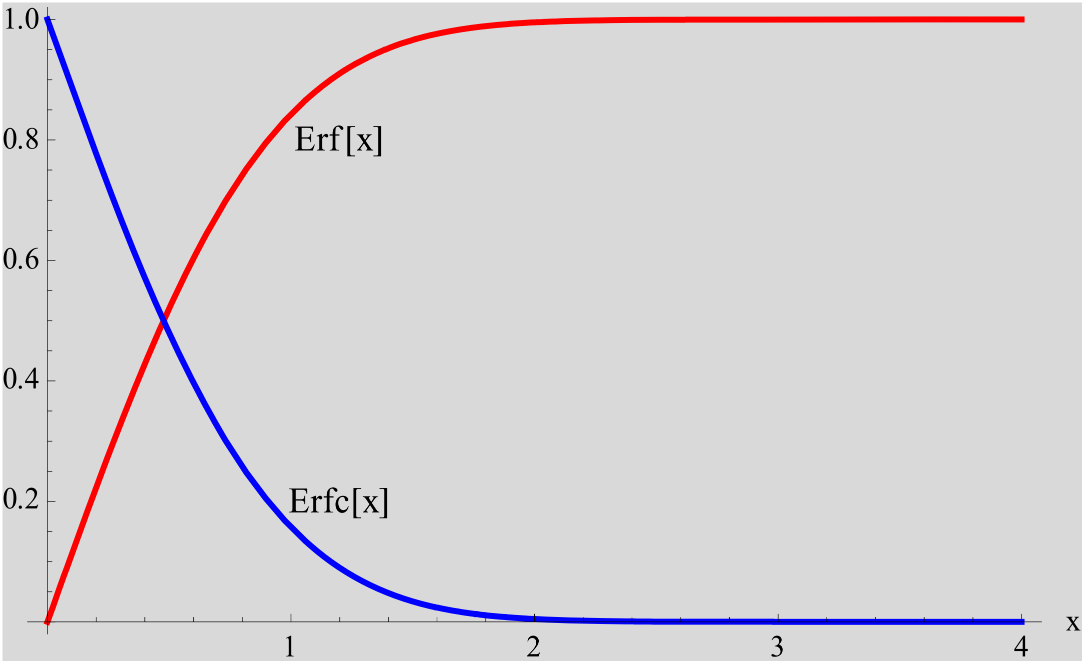


Fig.4 Plot of erf(x) and erfc(x) as a function of x.

12.6.4 Example-4: inverse Laplace transformation

```
Clear["Global`*"]
```

```
InverseLaplaceTransform[s^(-1/2) Exp[1/s], s, t]
```

$$\frac{\text{Cosh}\left[2\sqrt{t}\right]}{\sqrt{\pi}\sqrt{t}}$$

```
InverseLaplaceTransform[Exp[a/s]/s, s, t]
```

$$\text{BesselI}\left[0, 2\sqrt{a}\sqrt{t}\right]$$

```
InverseLaplaceTransform[Exp[-a/s]/s, s, t]
```

$$\text{BesselJ}\left[0, 2\sqrt{a}\sqrt{t}\right]$$

$$\text{InverseLaplaceTransform}\left[\frac{1}{s\sqrt{s}}, s, t\right]$$

$$\frac{2\sqrt{t}}{\sqrt{\pi}}$$

$$\text{InverseLaplaceTransform}\left[\frac{1}{s\left(1+\sqrt{s}\right)}, s, t\right]$$

$$1 - e^{t}\,\text{Erfc}\left[\sqrt{t}\right]$$

$$\text{InverseLaplaceTransform}\left[\frac{1}{\left(1+\sqrt{s}\right)}, s, t\right]$$

$$\frac{1}{\sqrt{\pi}\sqrt{t}} - e^{t}\,\text{Erfc}\left[\sqrt{t}\right]$$

$$\text{InverseLaplaceTransform}\left[\frac{1}{\sqrt{s}\left(1+\sqrt{s}\right)}, s, t\right]$$

$$e^{t}\sqrt{\frac{1}{t}}\sqrt{t}\,\text{Erfc}\left[\frac{1}{\sqrt{\frac{1}{t}}}\right]$$

$$\text{InverseLaplaceTransform}\left[\text{Log}\left[\frac{s+a}{s+b}\right], s, t\right]$$

$$\frac{-e^{-a\,t}+e^{-b\,t}}{t}$$

$$\text{InverseLaplaceTransform}\left[\frac{\sqrt{s}}{s+1}, s, t\right]$$

$$\frac{1}{\sqrt{\pi}\sqrt{t}}+i\,e^{-t}\,\text{Erf}\left[\sqrt{-t}\right]$$

$$\text{InverseLaplaceTransform}\left[\frac{1}{s\left(b+\sqrt{s}\right)}, s, t\right]$$

$$\frac{1-e^{b^2 t}\,\text{Erfc}\left[b\sqrt{t}\right]}{b}$$

$$\text{InverseLaplaceTransform}\left[\frac{1}{\sqrt{s}}, s, t\right]$$

$$\frac{1}{\sqrt{\pi}\sqrt{t}}$$

$$\text{InverseLaplaceTransform}\left[s^{-3/2}\frac{\text{Exp}[1/s]}{\sqrt{s}}, s, t\right]$$

$$\sqrt{t}\,\text{BesselI}\left[1, 2\sqrt{t}\right]$$

12.7 Laplace transformation of the Bessel function

The Bessel function of the first kind can be expressed by

$$J_n(at) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(i(n\theta - at\sin\theta)d\theta \,.$$

The Laplace transform of $J_n(at)$ is

$$\begin{aligned}
L[J_n(at)] &= \frac{1}{2\pi}\int_0^{\infty} \exp(-st)dt \int_{-\pi}^{\pi} \exp(i(n\theta - at\sin\theta)d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(in\theta)d\theta \int_0^{\infty} \exp[-(s + ia\sin\theta)t]dt \,, \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(in\theta)d\theta \frac{1}{s + ia\sin\theta}
\end{aligned}$$

for Re[s]>0. When $z = ae^{i\theta}$, $dz = ae^{i\theta} id\theta = izd\theta$,

and

$$ia\sin\theta = \frac{1}{2}(z - \frac{a^2}{z}) \,.$$

we have

$$\begin{aligned}
L[J_n(at)] &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{z^{n-1}}{a^n} dz \frac{1}{s + \frac{1}{2}(z - \frac{a^2}{z})} \,. \\
&= \frac{1}{2\pi i}\frac{1}{a^n}\oint_C dz \frac{2z^n}{z^2 + 2sz - a^2}
\end{aligned}$$

There is a simple pole at $z = -s + \sqrt{s^2 + a^2}$ inside the closed path (|z|<a). From the Residue theorem,

$$L[J_n(at)] = \frac{1}{\sqrt{s^2 + a^2}} \frac{(\sqrt{s^2 + a^2} - s)^n}{a^n} \,.$$

For $n = 1$,

$$L[J_0(at)] = \frac{1}{\sqrt{s^2 + a^2}}.$$

12.8 Laplace transform of the Bessel differential equation

(a)

$$L[J_1(at)] = -\frac{1}{a} L[\frac{d}{dt} J_0(at)],$$

with $J_0(0)=1$.

$$L[J_1(at)] = -\frac{1}{a}[sL[J_0(at)] - J_0(0)]$$
$$= -\frac{1}{a}(s\frac{1}{\sqrt{s^2 + a^2}} - 1) \qquad .$$
$$= \frac{1}{\sqrt{s^2 + a^2}} \frac{\sqrt{s^2 + a^2} - s}{a}$$

(b)

$$L[\frac{J_n(t)}{t}] = \frac{1}{2n}\{L[J_{n-1}(t)] + L[J_{n+1}(t)]\}$$
$$= \frac{1}{2n\sqrt{s^2 + 1}}\{(\sqrt{s^2 + 1} - s)^{n-1} + (\sqrt{s^2 + 1} - s)^{n+1}\}.$$
$$= \frac{1}{n}(\sqrt{s^2 + 1} - s)^n$$

(c) Laplace transform of the modified Bessel function of the second kind

$$L[I_n(at)] = i^{-n}\{L[J_n(iat)]$$
$$= i^{-n}\frac{1}{\sqrt{s^2 - a^2}} \frac{(\sqrt{s^2 - a^2} - s)^n}{a^n i^n}.$$
$$= \frac{(s - \sqrt{s^2 - a^2})^n}{a^n\sqrt{s^2 - a^2}}$$

12.9 Laplace transform of Bessel functions (Mathematica)

Laplace transformation of the Bessel function

```
Clear["Global`*"]
```

```
LaplaceTransform[BesselJ[0, t], t, s]
```

$$\frac{1}{\sqrt{1+s^2}}$$

```
LaplaceTransform[BesselJ[1, t], t, s]
```

$$\frac{1}{\sqrt{1+s^2}\left(s+\sqrt{1+s^2}\right)}$$

```
LaplaceTransform[BesselJ[2, t], t, s]
```

$$\frac{1}{\sqrt{1+s^2}\left(s+\sqrt{1+s^2}\right)^2}$$

```
LaplaceTransform[BesselJ[n, t], t, s]
```

$$\frac{\left(s+\sqrt{1+s^2}\right)^{-n}}{\sqrt{1+s^2}}$$

```
LaplaceTransform[BesselI[0, t], t, s] //
  Simplify[#, s > 1] &
```

$$\frac{1}{\sqrt{-1+s^2}}$$

```
LaplaceTransform[BesselI[1, t], t, s] //
  Simplify[#, s > 1] &
```

$$-1+\frac{s}{\sqrt{-1+s^2}}$$

```
LaplaceTransform[BesselI[2, t], t, s] //
  Simplify[#, s > 1] &
```

$$\frac{-1+2\,s^2-2\,s\,\sqrt{-1+s^2}}{\sqrt{-1+s^2}}$$

```
LaplaceTransform[BesselI[n, t], t, s]
```

$$\frac{\left(s+\sqrt{-1+s^2}\right)^{-n}}{\sqrt{-1+s^2}}$$

12.10 Faltung (convolution): Example-1

The Faltung (convolution) can be used for solving the integral equations.

(a) Example-1

$$\int_0^t \frac{g(\tau)}{\sqrt{t-\tau}} d\tau = t^n . \qquad (1)$$

The Laplace transforms:

$$G(s) = L[g(t)], \qquad K[s] = L[\frac{1}{\sqrt{t}}] = \sqrt{\frac{\pi}{s}},$$

$$L[t^n] = \frac{n!}{s^{n+1}} .$$

From the property of the Faltung (convolution), the Laplace transform of Eq.(1) can be obtained as

$$G(s)\sqrt{\frac{\pi}{s}} = \frac{n!}{s^{n+1}},$$

or

$$G(s) = \frac{n!}{\sqrt{\pi} s^{n+1/2}}.$$

Then we have

$$g(t) = L^{-1}[G(s)] = \frac{n!}{\sqrt{\pi}\Gamma(n+\frac{1}{2})} t^{n-\frac{1}{2}}.$$

((**Mathematica**))

```
K[s_] = LaplaceTransform[1/Sqrt[t], t, s]
```

$$\frac{\sqrt{\pi}}{\sqrt{s}}$$

```
G[s_] = n!/(Sqrt[π] s^(n+1/2));

InverseLaplaceTransform[G[s], s, t]
```

$$\frac{t^{-\frac{1}{2}+n}\, n!}{\sqrt{\pi}\ \mathrm{Gamma}\left[\frac{1}{2}+n\right]}$$

12.11 Faltung (convolution): Example-2

We solve

$$g(t) - \int_0^t g(\tau) d\tau = 1,$$

by using the Laplace transform. Laplace transform of this equation is obtained as

$$G[s] - \frac{1}{s}G(s) = \frac{1}{s},$$

or

$$G(s) = \frac{1}{s-1}.$$

Then we have

$$g(t) = L^{-1}[G(s)] = e^{t}$$

12.12 Faltung (convolution): Example-3

$$g(t) - \int_0^t (t-\tau)g(\tau)d\tau = 2(\cos t + \sin t)$$

Laplace transform of this equation:

$$G(s) - \frac{1}{s^2}G(s) = 2(\frac{s+1}{s^2})$$

or

$$G(s) = \frac{2s^2}{(s-1)(s^2+1)}$$

The inverse Laplace transform (Bromwich iontegral):

$$\begin{aligned} g(t) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} G(s)e^{st}ds \\ &= \frac{1}{2\pi i}\oint_C G(s)e^{st}ds = \mathrm{Re}\,s(s=1) + \mathrm{Re}\,s(s=i) + \mathrm{Re}\,s(s=-i) \\ &= e^{t} + \sin t + \cos t \end{aligned}$$

