Chapter 13

Calculus of variations

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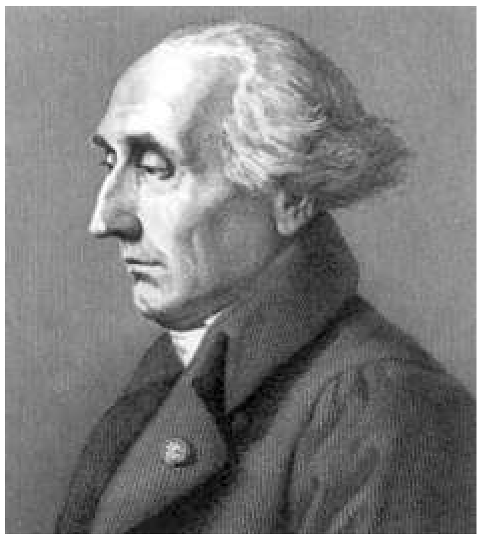
(Date: November 3, 2010)

Leonhard Euler (15 April 1707 – 18 September 1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function. He is also renowned for his work in mechanics, fluid dynamics, optics, and astronomy.

http://en.wikipedia.org/wiki/Leonhard_Euler

Joseph-Louis Lagrange (25 January 1736, Turin, Piedmont – 10 April 1813, Paris), born **Giuseppe Lodovico (Luigi) Lagrangia**, was an Italian-born mathematician and astronomer, who lived part of his life in Prussia and part in France, making significant contributions to all fields of analysis, to number theory, and to classical and celestial mechanics. On the recommendation of Euler and d'Alembert, in 1766 Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, where he stayed for over twenty years, producing a large body of work and winning several prizes of the French Academy of Sciences..



http://en.wikipedia.org/wiki/Joseph_Louis_Lagrange

This note was written in part for the Classical mechanics, and is revised for the present course.

13.1 Line integral

We start to discuss the calculus of variations with an integral given by the form

$$J = \int_{x_1}^{x_2} f(y, y', x)dx ,$$

where $y' = dy/dx$. The problem is to find has a stationary function $\bar{y}(x)$ so as to minimize the value of the integral J. The minimization process can be accomplished by introducing a parameter ε.

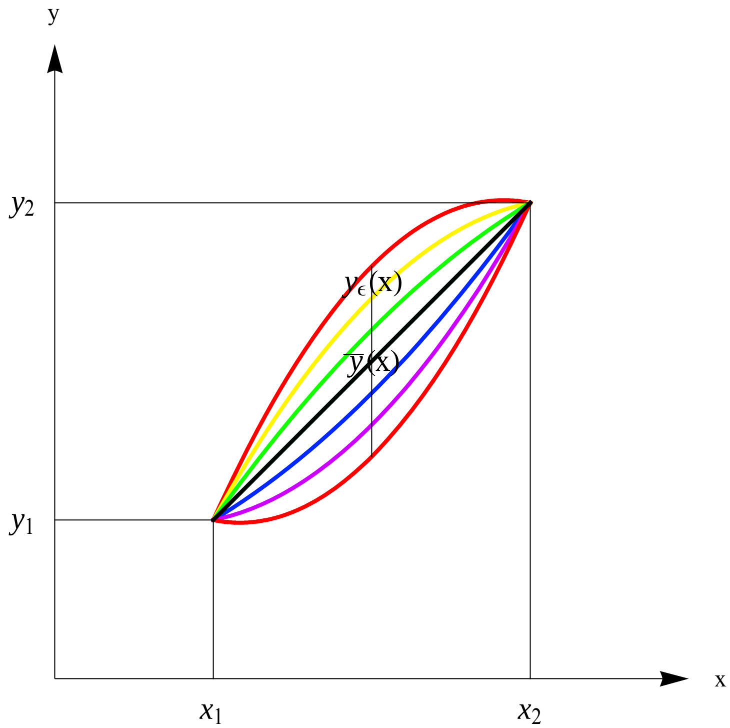


Fig.