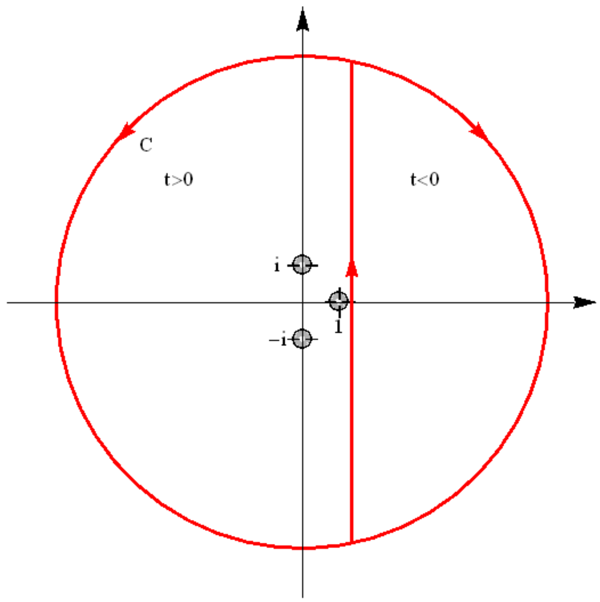


Fig.5

((**Mathematica**))

```
Residue[F[s], {s, 1}]
```

e^t

```
Residue[F[s], {s, i}] + Residue[F[s], {s, -i}] // FullSimplify
```

```
Cos[t] + Sin[t]
```

12.13 Faltung (convolution): Example-4

$$g(t) - a\int_0^t J_1[a(t-\tau)]g(\tau)d\tau = \cos(at)$$

The Laplace transform of this equation:

$$G(s)[1 - (\frac{\sqrt{s^2+a^2} - s}{\sqrt{s^2+a^2}})] = \frac{s}{s^2+a^2}$$

or

$$G(s) = \frac{1}{\sqrt{s^2+a^2}}.$$

Then we have

$$g(t) = J_0(at)$$

12.14 Arfken: Example 15-10-3, p.983

We consider the Laplace transform of the Bessel equation with $n = 0$

$$xy'' + y' + xy$$

We put $x = t$, $y = f(t)$

Initial condition; $f(0) = 1$ and $f'(0) = 0$

$$t\ddot{f} + \dot{f} + tf = 0$$

Noting that

$$L[t\ddot{f}] = -\frac{d}{ds}L[\ddot{f}] = -\frac{d}{ds}[s^2F(s) - sf(0) - f'(0)] = -\frac{d}{ds}[s^2F(s) - s]$$

$$L[tf] = -\frac{d}{ds}L[f] = -\frac{d}{ds}F(s)$$

we have

$$-\frac{d}{ds}[s^2F(s) - s] + sF(s) - 1 - \frac{d}{ds}F(s) = 0$$

or

$$-[s^2F'(s) + 2sF(s) - 1] + sF(s) - 1 - F'(s) = 0$$

or

$$(s^2 + 1)F'(s) + sF(s) = 0$$

or

$$F(s) = \frac{C}{\sqrt{s^2 + 1}}$$

When $C = 1$, $f(t) = J_0(t)$;

$$L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$$

where J_0(t) is the Bessel function of the first kind.

((**Mathematica**))

```
InverseLaplaceTransform[1/Sqrt[s^2 + 1], s, t]
```

```
BesselJ[0, t]
```

12.15 Bromwich integral Arfken 15-12-4

$$L^{-1}[\frac{s}{s^2-k^2}]$$

(a) Partial fraction expansion

$$F(s) = \frac{s}{s^2-k^2} = \frac{s}{(s+k)(s-k)} = \frac{1}{2}(\frac{1}{s+k} + \frac{1}{s-k})$$

$$f(t) = L^{-1}[F(s)] = \frac{1}{2}(e^{-kt} + e^{kt})$$

(b) Bromwich integral

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{s}{s^2-k^2}e^{st}ds \\ &= \frac{1}{2\pi i}\int_C \frac{s}{s^2-k^2}e^{st}ds \\ &= \operatorname{Re}s(s=k) + \operatorname{Re}s(s=-k) \\ &= \frac{1}{2}(e^{kt} + e^{-kt}) \end{aligned}$$

where we use the Jordan's lemma.

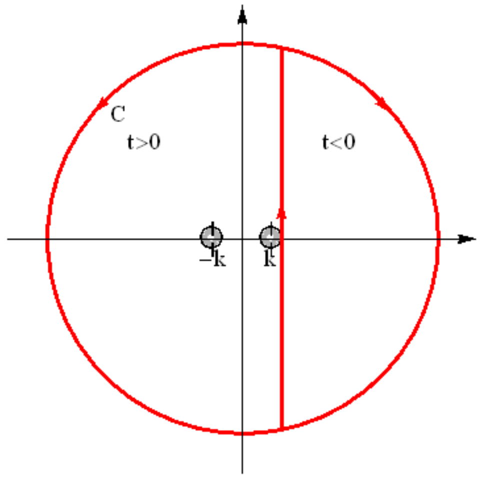


Fig.6

12.16 Bromwich integral Arfken 15-12-5c

Bromwich integral

$$L^{-1}[\frac{1}{s(s^2+1)}],$$

$$F(s) = \frac{1}{s(s^2+1)},$$

$$\begin{aligned}
f(t) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{s(s^2+1)}e^{st}ds \\
&= \frac{1}{2\pi i}\int_C \frac{1}{s(s^2+1)}e^{st}ds \\
&= \mathrm{Re}\,s(s=i) + \mathrm{Re}\,s(s=-i) + \mathrm{Re}\,s(s=0) \quad . \\
&= e^{0t} + \frac{e^{it}}{i(2i)} + \frac{e^{-it}}{(-i)(-2i)} \\
&= 1 - \cos t
\end{aligned}$$

where we use the Jordan's lemma.

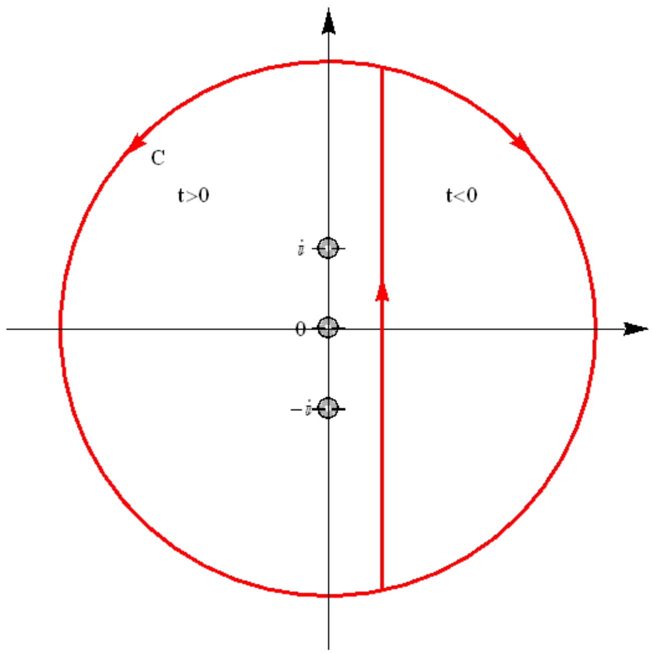


Fig.7

12.17 Bromwich integral Arfken 15-12-6

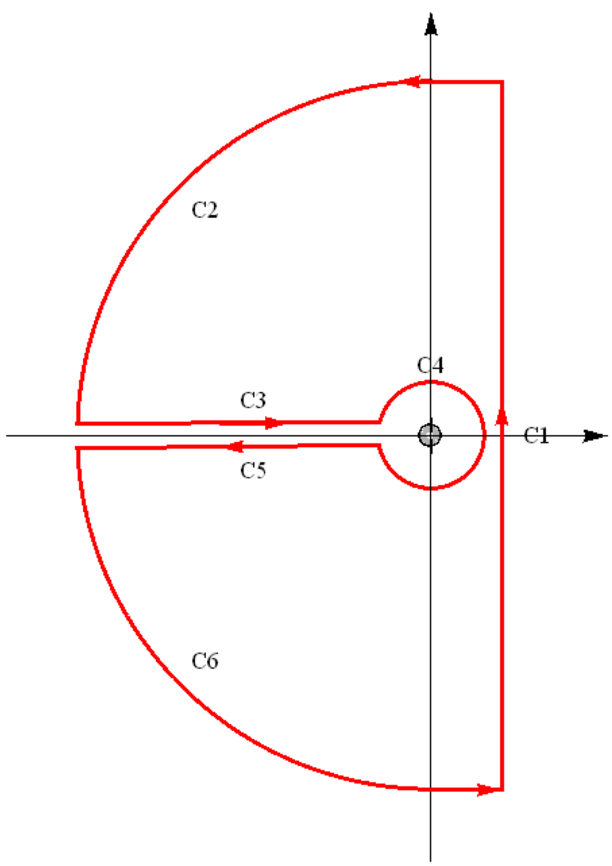


Fig.8

$$F(s) = \frac{1}{\sqrt{s}}.$$

$s = 0$, $s = \infty \rightarrow$ branch point.

$$L^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) ds$$
$$= \frac{1}{2\pi i}\{\oint e^{st} F(s) ds - [\int_{C2} e^{st} F(s) ds + \int_{C3} e^{st} F(s) ds.$$
$$+ \int_{C4} e^{st} F(s) ds + \int_{C5} e^{st} F(s) ds + \int_{C6} e^{st} F(s) ds]\}$$

Since there is no pole, we get

$$\oint e^{st} F(s) ds = 0,$$

from the Cauchy's theorem.

$$L^{-1}[F(s)] = -\frac{1}{2\pi i}[\int_{C2} e^{st} F(s) ds + \int_{C3} e^{st} F(s) ds$$
$$+ \int_{C4} e^{st} F(s) ds + \int_{C5} e^{st} F(s) ds + \int_{C6} e^{st} F(s) ds].$$

The integral along the line C5 (path 5 $\rightarrow$ 4)

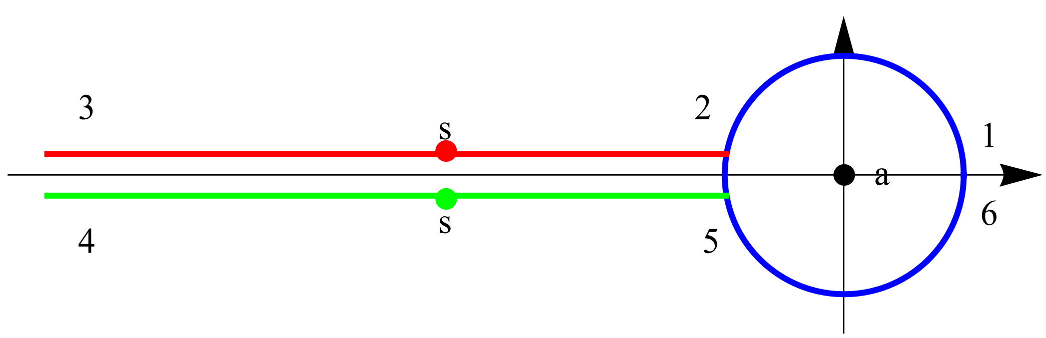


Fig.9

$s - 0 = qe^{-i\pi}$ (q>0) on the line between the points 5 and 4.

$$I_5 = -\frac{1}{2\pi i}\int_{C5} ds e^{st} F(s)$$
$$= -\frac{1}{2\pi i}\int_0^\infty e^{-qt}\frac{1}{\sqrt{qe^{-i\pi}}} e^{-i\pi} dq$$
$$= \frac{1}{2\pi i}\int_0^\infty e^{-qt}\frac{1}{\sqrt{q}} e^{i\pi/2} dq$$
$$= \frac{1}{2\pi}\int_0^\infty e^{-qt}\frac{1}{\sqrt{q}} dq = \frac{1}{2\pi}\frac{\Gamma(\frac{1}{2})}{\sqrt{t}} = \frac{1}{2\sqrt{\pi t}}$$

The integral along the line C3 (path 3 → 2):

$$s - 0 = qe^{i\pi} \qquad (q>0).$$

$$I_3 = -\frac{1}{2\pi i}\int_{C3} ds e^{st} F(s)$$
$$= -\frac{1}{2\pi i}\int_\infty^0 e^{-qt}\frac{1}{\sqrt{qe^{i\pi}}} e^{i\pi} dq$$
$$= -\frac{1}{2\pi i}\int_0^\infty e^{-qt}\frac{1}{\sqrt{q}} e^{-i\pi/2} dq$$
$$= \frac{1}{2\pi}\int_0^\infty e^{-qt}\frac{1}{\sqrt{q}} dq = \frac{1}{2\pi}\frac{\Gamma(\frac{1}{2})}{\sqrt{t}} = \frac{1}{2\sqrt{\pi t}}$$

From the Jordan's lemma,

$$\lim_{R\to\infty}\int_{C2} e^{st}F(s)ds = 0, \quad \text{and} \qquad \lim_{R\to\infty}\int_{C6} e^{st}F(s)ds = 0,$$

when

$$\lim_{R\to\infty} F(s) = 0,$$

where R (→∞) is the radius of the path C2 and the path C6.