$$y(x_1) = y_1, \qquad y(x_2) = y_2,$$

$$y_\varepsilon(x) = \bar{y}(x) + \varepsilon\eta(x)$$

where ε is a real number and

$$\bar{y}(x_1) = y_1, \qquad \bar{y}(x_2) = y_2,$$

$$\eta(x_1) = 0, \qquad \eta(x_2) = 0$$

$$\delta y = \left(\frac{\partial y}{\partial \varepsilon}\right)_{\varepsilon=0} d\varepsilon = \eta(x)d\varepsilon$$

$$J[y_\varepsilon] = \int_{x_1}^{x_2} f(\bar{y}(x) + \varepsilon\eta(x), \bar{y}'(x) + \varepsilon\eta'(x), x)dx$$

has a minimum at $\varepsilon = 0$

$$\left(\frac{\partial J[y_\varepsilon]}{\partial \varepsilon}\right)_{\varepsilon=0} = 0, \qquad \delta J = \frac{\partial J}{\partial \varepsilon}|_{\varepsilon=0}\, d\varepsilon$$

$$\begin{aligned}
\frac{\partial J[y_\varepsilon]}{\partial \varepsilon} &= \int_{x_1}^{x_2} \{\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\}dx \\
&= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)dx + \frac{\partial f}{\partial y'}\eta(x)\}|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}(\frac{\partial f}{\partial y'})\eta(x)dx \\
&= \int_{x_1}^{x_2} [\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'})]\eta(x)dx
\end{aligned}$$

(1)

((Fundamental lemma))

If

$$\int_{x_1}^{x_2} M(x)\eta(x)dx = 0$$

for all arbitrary function $\eta(x)$ continuous through the second derivative, then $M(x)$ must identically vanish in the interval $x_1 \leq x \leq x_2$.

From this fundamental lemma of variational and Eq.(1), we have Euler equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'}) = 0. \qquad (2)$$

J can have an stationary value only if the Euler equation is valid. The Euler equation clearly resembles the Lagrange's equation.

In summary,

$$J = \int_{x_1}^{x_2} f(y, y' x) dx .$$

$$\delta J = 0 \Leftrightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial y'}) = 0 .$$

13.2 Euler- Lagrange's equations

Now we consider the calculus of variation for the integral

$$J = \int_{x_1}^{x_2} f(y_1, y_2, ..., y_n, y_1', y_2', ..., y_n', x) dx$$

We may introduce ε by setting

$$y_1 = \bar{y}_1(x) + \varepsilon \eta_1(x) ,$$
$$y_2 = \bar{y}_2(x) + \varepsilon \eta_2(x) ,$$
..............................,
$$y_n = \bar{y}_n(x) + \varepsilon \eta_n(x)$$

where $\bar{y}_1(x)$, $\bar{y}_2(x)$, ..., are the solutions of the problem,

$$\eta_1(x_1) = \eta_2(x_1) = ... = \eta_n(x_1) = 0 ,$$
$$\eta_1(x_2) = \eta_2(x_2) = ... = \eta_n(x_2) = 0 .$$

J has a minimum at $\varepsilon = 0$,

$$\frac{\partial J(y_\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0,$$

$$\delta J = \left(\frac{\partial J}{\partial \varepsilon}\right)\Big|_{\varepsilon=0} d\varepsilon, \qquad\qquad \delta y_i = \eta_i d\varepsilon,$$

$$\begin{aligned}\frac{\partial J}{\partial \varepsilon} &= \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i}\eta_i(x) + \frac{\partial f}{\partial y_i'}\eta_i'(x)\right]dx \\ &= \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)\right]\eta_i(x)]dx\end{aligned},$$

at $\varepsilon = 0$.

$$\begin{aligned}\frac{\partial J}{\partial \varepsilon} &= \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i}\eta_i(x) + \frac{\partial f}{\partial y_i'}\eta_i'(x)\right]dx \\ &= \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)\right]\eta_i(x)dx\end{aligned}$$

Formally, this can be written as

$$\delta J = \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)\right]\delta y_i dx = 0$$

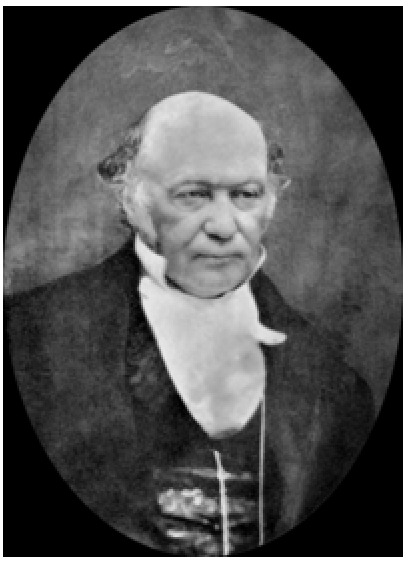
This is the assertion that J is stationary for the correct path. δy_i is the virtual displacement. By an obvious extension of the fundamental lemma, we have

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0,$$

($i = 1, 2, ..., n$).

13.3 Hamilton's principle

William Rowan Hamilton (4 August 1805 – 2 September 1865) was an Irish physicist, astronomer, and mathematician, who made important contributions to classical mechanics, optics, and algebra. His studies of mechanical and optical systems led him to discover new mathematical concepts and techniques. His greatest contribution is perhaps the reformulation of Newtonian mechanics, now called Hamiltonian mechanics. This work has proven central to the modern study of classical field theories such as electromagnetism, and to the development of quantum mechanics.



http://en.wikipedia.org/wiki/William_Rowan_Hamilton

Hamilton's principle

Hamilton's principle states that the physical path taken by a particle system moving between two fixed points in configuration space is one for which the action integral is stationary under a virtual variation of the path. The action (action integral) is defined as

$$\int_1^2 L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)dt \,.$$

The Hamilton's principle is sometimes also called *the principle of least action*.

The instantaneous configuration of a system is described by the values of the n generated co-ordinates

$$(q_1, q_2, q_3, ..., q_n)$$

and corresponds to a particular point in the configuration space.

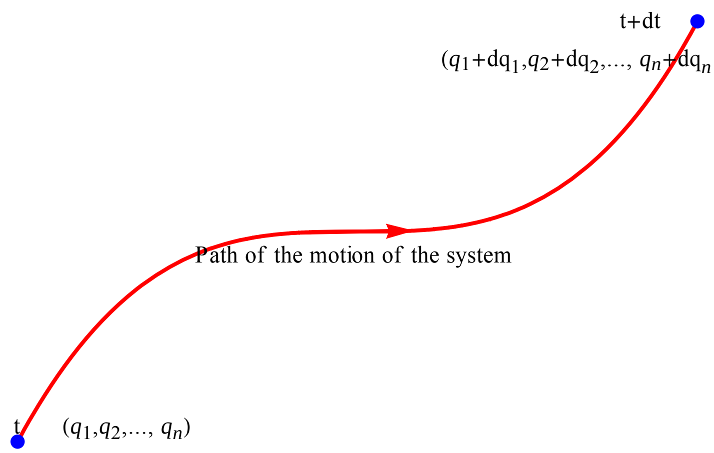


Fig.   Configurational space

The Lagrangian of monogenic system is defined by

$$L = T - V$$

where T is the kinetic energy and V is a potential energy of the system. Here we define the line integral as

$$I = \int_{t_1}^{t_2} L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)dt \, .$$

The motion of the system from time t_1 to time t_2 is such that the line integral has a stationary value for the actual path of the motion. We can summarize the Hamilton's principle by saying that the motion is such that the variation of the line integral *I* for fixed t_1 and t_2 is zero,

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n) dt = 0 .$$

When the system constraints are holonomic, Hamiltonian's principle is both a necessary and sufficient condition for Lagrange's equation,

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0 .$$

Hamilton's principle $\Leftrightarrow$ Lagrange's equation

$$\delta \int_{t_1}^{t_2} L dt = 0 , \qquad \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0$$

Then the Euler-Lagrange's equation corresponding to *I* becomes the Lagrange's equation of motion.

13.4 Derivation of Lagrange's equation.

We consider the Hamilton's principle with

$$I = \int_{t_1}^{t_2} L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t) dq ,$$

$$\delta I = \frac{\partial I}{\partial \alpha} d\alpha = \delta \int_{t_1}^{t_2} L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t) dt = 0 ,$$