The integral around the small circle (C4) centered at $s = 0$:

$$s - 0 = \varepsilon e^{i\theta}, \qquad ds = \varepsilon i e^{i\theta} d\theta,$$

$$
\begin{aligned}
-\frac{1}{2\pi i}\int_{C4} e^{st}F(s)ds &= -\frac{1}{2\pi i}\int_{\pi}^{-\pi} e^{st}F(s)ds \\
&= \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{\varepsilon e^{i\theta}t}\frac{1}{\sqrt{\varepsilon e^{i\theta}}}\varepsilon i e^{i\theta}d\theta \\
&= \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{\varepsilon e^{i\theta}t}\sqrt{\varepsilon} i e^{i\theta/2}d\theta \\
&= \frac{\sqrt{\varepsilon}}{2\pi}\int_{-\pi}^{\pi} e^{i\theta/2}d\theta = 0
\end{aligned},
$$

as $\varepsilon \to 0$. Then we have

$$L^{-1}[F(s)] = I_3 + I_5 = \frac{1}{\sqrt{\pi t}}.$$

((**Mathematica**))

```
InverseLaplaceTransform[1/Sqrt[s], s, t]
```

$$\frac{1}{\sqrt{\pi}\sqrt{t}}$$

12.18 Bromwich integral Arfken 15-12-7

Show that

$$L^{-1}[F(s)] = L^{-1}[\frac{1}{\sqrt{s^2+a^2}}] = J_0(at).$$

by evaluation of the Bromwich integral, where

$$F(s) = \frac{1}{\sqrt{s^2+a^2}}$$

We note that the line connecting between s = ia, s = -ia, form a cut line.

$$L^{-1}[F(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds$$

$$= \frac{1}{2\pi i}\{\oint e^{st}F(s)ds - [\int_{C2} e^{st}F(s)ds + \int_{C3} e^{st}F(s)ds$$

$$+ \int_{C4} e^{st}F(s)ds + \int_{C5} e^{st}F(s)ds + \int_{C6} e^{st}F(s)ds$$

$$+ \int_{C7} e^{st}F(s)ds + \int_{C8} e^{st}F(s)ds + \int_{C9} e^{st}F(s)ds + \int_{C10} e^{st}F(s)ds]\}$$

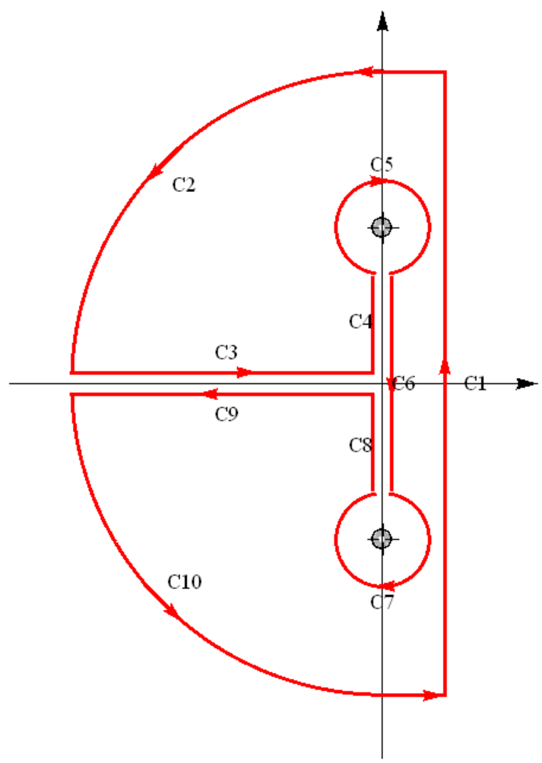


Fig.10

Since there is no pole inside the closed path, we have

$$\oint e^{st}F(s)ds = 0 . \qquad \text{(Cauchy's theorem).}$$

From the Jordan's lemma,

$$\int_{C2} e^{st}F(s)ds = 0 \text{, and} \quad \int_{C10} e^{st}F(s)ds \quad \text{for t>0.}$$

We note that $F(s)$ is continuous when crossing between the lines C3 and C9. In other words, the integrals along the C3 line and the line C9 cancel out each other.

$$\int_{C3} e^{st}F(s)ds + \int_{C9} e^{st}F(s)ds = 0\,.$$

On the circle C5 with the radius $\varepsilon\,(\to 0)$,

$$s - ia = \varepsilon e^{i\theta} \qquad (\theta = 3\pi/2 \to -\pi/2).$$

$$\begin{aligned}
\int_{C5} e^{st}F(s)ds &= \int_{C5} \frac{e^{st}}{\sqrt{(s+ia)(s-ia)}}ds \\
&= \int_{3\pi/2}^{-\pi/2} \frac{e^{\varepsilon e^{i\theta}t}}{\sqrt{(2ia+\varepsilon e^{i\theta})\varepsilon e^{i\theta}}}\varepsilon e^{i\theta} i d\theta \\
&= -i\int_{-3\pi/2}^{\pi/2} \frac{e^{\varepsilon e^{i\theta}t}}{\sqrt{(2ia+\varepsilon e^{i\theta})}}\sqrt{\varepsilon}e^{i\theta/2}d\theta \\
&= -i\sqrt{\varepsilon}\int_{-3\pi/2}^{\pi/2} \frac{1}{\sqrt{(2ia)}}e^{i\theta/2}d\theta = 0
\end{aligned},$$

On the circle C7 with the radius $\varepsilon\,(\to 0)$,

$$s + ia = \varepsilon e^{i\theta} \qquad (\theta = \pi/2 \to -3\pi/2)$$

$$\begin{aligned}
\int_{C7} e^{st}F(s)ds &= \int_{C7} \frac{e^{st}}{\sqrt{(s+ia)(s-ia)}}ds \\
&= \int_{\pi/2}^{-3\pi/2} \frac{e^{\varepsilon e^{i\theta}t}}{\sqrt{(-2ia+\varepsilon e^{i\theta})\varepsilon e^{i\theta}}}\varepsilon e^{i\theta} i d\theta \\
&= -i\int_{-3\pi/2}^{\pi/2} \frac{e^{\varepsilon e^{i\theta}t}}{\sqrt{(-2ia+\varepsilon e^{i\theta})}}\sqrt{\varepsilon}e^{i\theta/2}d\theta \\
&= -i\sqrt{\varepsilon}\int_{-3\pi/2}^{\pi/2} \frac{1}{\sqrt{(-2ia)}}e^{i\theta/2}d\theta = 0
\end{aligned}$$

as $\varepsilon \to 0$.

$$\int_{C7} e^{st}F(s)ds = 0\,.$$

Then we have

$$L^{-1}[F(s)] = -\frac{1}{2\pi i}[\int_{C4} e^{st}F(s)ds + \int_{C6} e^{st}F(s)ds + \int_{C8} e^{st}F(s)ds].$$

On the line C4 and C8:

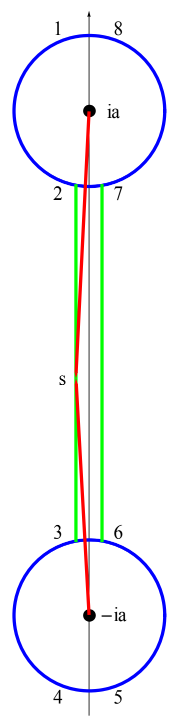


Fig.11

$$s + ia = qe^{i\pi/2}, \qquad (0 \leq q < 2a)$$

$$s - ia = (2a - q)e^{i3\pi/2}$$

$$\begin{aligned}\int_{C4+C8} \frac{e^{st}}{\sqrt{s^2 + a^2}} ds &= \int_{C4+C8} \frac{e^{st}}{\sqrt{(s+ia)(s-ia)}} ds \\ &= \int_0^{2a} \frac{e^{(-ia+qe^{i\pi/2})t} dqe^{i\pi/2}}{\sqrt{qe^{i\pi/2}(2a-q)e^{i3\pi/2}}} \quad . \\ &= -i\int_0^{2a} \frac{e^{-i(a-q)t} dq}{\sqrt{q(2a-q)}}\end{aligned}$$

On the line C6:

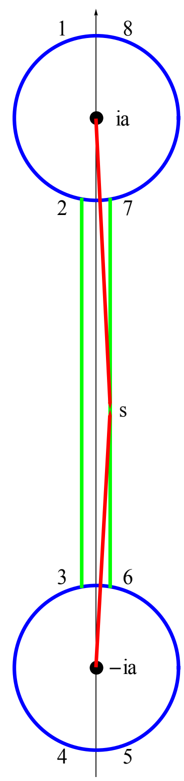


Fig.12

$$s + ia = qe^{i5\pi/2}, \qquad (0 \le q < 2a)$$

$$s - ia = (2a - q)e^{i3\pi/2}.$$

Then we have

$$\begin{aligned}\int_{C6} \frac{e^{st}}{\sqrt{s^2 + a^2}} ds &= \int_{C6} \frac{e^{st}}{\sqrt{(s+ia)(s-ia)}} ds \\ &= \int_{2a}^{0} \frac{e^{(-ia + qe^{i5\pi/2})t} dqe^{i5\pi/2}}{\sqrt{qe^{i5\pi/2}(2a-q)e^{i3\pi/2}}}. \\ &= -i \int_{0}^{2a} \frac{e^{-i(a-q)t} dq}{\sqrt{q(2a-q)}}\end{aligned}$$

Finally we have

$$L^{-1}[F(s)] = -\frac{1}{2\pi i}[-i\int_0^{2a}\frac{e^{-i(a-q)t}dq}{\sqrt{q(2a-q)}} - i\int_0^{2a}\frac{e^{-i(a-q)t}dq}{\sqrt{q(2a-q)}}]$$
$$= \frac{1}{\pi}\int_0^{2a}\frac{e^{i(q-a)t}dq}{\sqrt{q(2a-q)}}.$$

We put

$$q - a = a\cos\theta,$$

where $0 \le \theta \le \pi$. Since $dq = a(-\sin\theta)d\theta$, we have

$$L^{-1}[F(s)] = \frac{1}{\pi}\int_\pi^0 \frac{e^{ia\cos\theta t}a(-\sin\theta)}{a\sin\theta}d\theta = \frac{1}{\pi}\int_0^\pi e^{ia\cos\theta t}d\theta = J_0(at).$$

((**Mathematica**))

```
InverseLaplaceTransform[1/Sqrt[s^2 + a^2], s, t]

BesselJ[0, a t]
```

12.18 Bromwich integral Arfken 15-12-10

Evaluate the Bromwich integral for

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F(s)e^{st}ds$$

with

$$F(s) = \frac{s}{(s^2+a^2)^2}$$

Bromwich integral:

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F(s)e^{st}ds = \frac{1}{2\pi i}\oint_C F(s)e^{st}ds$$

from the Jordan's lemma (t>0). There are two poles at $s = ia$ and $-ia$. From the Residue thorem, we have

$$f(t) = \frac{1}{2\pi i}\oint_C F(s)e^{st}ds = \mathrm{Res}(s = ia) + \mathrm{Res}(s = -ia) = \frac{t\sin(at)}{2a}$$

((Mathematica))

```
Residue[G[s], {s, ⅈ a}] +
  Residue[G[s], {s, -ⅈ a}] // ExpToTrig
```

$$\frac{t\,\mathrm{Sin}[a\,t]}{2\,a}$$

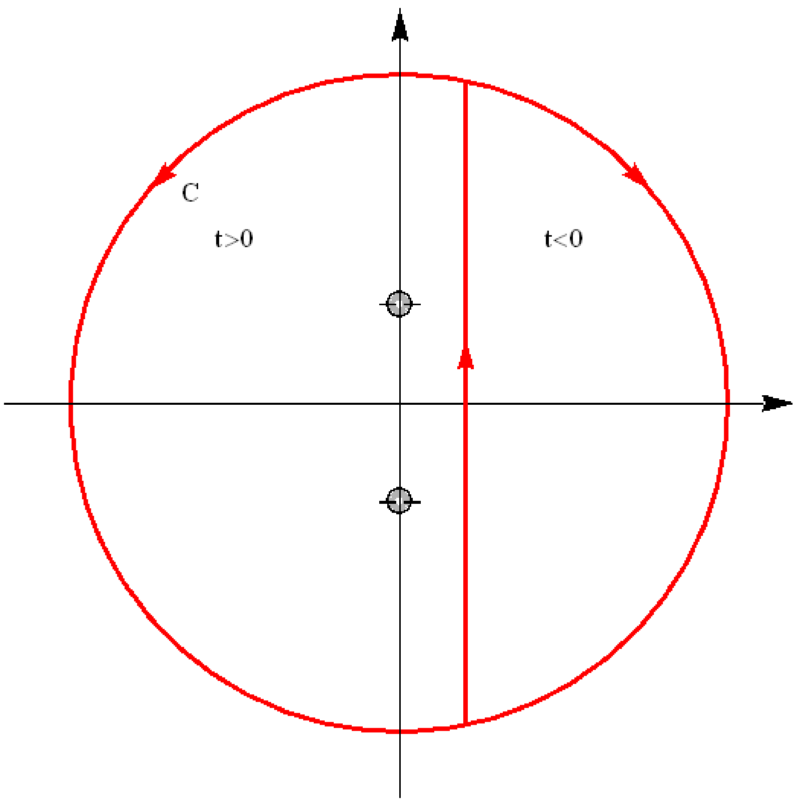


Fig.13

Here note that

$$\frac{d}{ds}\left[\frac{se^{st}}{(s \pm ia)^2}\right] = \frac{(s \pm ia)(e^{st} + ste^{st}) - 2se^{st}}{(s \pm ia)^3},$$

$$\mathrm{Res}(s = ia) = \frac{2ia(e^{iat} + iate^{iat}) - 2iae^{iat}}{(2ia)^3} = \frac{t}{4ia}e^{iat},$$

$$\mathrm{Re}\,s(s=-ia)=\frac{(-2ia)(e^{-iat}-iate^{-iat})+2iae^{-iat}}{(-2ia)^3}=-\frac{t}{4ia}e^{-iat}.$$

Therefore

$$f(t)=\frac{t}{4ia}(e^{iat}-e^{-iat})=\frac{t}{4ia}2i\sin(at)=\frac{t}{2a}\sin(at).$$

12.19 Bromwich integral; Arfken 15-12-13

You have a Laplace transform;

$$F(s)=\frac{1}{(s+a)(s+b)}. \qquad ((a\neq b)$$

Invert this transform by each of three methods.