where

$$q_1 = \bar{q}_1(t) + \alpha\eta_1(t),$$

$$q_2 = \bar{q}_2(t) + \alpha\eta_2(t),$$

..............................,

$$q_n = \bar{q}_n(t) + \alpha\eta_n(t),$$

and

$$\dot{q}_1 = \dot{\bar{q}}(t) + \alpha\eta_1'(t),$$

$$\dot{q}_2 = \dot{\bar{q}}_2(t) + \alpha\eta_2'(t),$$

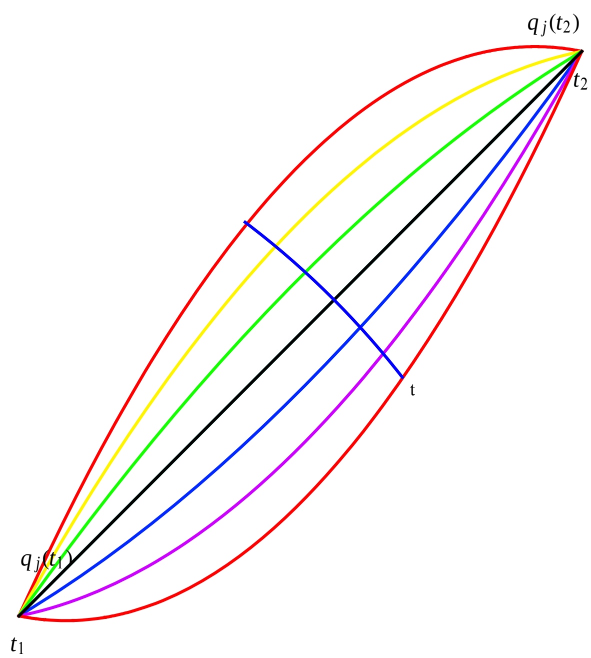
..............................,

$$\dot{q}_n = \dot{\bar{q}}_n(t) + \alpha\eta_n'(t),$$

with

$$\eta_1(t_1) = \eta_2(t_1) = ... = \eta_n(t_1) = 0$$

$$\eta_1(t_2) = \eta_2(t_2) = ... = \eta_n(t_2) = 0$$



I has a minimum at $\alpha = 0$.

$$\left(\frac{\partial I}{\partial \alpha}\right) = \int_{t_1}^{t_2} \sum_i [\frac{\partial L}{\partial q_i}\eta_i(t) + \frac{\partial L}{\partial \dot{q}_i}\eta_i'(t)]dt$$
$$= \int_{t_1}^{t_2} \sum_i [\frac{\partial L}{\partial q_i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})]\eta_i(t)dt = 0$$

at $\alpha = 0$.

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0, \qquad (i = 1, 2, ..., n).$$

Here we note that

$$\eta_i(t) = \left(\frac{\partial q_i}{\partial \alpha}\right)_{\alpha=0}, \qquad \delta q_i = \left(\frac{\partial q_i}{\partial \alpha}\right)_{\alpha=0} d\alpha = \eta_i d\alpha$$

$$\delta I = \left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} d\alpha$$

or simply

$$\delta I = \frac{\partial I}{\partial \alpha} d\alpha$$

Then we have the form of

$$\delta I = \left(\frac{\partial I}{\partial \alpha}\right)_{\alpha=0} d\alpha = \int_{t_1}^{t_2} \sum_i [\frac{\partial L}{\partial q_i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})]\delta q_i dt = 0$$

where δq_i is a virtual displacement.

((**Note**)) Formulation

Formally we can describe the Hamilton's principle as follows (formulation).

$$\begin{aligned}\delta I &= \delta\int_{t_1}^{t_2} L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)dt \\ &= d\alpha \frac{\partial}{\partial \alpha}\int_{t_1}^{t_2} L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)dt \\ &= d\alpha \int_{t_1}^{t_2} \sum_i [\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha}]dt\end{aligned}$$

Here

$$\begin{aligned}\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha} dt &= \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right)dt \\ &= [\frac{\partial q_i}{\partial \alpha}\frac{\partial L}{\partial \dot{q}_i}]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial q_i}{\partial \alpha}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)dt \\ &= -\int_{t_1}^{t_2} \frac{\partial q_i}{\partial \alpha}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)dt\end{aligned}$$

with

$$d\alpha \frac{\partial q_i}{\partial \alpha} = \delta q_i, \qquad \delta q_i(t_1) = \delta q_i(t_2) = 0 .$$

Then we have

$$\delta I = d\alpha \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \alpha} dt$$

$$= \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right]\delta q_i dt$$

leading to the Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 .$$

13.5 Definition of cyclic

If the Lagrangian of a system does not contain a given co-ordinate q_j, then the coordinate is said to be "cyclic" or "igonorable".