(a) Partial fractions and use of tables.

$$F(s)=\frac{1}{(s+a)(s+b)}=\frac{1}{b-a}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)$$

$$f(t)=L^{-1}[F(s)]=\frac{1}{b-a}(e^{-at}-e^{-bt}) \qquad \text{for } a\neq b$$

(2) Faltung (convolution).

$$L^{-1}[F(s)]=\int_0^t e^{-a(t-\tau)}e^{-b\tau}d\tau=e^{-at}\int_0^t e^{(a-b)\tau}d\tau$$
$$=e^{-at}\frac{1}{a-b}[e^{(a-b)\tau}]|_0^t$$
$$=\frac{1}{b-a}(e^{-at}-e^{-bt})$$

(3) Bromwich integral.

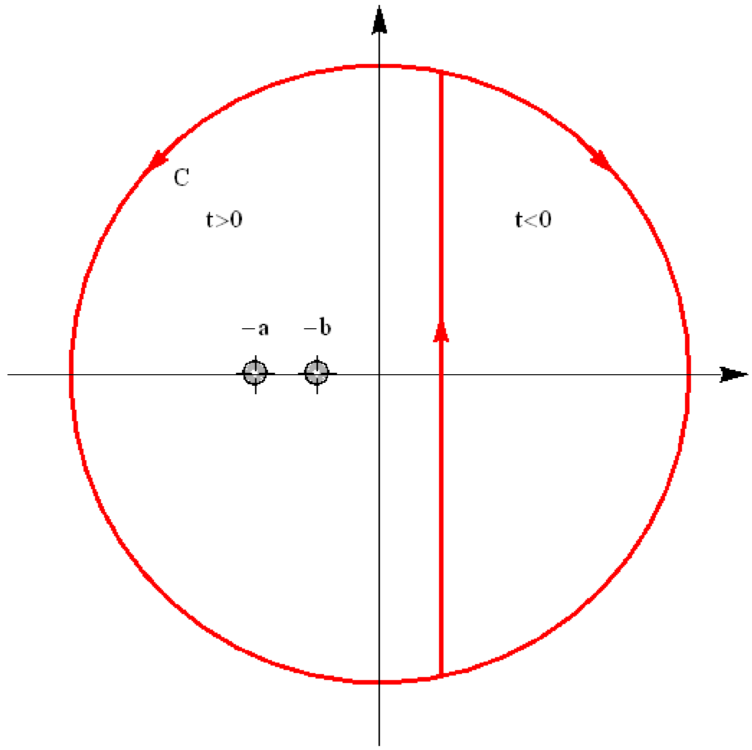


Fig.14

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}ds = \frac{1}{2\pi i}\oint_C F(s)e^{st}ds$$
$$= \mathrm{Re}\,s(s=-a) + \mathrm{Re}\,s(s=-b) \qquad ,$$
$$= \frac{1}{b-a}(e^{-at} - e^{-bt})$$

from the Jordan's lemma and the residue theorem.

((**Mathematica**))

```
F[s_] = 1/((s + a) (s + b));

Residue[F[s] e^(s t), {s, - a}] +
  Residue[F[s] e^(s t), {s, -b}] // Simplify
```

$$-\frac{e^{-a t} - e^{-b t}}{a - b}$$

12.20 Cut lines

The choice of variables in the path integrals when the multi-valued function is included

(a) $\sqrt{s}$

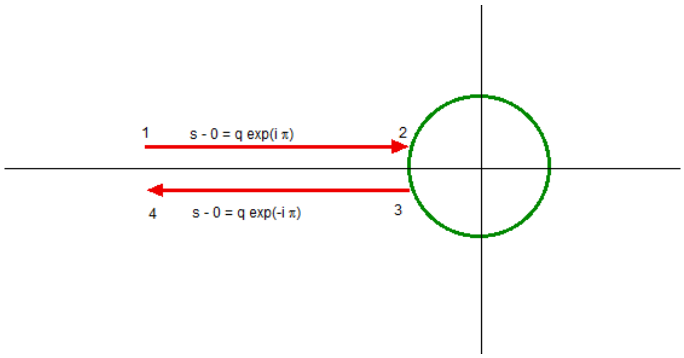


Fig.15

Path (1→2):
$s - 0 = qe^{i\pi}$

Path (3→4):
$s - 0 = qe^{-i\pi}$

(b) $\sqrt{(s+1)(s-1)}$

Check on the possibility of taking the line segment joining s+1 and s-1 as a cut line

$$s + 1 = re^{i\theta}$$
$$s - 1 = \rho e^{i\phi}$$

where $0 \le \theta < 2\pi$ and $0 \le \phi < 2\pi$, depending on the position of s in the complex plane.

$$f(s) = \sqrt{r\rho}e^{i(\theta+\phi)/2}$$

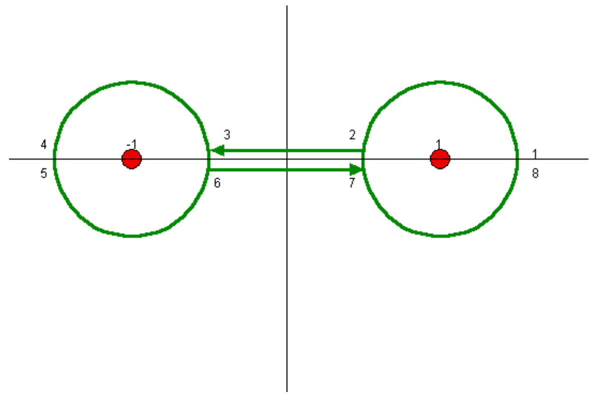


Fig.16

points	θ	ϕ	$(\theta+\phi)/2$
1	0	0	0
2	0	π	π/2
3	0	π	π/2
4	π	π	π
5	π	π	π
6	2π	π	3π/2
7	2π	π	3π/2
8	2π	2π	2π

The phase at points 6 and 7 is not the same as the points 2 and 3. This behavior can be expected at a branch point cut line.

(b) $\sqrt{(s+i)(s-i)}$

Check on the possibility of taking the line segment joining *s*+i and *s*-i as a cut line

$$s+i = re^{i\theta}$$
$$s-i = \rho e^{i\phi}$$

where $\frac{\pi}{2} \le \theta < 2\pi + \frac{\pi}{2}$ and $\frac{\pi}{2} \le \phi < 2\pi + \frac{\pi}{2}$, and the values of θ and ϕ are discrete but not continuous.

$$f(s) = \sqrt{r\rho}e^{i(\theta+\phi)/2}$$

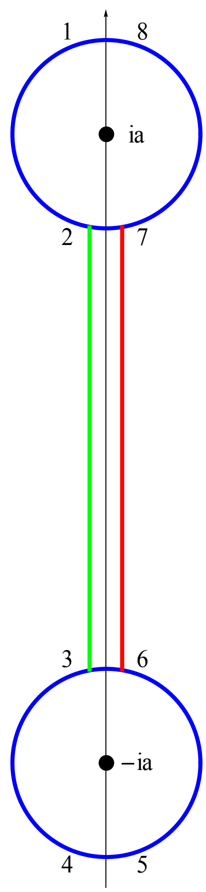


Fig.17

Check on the possibility of taking the line segment joining *s*+i and *s*-i as a cut line

points	θ	ϕ	$(\theta+\phi)/2$
1	$\pi/2$	$\pi/2$	$\pi/2$
2	$\pi/2$	$3\pi/2$	π
3	$\pi/2$	$3\pi/2$	π
4	$3\pi/2$	$3\pi/2$	$3\pi/2$
5	$3\pi/2$	$3\pi/2$	$3\pi/2$
6	$5\pi/2$	$3\pi/2$	2π
7	$5\pi/2$	$3\pi/2$	2π
8	$5\pi/2$	$5\pi/2$	$5\pi/2$

The phase at points 6 and 7 is not the same as the points 2 and 3. This behavior can be expected at a branch point cut line.

((Note))

Path (3→2):

$$s+i=qe^{i\frac{\pi}{2}}, \qquad s-i=(2-q)e^{i\frac{3\pi}{2}}$$

Path (7→6):

$$s-i=qe^{i\frac{3\pi}{2}}, \qquad s+i=(2-q)e^{i\frac{5\pi}{2}}$$

12.21 Diffusion with constant boundary condition

Suppose that $\psi(x,t)$ satisfies a diffusion equation

$$\frac{\partial^2\psi}{\partial x^2}=\frac{1}{\kappa}\frac{\partial\psi}{\partial t}$$

for x>0, subject to the boundary condition

$\psi=0$ for x>0 and t<0

$\psi(x=0,t)=\psi_0\Theta(t)$ for $x = 0$ and t>0,

where ψ_0 is constant.

Laplace transformation:

$$\frac{\partial^2\overline{\psi}(x,s)}{\partial x^2}=\frac{1}{\kappa}[s\overline{\psi}(x,s)-\psi(x,0)]=\frac{1}{\kappa}s\overline{\psi}(x,s)$$

with

$$\overline{\psi}(x=0,s)=\frac{\psi_0}{s}$$

$$\frac{\partial^2\overline{\psi}(x,s)}{\partial x^2}=\frac{1}{\kappa}s\overline{\psi}(x,s)$$

$$\overline{\psi}(x,s)=\exp(-x\sqrt{\frac{s}{\kappa}})\overline{\psi}(x=0,s)=\frac{\psi_0}{s}\exp(-x\sqrt{\frac{s}{\kappa}})$$

The inverse Laplace transformation (Bromwich integral)

$$\psi(x,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}\overline{\psi}(x,s)ds$$

$$=\frac{\psi_0}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}\frac{1}{s}\exp(-x\sqrt{\frac{s}{\kappa}})ds$$

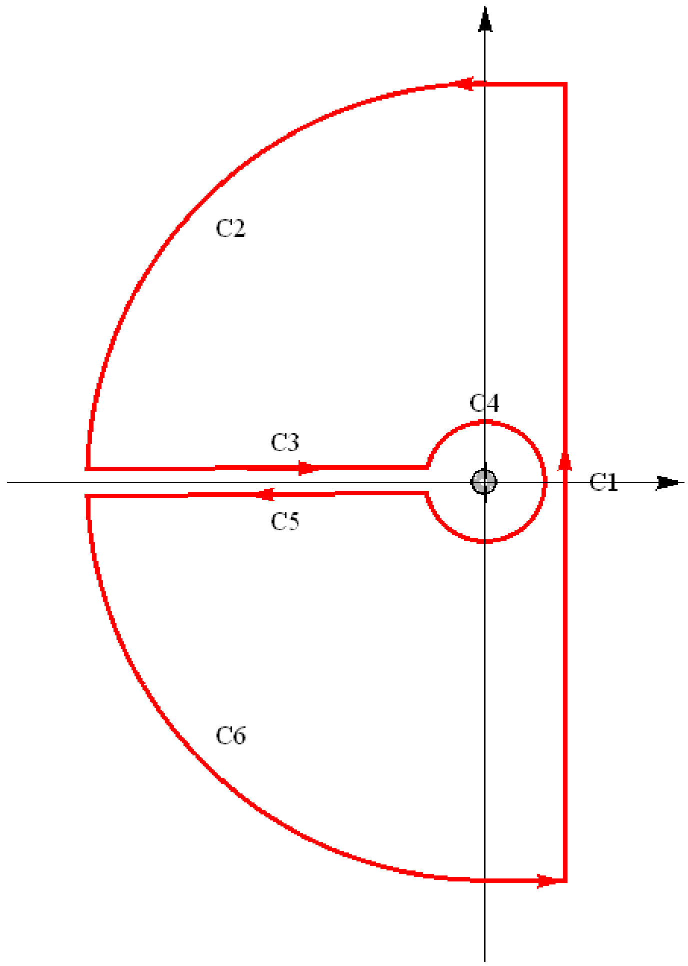


Fig.18

Using the Jordan's lemma, we have

$$\oint_C e^{st}\frac{1}{s}\exp(-x\sqrt{\frac{s}{\kappa}})ds = 0$$

since there is no pole inside the path C. Then we have

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{1}{s}\exp(-x\sqrt{\frac{s}{\kappa}})ds = -(\int_{C_3} + \oint_{C_4} + \int_{C_5}) = I_3 + I_4 + I_5$$

<u>Calculation of I_4</u>

$$I_4 = -\oint_{C_4} e^{st}\frac{1}{s}\exp(-x\sqrt{\frac{s}{\kappa}})ds = -\int_{\pi}^{-\pi} id\theta = \int_{-\pi}^{\pi} id\theta = 2\pi i$$

Calculation of I_3

$$I_3 = -\int_{C_3} e^{st} \frac{1}{s} \exp(-x\sqrt{\frac{s}{\kappa}}) ds = \int_0^\infty \frac{e^{-qt} e^{-ix\sqrt{\frac{q}{\kappa}}}}{q} dq = 2\int_0^\infty \frac{e^{-\kappa t u^2} e^{-ixu}}{u} du$$

where

$$s - 0 = qe^{i\pi}, \qquad q = \kappa u^2$$

Calculation of I_5

$$I_5 = -\int_{C_5} e^{st} \frac{1}{s} \exp(-x\sqrt{\frac{s}{\kappa}}) ds = -\int_0^\infty \frac{e^{-qt} e^{ix\sqrt{\frac{q}{\kappa}}}}{q} dq = -2\int_0^\infty \frac{e^{-\kappa t u^2} e^{ixu}}{u} du,$$

where

$$s - 0 = qe^{-i\pi}, \qquad q = \kappa u^2,$$

So we have

$$\begin{aligned} I_3 + I_4 + I_5 &= 2\pi i - 2(\int_0^\infty \frac{e^{-\kappa t u^2} e^{ixu}}{u} du - \int_0^\infty \frac{e^{-\kappa t u^2} e^{-ixu}}{u} du) \\ &= 2\pi i - 4i \int_0^\infty e^{-\kappa t u^2} \sin(xu) \frac{du}{u} \end{aligned}.$$

$$\begin{aligned} \psi(x,t) &= \frac{\psi_0}{2\pi i}(I_3 + I_4 + I_5) \\ &= \frac{\psi_0}{2\pi i}[2\pi i - 4i \int_0^\infty e^{-\kappa t u^2} \sin(xu) \frac{du}{u}] \\ &= \psi_0 [1 - \frac{2}{\pi} \int_0^\infty e^{-\kappa t u^2} \sin(xu) \frac{du}{u}] \\ &= \psi_0 [1 - Erf[\frac{x}{2\sqrt{\kappa t}}]] \\ &= \psi_0 Erfc[\frac{x}{2\sqrt{\kappa t}}] \end{aligned}.$$

where Erfc is the complementary error function.

12.22 Ladder circuit

Frequency domain

The impedance of inductance is Ls.
The impedance of capacitance is $1/(Cs)$.
The impedance of resistance is R.

The source is the Laplace transformation of the time-dependent voltage source.

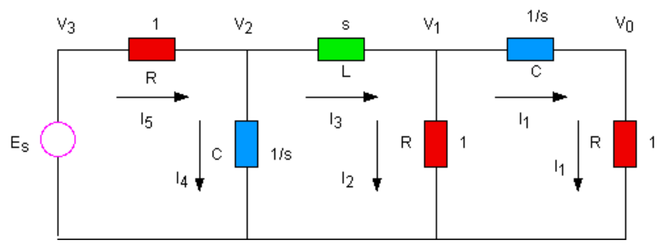


Fig.19

We assume that

$$I_1 = \frac{V_0}{1} = V_0$$

$$V_1 = V_0 + \frac{1}{s} I_1 = \left(1 + \frac{1}{s}\right) V_0$$

$$I_2 = \frac{V_1}{1} = \left(1 + \frac{1}{s}\right) V_0$$

$$I_3 = I_1 + I_2 = \left[1 + (1 + \frac{1}{s})\right] V_0 = \left(2 + \frac{1}{s}\right) V_0$$

$$V_2 = V_1 + sI_3 = \left[1 + \frac{1}{s} + s(2 + \frac{1}{s})\right] V_0 = \left(2 + 2s + \frac{1}{s}\right) V_0$$

$$I_4 = \frac{V_2}{\frac{1}{s}} = sV_2 = \left[s(2 + 2s + \frac{1}{s})\right] V_0 = \left(2s + 2s^2 + 1\right) V_0$$

$$I_5 = I_3 + I_4 = \left(2 + \frac{1}{s} + 2s + 2s^2 + 1\right) V_0 = (3 + 2s + 2s^2 + \frac{1}{s}) V_0$$

$$V_3 = 1I_5 + V_2 = (3 + 2s + 2s^2 + \frac{1}{s} + 2 + 2s + \frac{1}{s}) V_0 = (5 + 4s + 2s^2 + \frac{2}{s}) V_0$$

$$E_s = V_3 = (5 + 4s + 2s^2 + \frac{2}{s}) V_0$$

$$V_0 = \frac{E_s}{5 + 4s + 2s^2 + \frac{2}{s}}$$

Then we can get the following expressions

$$I_5 = (3 + 2s + 2s^2 + \frac{1}{s})V_0 = \left(\frac{3 + 2s + 2s^2 + \frac{1}{s}}{5 + 4s + 2s^2 + \frac{2}{s}} \right) E_s$$

$$I_4 = \left(2s + 2s^2 + 1\right)V_0 = \left(\frac{2s + 2s^2 + 1}{5 + 4s + 2s^2 + \frac{2}{s}} \right) E_s$$

$$I_3 = \left(2 + \frac{1}{s} \right) V_0 = \left(\frac{2 + \frac{1}{s}}{5 + 4s + 2s^2 + \frac{2}{s}} \right) E_s$$

$$I_2 = \left(1 + \frac{1}{s} \right) V_0 = \left(\frac{1 + \frac{1}{s}}{5 + 4s + 2s^2 + \frac{2}{s}} \right) E_s$$

$$I_1 = V_0 = \left(\frac{1}{5 + 4s + 2s^2 + \frac{2}{s}} \right) E_s$$

$$V_3 = E_s$$

$$V_2 = \left(2 + 2s + \frac{1}{s} \right) V_0 = \left(\frac{2 + 2s + \frac{1}{s}}{5 + 4s + 2s^2 + \frac{2}{s}} \right) E_s$$

$$V_1 = \left(1+\frac{1}{s}\right)V_0 = \left(\frac{1+\frac{1}{s}}{5+4s+2s^2+\frac{2}{s}}\right)E_s$$

$$V_0 = \frac{E_s}{5+4s+2s^2+\frac{2}{s}}$$

((**Mathematica**))

```
Clear["Global`*"]
```

$$\texttt{V0} = \frac{\texttt{E1[s]}}{\left(5+4\,s+2\,s^2+\frac{2}{s}\right)} \;//\; \texttt{Simplify}$$

$$\frac{s\,\texttt{E1[s]}}{2+5\,s+4\,s^2+2\,s^3}$$

$$\texttt{V1} = \frac{\left(1+\frac{1}{s}\right)\texttt{E1[s]}}{\left(5+4\,s+2\,s^2+\frac{2}{s}\right)} \;//\; \texttt{Simplify}$$

$$\frac{(1+s)\,\texttt{E1[s]}}{2+5\,s+4\,s^2+2\,s^3}$$

$$\texttt{V2} = \frac{\left(2+2\,s+\frac{1}{s}\right)\texttt{E1[s]}}{\left(5+4\,s+2\,s^2+\frac{2}{s}\right)} \;//\; \texttt{Simplify}$$

$$\frac{\left(1+2\,s+2\,s^2\right)\texttt{E1[s]}}{2+5\,s+4\,s^2+2\,s^3}$$

```
V3 = E1[s]
```

```
E1[s]
```

$$\mathbf{I1} = \frac{\mathbf{E1[s]}}{\left(5 + 4\,s + 2\,s^2 + \frac{2}{s}\right)} \; \mathbf{//\;Simplify}$$

$$\frac{s\,\mathrm{E1}[s]}{2 + 5\,s + 4\,s^2 + 2\,s^3}$$

$$\mathbf{I2} = \frac{\left(1 + \frac{1}{s}\right)\mathbf{E1[s]}}{\left(5 + 4\,s + 2\,s^2 + \frac{2}{s}\right)} \; \mathbf{//\;Simplify}$$

$$\frac{(1 + s)\,\mathrm{E1}[s]}{2 + 5\,s + 4\,s^2 + 2\,s^3}$$

$$\mathbf{I3} = \frac{\left(2 + \frac{1}{s}\right)\mathbf{E1[s]}}{\left(5 + 4\,s + 2\,s^2 + \frac{2}{s}\right)} \; \mathbf{//\;Simplify}$$

$$\frac{(1 + 2\,s)\,\mathrm{E1}[s]}{2 + 5\,s + 4\,s^2 + 2\,s^3}$$

$$\mathbf{I4} = \frac{\left(2\,s + 2\,s^2 + 1\right)\mathbf{E1[s]}}{\left(5 + 4\,s + 2\,s^2 + \frac{2}{s}\right)} \; \mathbf{//\;Simplify}$$

$$\frac{s\left(1 + 2\,s + 2\,s^2\right)\mathrm{E1}[s]}{2 + 5\,s + 4\,s^2 + 2\,s^3}$$

```
I5 = ((3 + 2 s + 2 s^2 + 1/s) E1[s]) / (5 + 4 s + 2 s^2 + 2/s) // Simplify
```

$$\frac{\left(1+3\,s+2\,s^2+2\,s^3\right)\,\mathrm{E1}[s]}{2+5\,s+4\,s^2+2\,s^3}$$

```
rule1 = {E1 → ((1 / #) &)}
```

$$\left\{\mathrm{E1}\to\left(\frac{1}{\#1}\;\&\right)\right\}$$

```
v0 = V0 /. rule1
```

$$\frac{1}{2+5\,s+4\,s^2+2\,s^3}$$

```
v0 = InverseLaplaceTransform[V0 /. rule1, s, t];
v1 = InverseLaplaceTransform[V1 /. rule1, s, t];
v2 = InverseLaplaceTransform[V2 /. rule1, s, t];
v3 = InverseLaplaceTransform[V3 /. rule1, s, t];
i1 = InverseLaplaceTransform[I1 /. rule1, s, t];
i2 = InverseLaplaceTransform[I2 /. rule1, s, t];
i3 = InverseLaplaceTransform[I3 /. rule1, s, t];
i4 = InverseLaplaceTransform[I4 /. rule1, s, t];
i5 = InverseLaplaceTransform[I5 /. rule1, s, t];
```

```
pl1 = Plot[Evaluate[{v3, v2, v1, v0}], {t, 0.1, 10},
  PlotRange → {{0.1, 10}, {-0.2, 1}},
  PlotStyle → Table[{Hue[0.2 i], Thick}, {i, 0, 4}],
  Background → LightGray]
```

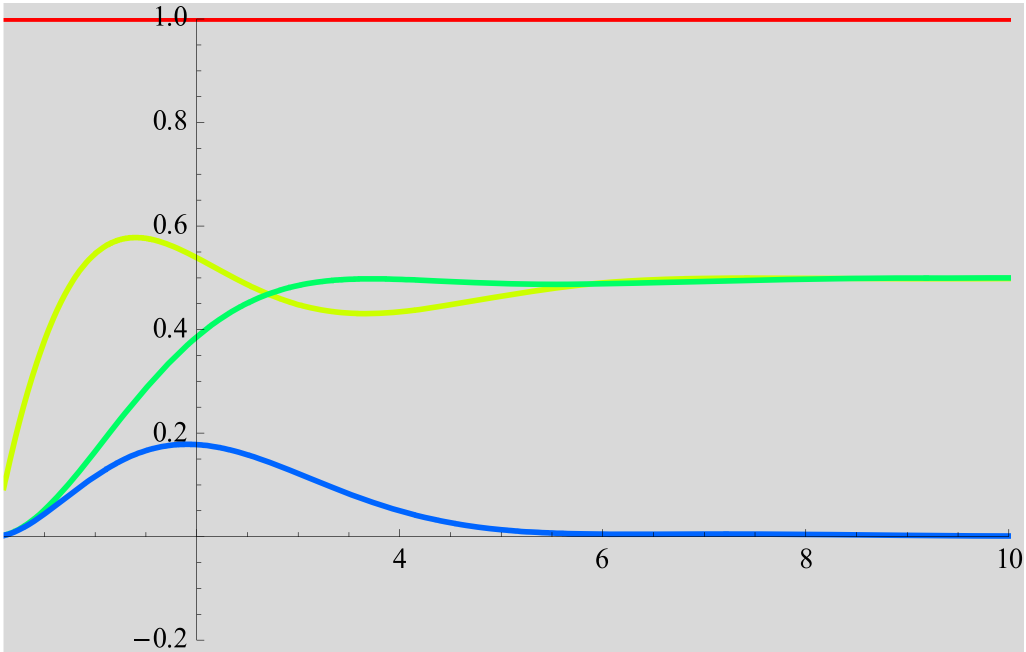


Fig.20

```
pl2 = Plot[Evaluate[{i5, i4, i3, i2, i1}], {t, 0.1, 20},
  PlotRange → {{0.1, 20}, {-0.2, 1}},
  PlotStyle → Table[{Hue[0.2 i], Thick}, {i, 0, 5}],
  Background → GrayLevel[0.7]]
```

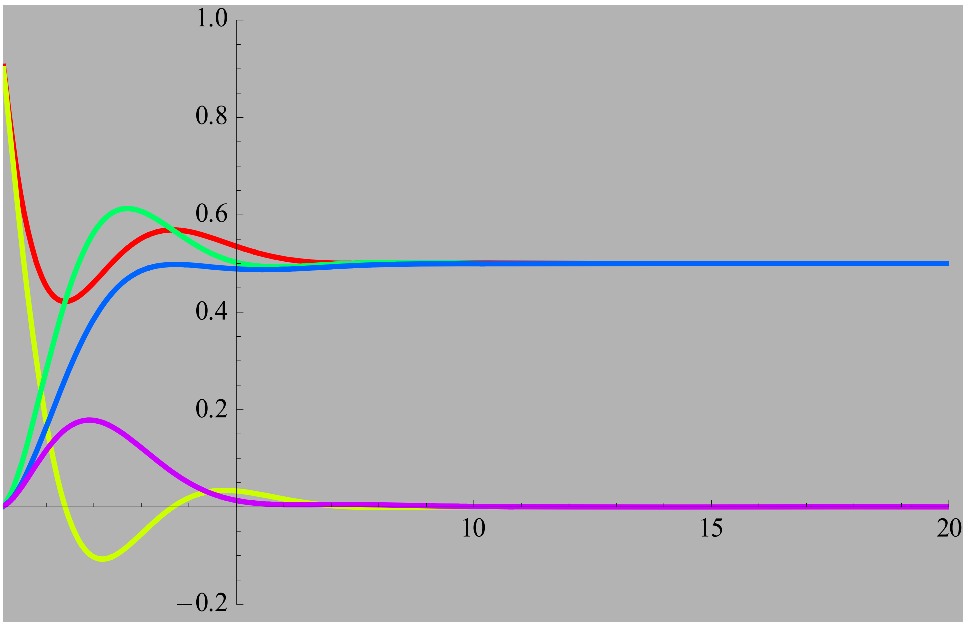
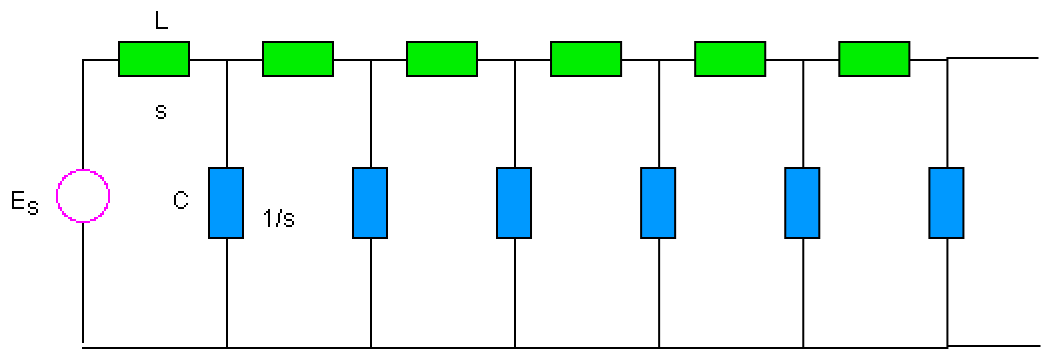


Fig.21

12.23 Ladder circuit: Example-2



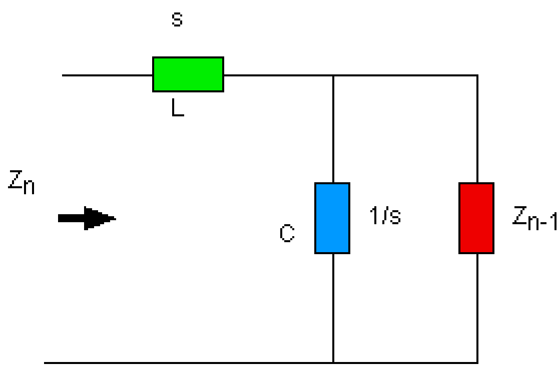


Fig.22

We now consider the infinite ladder circuit consisting of L = 1 H and C = 1 F.

Recurrence relation

$$Z_n = s + \frac{\frac{1}{s} Z_{n-1}}{\frac{1}{s} + Z_{n-1}} = s + \frac{Z_{n-1}}{1 + s Z_{n-1}}$$

In the limit of $n \to \infty$

$$Z = s + \frac{Z}{1 + sZ}$$

or

$$Z(s) = \frac{s + \sqrt{4 + s^2}}{2}$$

The current $I(s)$ is given by

$$I(s) = \frac{E(s)}{Z(s)} = \frac{2E(s)}{s+\sqrt{s^2}+4}$$

$$i(t) = L^{-1}[I(s)]$$

((Mathematikca))

Ladder circuit second exampl

We consider a infinite ladder consisting of L = 1 H and C = 1 F. What happens to the current when the unitstep is applied at t = 0.

$$\text{K1} = \text{Solve}\left[x == s + \frac{x}{1+s\,x},\ x\right]$$

$$\left\{\left\{x \to \frac{1}{2}\left(s-\sqrt{4+s^2}\right)\right\},\ \left\{x \to \frac{1}{2}\left(s+\sqrt{4+s^2}\right)\right\}\right\}$$

```
K2 = x /. K1[[2]];
```

$$\text{I1} = \frac{1/s}{\text{K2}}$$

$$\frac{2}{s\left(s+\sqrt{4+s^2}\right)}$$

```
eq1 = InverseLaplaceTransform[I1, s, t]
```

$$t\ \text{HypergeometricPFQ}\left[\left\{\frac{1}{2}\right\},\ \left\{\frac{3}{2},\ 2\right\},\ -t^2\right]$$

```
Plot[eq1, {t, 0, 20}, PlotStyle → {Thick, Red}, Background → GrayLevel[0.7],
 PlotRange → {{0, 20}, {-0.2, 1.2}}, AxesLabel → {"t", "i(t)"}]
```

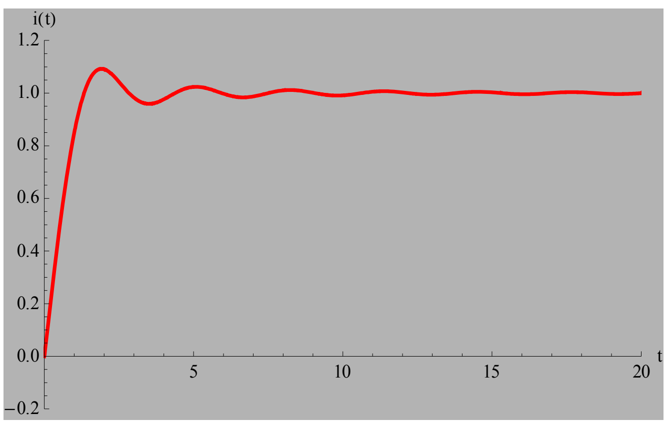


Fig.23

When the Dirac Delta function is applied at t = 0, what happens?

```
LaplaceTransform[DiracDelta[t], t, s]
```

1

$$I2 = \frac{1}{K2}$$

$$\frac{2}{s + \sqrt{4 + s^2}}$$

```
eq4 = InverseLaplaceTransform[I2, s, t]
```

$$\frac{\text{BesselJ}[1, 2\,t]}{t}$$

```
Plot[eq4, {t, 0, 20}, PlotStyle → {Red, Thick}, Prolog → AbsoluteThickness[1.5],
 Background → GrayLevel[0.7], PlotRange → {{0, 20}, {-0.2, 1}},
 AxesLabel → {"t", "i(t)"}]
```

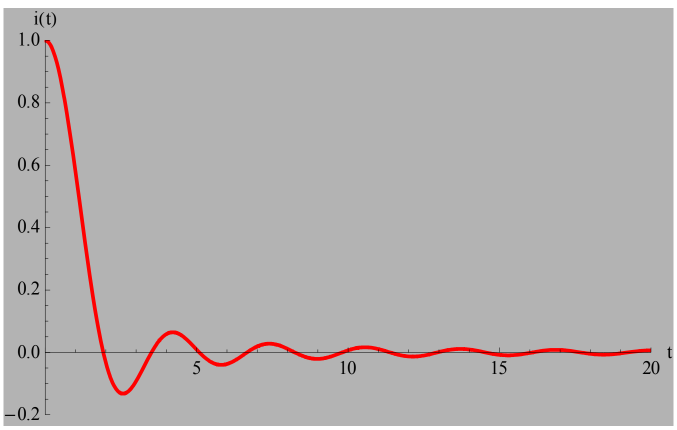


Fig.24

12.24 Laplace transform of periodic functions

Suppose that $f(t)$ is a periodic function with a period T. The Laplace transform of $f(t)$ is given by

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \int_{2T}^{3T} f(t)e^{-st}dt + ...$$

or

$$F(s) = \sum_{n=1}^{\infty} \int_{(n-1)T}^{nT} f(t)e^{-st}dt = \sum_{n=1}^{\infty} e^{-s(n-1)T} \int_0^T f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T f(t)e^{-st}dt$$

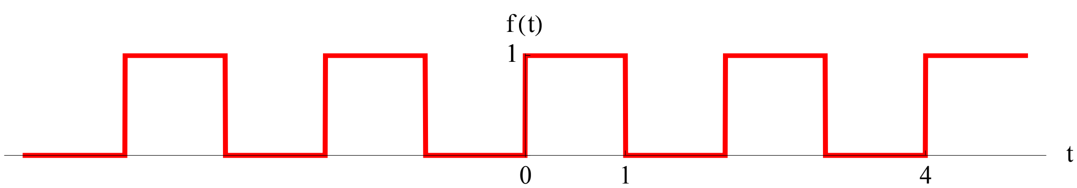
12.24.1 Laplace transform of periodic function: Example-1

```
Clear["Global`*"]

f[t_] := 1 /; 0 ≤ t ≤ 1; f[t_] := 0 /; 1 ≤ t ≤ 2;

extens[t_] := f[t] /; 0 ≤ t ≤ 2; extens[t_] := extens[t - 2] /; t > 2;
extens[t_] := extens[t + 2] /; t < 0;

Plot[extens[t], {t, -5, 5}, PlotStyle → {{Red, Thick}},
 AspectRatio → Automatic, AxesLabel → {"t", "f(t)"},
 Ticks → {{0, 4, 1}, {0, 1, 1}}]
```



T = 2;
f[t] := 1 /; 0 ≤ t ≤ 1;

```
eq1 = Integrate[1 Exp[-s t], {t, 0, 1}]
```

$$\frac{1 - e^{-s}}{s}$$

```
F[s_] = eq1 / (1 - Exp[-2 s]) // Simplify
```

$$\frac{e^{s}}{s + e^{s} s}$$

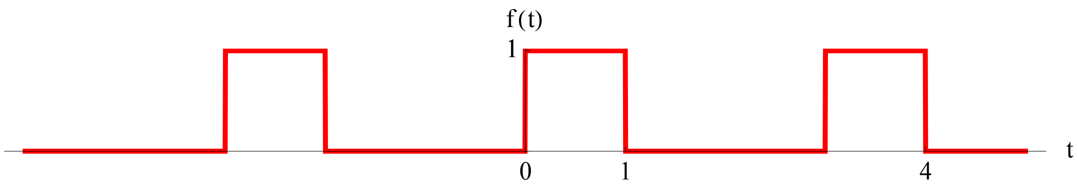
12.24.2 Laplace transform of periodic function: Example-2

```
Clear["Global`*"]

f[t_] := 1 /; 0 ≤ t ≤ 1; f[t_] := 0 /; 1 ≤ t ≤ 3;

extens[t_] := f[t] /; 0 ⩽ t ≤ 3; extens[t_] := extens[t - 3] /; t > 3;
extens[t_] := extens[t + 3] /; t < 0;

Plot[extens[t], {t, -5, 5}, PlotStyle → {{Red, Thick}},
 AspectRatio → Automatic, AxesLabel → {"t", "f(t)"},
 Ticks → {{0, 4, 1}, {0, 1, 1}}]
```



T = 3;
f[t] := 1 /; 0 ≤ t ≤ 1;

```
eq1 = Integrate[ 1 Exp[-s t], {t, 0, 1}]
```

$$\frac{1 - e^{-s}}{s}$$

```
F[s_] = eq1 / (1 - Exp[-3 s]) // Simplify
```

$$\frac{e^{2 s}}{s + e^{s} s + e^{2 s} s}$$

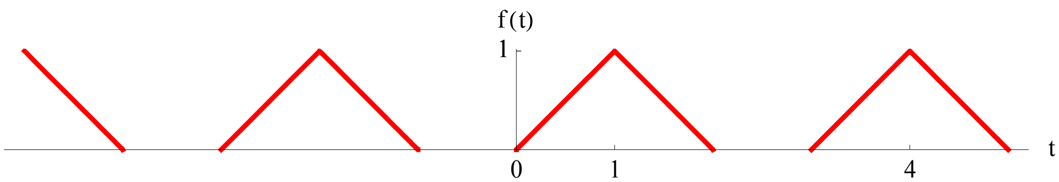
12.24.3 Laplace transform of periodic function: Example-3

```
Clear["Global`*"]

f[t_] := 0 /. -1 ≤ t ≤ 0; f[t_] := t /; 0 ≤ t ≤ 1;
f[t_] := -t + 2 /; 1 ≤ t ≤ 2;

extens[t_] := f[t] /; -1 ⩽ t ≤ 2; extens[t_] := extens[t - 3] /; t > 2;
extens[t_] := extens[t + 3] /; t < -1;

Plot[extens[t], {t, -5, 5}, PlotStyle → {{Red, Thick}},
 AspectRatio → Automatic, AxesLabel → {"t", "f(t)"},
 Ticks → {{0, 4, 1}, {0, 1, 1}}]
```



T = 3;
f[t] := 1 /; 0 ≤ t ≤ 1; f[t] := 2 - t /; 1 ≤ t ≤2;

```
eq1 =
 Integrate[ 1 Exp[-s t], {t, 0, 1}] +
  Integrate[(2 - t) Exp[-s t], {t, 1, 2}] // Simplify
```

$$\frac{e^{-2 s} - e^{-s} + s}{s^2}$$

$$F[s\_] = \frac{eq1}{1 - Exp[-3 s]} \; // \; Simplify$$

$$\frac{e^{s} \left(1 - e^{s} + e^{2 s} s\right)}{\left(-1 + e^{3 s}\right) s^2}$$

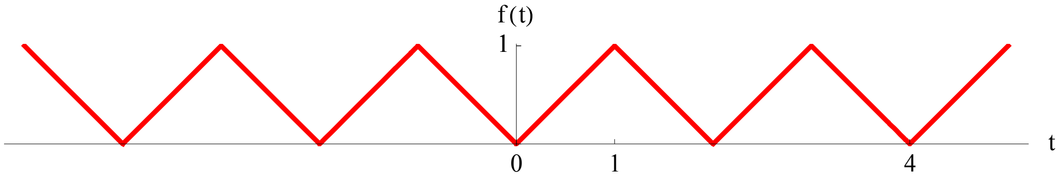
12.24.4 Laplace transform of periodic function: Example-4

```
Clear["Global`*"]

f[t_] := -t /; -1 ≤ t ≤ 0; f[t_] := t /; 0 ≤ t ≤ 1;

extens[t_] := f[t] /; -1 ≤ t ≤ 1; extens[t_] := extens[t - 2] /; t > 1;
extens[t_] := extens[t + 2] /; t < -1;

Plot[extens[t], {t, -5, 5}, PlotStyle → {{Red, Thick}},
 AspectRatio → Automatic, AxesLabel → {"t", "f(t)"},
 Ticks → {{0, 4, 1}, {0, 1, 1}}]
```



T = 2;
f[t] := t /; 0 ≤ t ≤ 1; f[t] := 2 - t /; 1 ≤ t ≤2;

```
eq1 =
 Integrate[ t Exp[-s t], {t, 0, 1}] +
   Integrate[(2 - t) Exp[-s t], {t, 1, 2}] // Simplify
```

$$\frac{e^{-2 s} \left(-1+e^s\right)^2}{s^2}$$

$$F[s\_] = \frac{\text{eq1}}{1 - \text{Exp}[-2\, s]} \; // \; \text{Simplify}$$

$$\frac{-1+e^s}{\left(1+e^s\right) s^2}$$

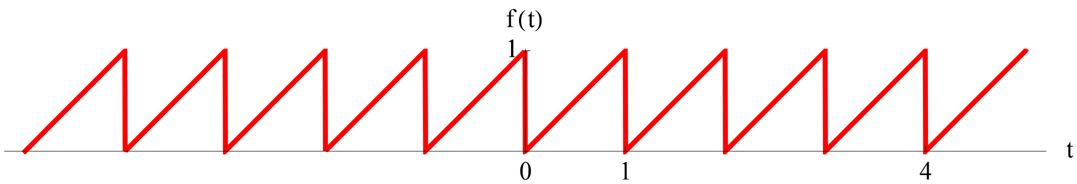
12.24.5 Laplace transform of periodic function: Example-5

```
Clear["Global`*"]

f[t_] := t /; 0 ≤ t ≤ 1;

extens[t_] := f[t] /; 0 ⩽ t ≤ 1; extens[t_] := extens[t - 1] /; t > 1;
extens[t_] := extens[t + 1] /; t < 0;

Plot[extens[t], {t, -5, 5}, PlotStyle → {{Red, Thick}},
 AspectRatio → Automatic, AxesLabel → {"t", "f(t)"},
 Ticks → {{0, 4, 1}, {0, 1, 1}}]
```



T = 1;
f[t] := t /; 0 ≤ t ≤ 1;

```
eq1 = Integrate[ t Exp[-s t], {t, 0, 1}] // Simplify
```

$$\frac{1 - e^{-s}\,(1 + s)}{s^2}$$

```
F[s_] = eq1 / (1 - Exp[- s]) // Simplify
```

$$\frac{-1 + e^{s} - s}{(-1 + e^{s})\,s^2}$$

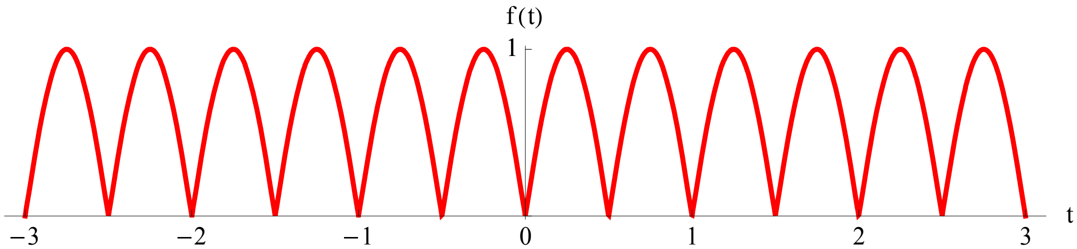
12.24.5 Laplace transform of periodic function: Example-5

```
Clear["Global`*"]

f[t_] := Sin[2 π t] /; 0 ≤ t ≤ 1/2;
f[t_] := -Sin[2 π t] /; 1/2 ≤ t ≤ 1

extens[t_] := f[t] /; 0 ≤ t ≤ 1; extens[t_] := extens[t - 1] /; t > 1;
extens[t_] := extens[t + 1] /; t < 0;

Plot[extens[t], {t, -3, 3}, PlotStyle → {{Red, Thick}},
 AspectRatio → Automatic, AxesLabel → {"t", "f(t)"},
 Ticks → {{-3, -2, -1, 0, 1, 2, 3}, {0, 1, 1}}]
```



T = 1/2;
f[t] := Sin[2 π t] /; 0 ≤ t ≤ 1/2; f[t]

```
eq1 = Integrate[ Sin[2 π t] Exp[-s t], {t, 0, 1/2}] // Simplify
```

$$\frac{2\left(1+e^{-s/2}\right)\pi}{4\pi^2+s^2}$$

```
F[s_] = eq1/(1 - Exp[- s/2]) // Simplify
```

$$\frac{2\left(1+e^{s/2}\right)\pi}{\left(-1+e^{s/2}\right)\left(4\pi^2+s^2\right)}$$

12.25 Solving of differential equations using Laplace transform (Mathematica)

((Example-1))

■

x'[t]+y[t]==t, 4 x[t]+ y'[t]==0

with the initial condition x[0]==1, y[0]==-1

```
Clear["Global`*"]

eq1 = x'[t] + y[t] == t;

eq2 = 4 x[t] + y'[t] == 0;

eq3 = LaplaceTransform[eq1, t, s] /. x[0] → 1
```

$$-1 + s\,\text{LaplaceTransform}[x[t], t, s] + \text{LaplaceTransform}[y[t], t, s] == \frac{1}{s^2}$$

```
eq4 = LaplaceTransform[eq2, t, s] /. y[0] → -1
```

$$1 + 4\,\text{LaplaceTransform}[x[t], t, s] + s\,\text{LaplaceTransform}[y[t], t, s] == 0$$

```
eq5 = Solve[{eq3, eq4},
  {LaplaceTransform[x[t], t, s], LaplaceTransform[y[t], t, s]}]
```

$$\left\{\left\{\text{LaplaceTransform}[x[t], t, s] \to -\frac{-1 - s - s^2}{s\left(-4 + s^2\right)}, \text{LaplaceTransform}[y[t], t, s] \to -\frac{4 + 4 s^2 + s^3}{s^2\left(-4 + s^2\right)}\right\}\right\}$$

```
X[s_] = LaplaceTransform[x[t], t, s] /. eq5[[1]]
```

$$-\frac{-1 - s - s^2}{s\left(-4 + s^2\right)}$$

```
Y[s_] = LaplaceTransform[y[t], t, s] /. eq5[[1]]
```

$$-\frac{4 + 4 s^2 + s^3}{s^2\left(-4 + s^2\right)}$$

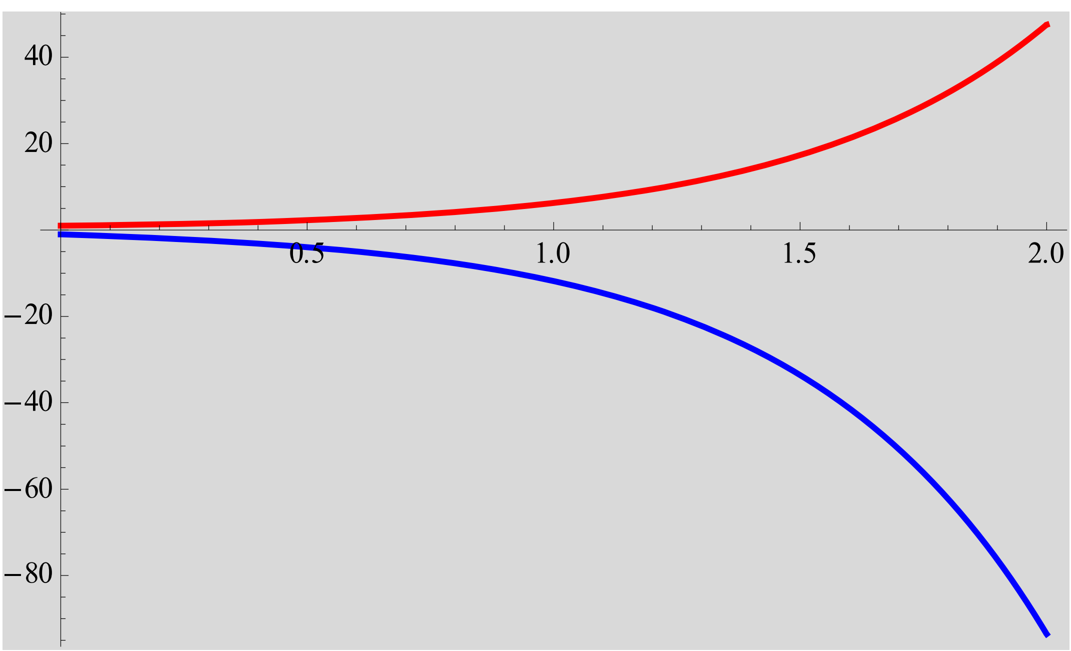
```
x1[t_] = InverseLaplaceTransform[X[s], s, t]
```

$$-\frac{1}{4} + \frac{3 e^{-2 t}}{8} + \frac{7 e^{2 t}}{8}$$

```
y1[t_] = InverseLaplaceTransform[Y[s], s, t]
```

$$\frac{3 e^{-2 t}}{4} - \frac{7 e^{2 t}}{4} + t$$

```
Plot[{x1[t], y1[t]}, {t, 0, 2},
 PlotStyle → {{Red, Thick}, {Blue, Thick}},
 Background → LightGray]
```



((**Exampl-2**))

■

Solve the differential equation

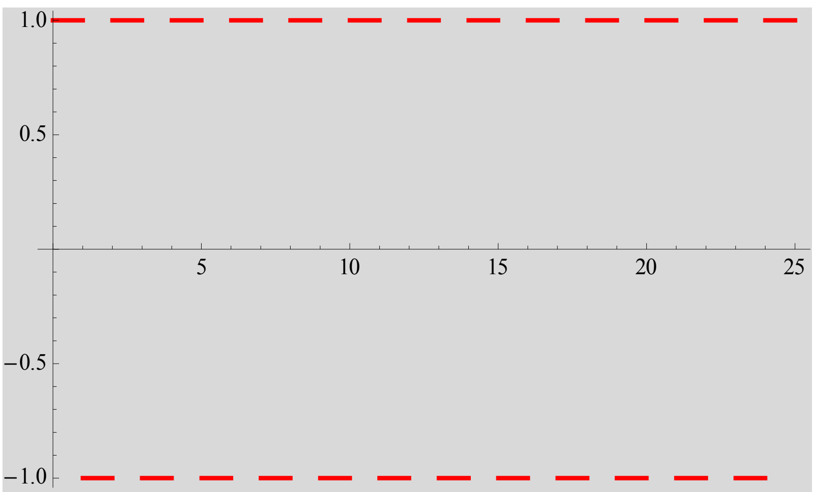
y''[t]+3 y'[t]+2 y[t]==f[t]

with
f[t]=periodic function: y[0]==0,y'[0]==0

```
Clear["Global`*"]

f[t_] =
  Sum[(UnitStep[t - 2 k] - UnitStep[t - 1 - 2 k] - UnitStep[t - 1 - 2 k] +
      UnitStep[t - 2 - 2 k]), {k, 0, 15}];

Plot[f[t], {t, 0, 25}, PlotStyle → {Red, Thick},
 Background → LightGray]
```




```
eq1 = y''[t] + 3 y'[t] + 2 y[t] == f[t];

eq2 = LaplaceTransform[eq1, t, s] /. {y[0] → 0, y'[0] → 0}
```

$$2\,\text{LaplaceTransform}[y[t],t,s]+3\,s\,\text{LaplaceTransform}[y[t],t,s]+s^2\,\text{LaplaceTransform}[y[t],t,s]==$$
$$\frac{1}{s}+\frac{e^{-32 s}}{s}-\frac{2e^{-31 s}}{s}+\frac{2e^{-30 s}}{s}-\frac{2e^{-29 s}}{s}+\frac{2e^{-28 s}}{s}-\frac{2e^{-27 s}}{s}+$$
$$\frac{2e^{-26 s}}{s}-\frac{2e^{-25 s}}{s}+\frac{2e^{-24 s}}{s}-\frac{2e^{-23 s}}{s}+\frac{2e^{-22 s}}{s}-\frac{2e^{-21 s}}{s}+$$
$$\frac{2e^{-20 s}}{s}-\frac{2e^{-19 s}}{s}+\frac{2e^{-18 s}}{s}-\frac{2e^{-17 s}}{s}+\frac{2e^{-16 s}}{s}-\frac{2e^{-15 s}}{s}+$$
$$\frac{2e^{-14 s}}{s}-\frac{2e^{-13 s}}{s}+\frac{2e^{-12 s}}{s}-\frac{2e^{-11 s}}{s}+\frac{2e^{-10 s}}{s}-\frac{2e^{-9 s}}{s}+\frac{2e^{-8 s}}{s}-$$
$$\frac{2e^{-7 s}}{s}+\frac{2e^{-6 s}}{s}-\frac{2e^{-5 s}}{s}+\frac{2e^{-4 s}}{s}-\frac{2e^{-3 s}}{s}+\frac{2e^{-2 s}}{s}-\frac{2e^{-s}}{s}$$

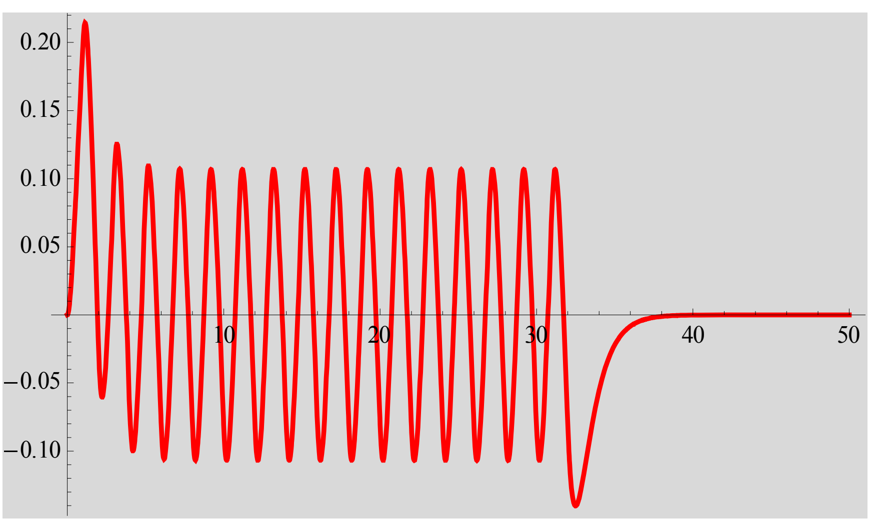
```
eq3 = Solve[eq2, LaplaceTransform[y[t], t, s]] // Simplify
```

$$\left\{\left\{\text{LaplaceTransform}[y[t],t,s]\to \frac{1}{s\left(2+3s+s^2\right)}\,e^{-32 s}\left(-1+e^{s}\right)^2\left(1+e^{2 s}+e^{4 s}+e^{6 s}+e^{8 s}+e^{10 s}+e^{12 s}+e^{14 s}+e^{16 s}+e^{18 s}+e^{20 s}+e^{22 s}+e^{24 s}+e^{26 s}+e^{28 s}+e^{30 s}\right)\right\}\right\}$$

```
eq4 = InverseLaplaceTransform[eq3, s, t] // Simplify;

y[t_] = y[t] /. eq4[[1]];

Plot[y[t], {t, 0, 50}, PlotStyle → {Red, Thick},
 Background → LightGray]
```



12.26 Damped simple harmonics (Mathematica)

We consider the motion of the damped oscillator using the Laplace transformation.

```
Clear["Global`*"]
```

```
eq11 = x''[t] + 2 γ x'[t] + ω0^2 x[t] == f[t]
```

$\omega 0^2\, x[t] + 2\,\gamma\, x'[t] + x''[t] == f[t]$

```
eq12 =
 LaplaceTransform[eq11, t, s] /.
   {LaplaceTransform[x[t], t, s] → X[s],
    LaplaceTransform[f[t], t, s] → F[s]} // Simplify
```

$F[s] + s\, x[0] + 2\,\gamma\, x[0] + x'[0] == \left(s^2 + 2\, s\,\gamma + \omega 0^2\right) X[s]$

```
eq13 = Solve[eq12, X[s]]
```

$\left\{\left\{X[s] \to \frac{F[s] + s\, x[0] + 2\,\gamma\, x[0] + x'[0]}{s^2 + 2\, s\,\gamma + \omega 0^2}\right\}\right\}$

```
X[s_] = X[s] /. eq13[[1]]
```

$\frac{F[s] + s\, x[0] + 2\,\gamma\, x[0] + x'[0]}{s^2 + 2\, s\,\gamma + \omega 0^2}$

f[t] = f0 UnitStep[t]; step pulse
Initial condition x[0] = 0; x'[0]=0

```
eq21 = F[s_] = LaplaceTransform[f0 UnitStep[t], t, s]
```

$$\frac{f0}{s}$$

```
Initial = {x[0] → 0, x'[0] → 0}
```

{x[0] → 0, x′[0] → 0}

```
X1[s_] = X[s] /. Initial // Simplify
```

$$\frac{f0}{s\left(s^2+2\,s\,\gamma+\omega 0^2\right)}$$

```
x1[t_] =
 InverseLaplaceTransform[X1[s], s, t] /.
```

$$\left\{\sqrt{\gamma^2-\omega 0^2}\to i\,\omega,\ \frac{1}{\sqrt{\gamma^2-\omega 0^2}}\to -i\,\omega\right\}\ //\ \text{ExpToTrig}\ //\ \text{Simplify}$$

$$\frac{1}{\omega 0^2}\,f0\,(1-\omega\,\text{Cosh}[t\,\gamma]\,(\omega\,\text{Cos}[t\,\omega]+\gamma\,\text{Sin}[t\,\omega])+\omega\,(\omega\,\text{Cos}[t\,\omega]+\gamma\,\text{Sin}[t\,\omega])\,\text{Sinh}[t\,\gamma])$$

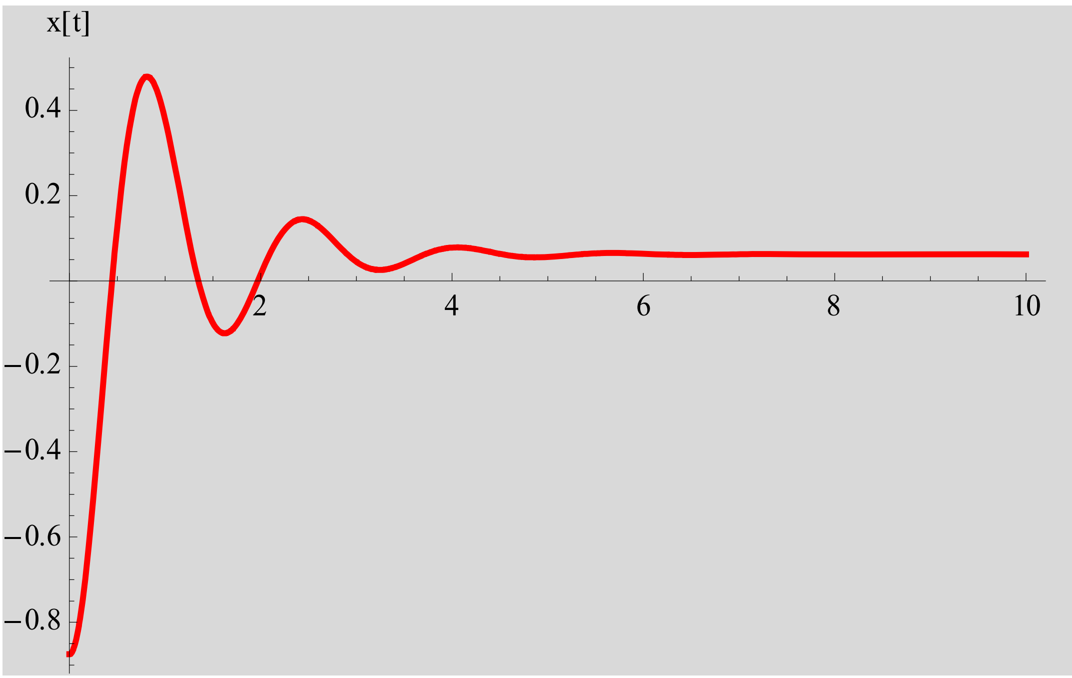
```
Limit[x1[t], t → 0, Direction → 1]
```

$$\frac{f0-f0\,\omega^2}{\omega 0^2}$$

```
rule1 = {γ → 1, ω0 → 4, f0 → 1, ω → √15}
```

$\{\gamma \to 1,\ \omega 0 \to 4,\ f0 \to 1,\ \omega \to \sqrt{15}\}$

```
Plot[x1[t] /. rule1, {t, 0, 10}, PlotRange → All,
 PlotStyle → {Thick, Red}, Background → LightGray,
 AxesLabel → {"t", "x[t]"}]
```



APPENDIX
Useful inverse Laplace transform

```
InverseLaplaceTransform[1/Sqrt[s] Exp[-a Sqrt[s]], s, t] //
 Simplify[#, a > 0] &
```

$$\frac{e^{-\frac{a^2}{4 t}}}{\sqrt{\pi} \sqrt{t}}$$

```
InverseLaplaceTransform[1/s Exp[-a Sqrt[s]], s, t] //
 Simplify[#, a > 0] &
```

$$\mathrm{Erfc}\left[\frac{a}{2 \sqrt{t}}\right]$$

```
InverseLaplaceTransform[Exp[-a Sqrt[s]], s, t] //
 Simplify[#, a > 0] &
```

$$\frac{a\, e^{-\frac{a^2}{4 t}}}{2 \sqrt{\pi}\, t^{3/2}}$$

```
InverseLaplaceTransform[s^(-3/2) E^(1/s), s, t] // Simplify
```

$$\frac{\mathrm{Sinh}\left[2 \sqrt{t}\right]}{\sqrt{\pi}}$$

```
InverseLaplaceTransform[s^(-1/2) E^(1/s), s, t] // Simplify
```

$$\frac{\mathrm{Cosh}\left[2 \sqrt{t}\right]}{\sqrt{\pi} \sqrt{t}}$$