The Lagrange equation of the motion is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 .$$

The momentum associated with the coordinate qj is defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j} .$$

The terms canonical momentum and conjugate momentum are often also used for p_j. using the expression for p_j, the Lagrange's equation can be rewritten as

$$\frac{dp_j}{dt} = \frac{\partial L}{\partial q_j} .$$

For a cyclic coordinate,

$$\frac{\partial L}{\partial q_j} = 0 \qquad (q_j \text{ is not included in } L)$$

$$\frac{dp_j}{dt} = 0 .$$

This means that

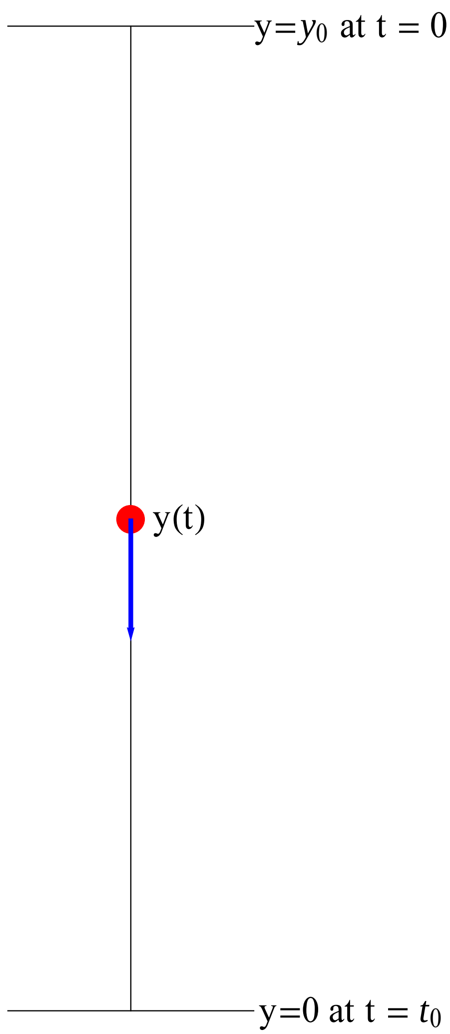
$$p_j = \text{const.}$$

((**Conservation theorem**))

The generalized momentum conjugate to a cyclic co-ordinate is conserved. If the system is invariant under the translation along a given direction, the corresponding linear momentum is conserved. If the system is invariant under the rotation about the given axis, the corresponding angular momentum is conserved. Thus the momentum conservation theorems are closely connected with the symmetry properties of the system.

13.6 Example

13.6.1 Free falling



At t = 0, the particle is at $y = y_0$. The particle starts to undergo the motion of free falling and reaches y = 0 at $t = t_0$. The value of y_0 is related to t_0 by

$$y_0 = \frac{1}{2} g t_0^{\ 2} .$$

What is the time dependence of $y(t)$ such that the line integral I over the Lagrangian L takes a minimum?

$$L = \frac{1}{2} m [\dot{y}(t)]^2 - mgy(t) ,$$

$$I = \int_0^{t_0} L dt \, .$$

((**Solution**))

We assume that

$$y(t) = a + bt + ct^2 \, ,$$

with

$$y(0) = y_0 = a = \frac{1}{2} g t_0^{\ 2} \, ,$$

$$y(t_0) = a + bt_0 + ct_0^{\ 2} = 0 \, .$$

The value of *I* can be calculated as

$$I(c) = \frac{m t_0^{\ 3}}{6} (c^2 + cg - \frac{3}{4} g^2) \, .$$

Taking the derivative of *I* with respect to *c*, we have a local minimum such that

$$\frac{\partial I(c)}{\partial c} = 0$$

or

$$c = -\frac{g}{2} \, ,$$

which leads to $b = 0$. Finally we have

$$y(t) = y_0 - \frac{g}{2}t^2$$

((**Mathematica**))

```
L = 1/2 m y'[t]^2 - m g y[t];

Y[t_] := A1 + B1 t + C1 t^2;

L1 = L /. y → Y // Simplify;

I1 = ∫_0^t0 L1 ⅆt // Simplify;

y[t_] = Y[t];

eq1 = Solve[{y[0] == g/2 t0^2, y[t0] == 0}, {A1, B1}]
```

$$\left\{\left\{B1 \to \frac{1}{2}\left(-2\,C1\,t0 - g\,t0\right),\ A1 \to \frac{g\,t0^2}{2}\right\}\right\}$$

```
I2 = I1 /. eq1[[1]] // Expand // Collect[#, C1] &
```

$$\frac{1}{6} C1^2\, m\, t0^3 + \frac{1}{6} C1\, g\, m\, t0^3 - \frac{1}{8} g^2\, m\, t0^3$$

```
eq2 = Solve[D[I2, C1] == 0, C1];

A1 = A1 /. eq1[[1]]
```

$$\frac{g\,t0^2}{2}$$

```
B1 = B1 /. eq1[[1]] /. eq2[[1]]
```

0

```
C1 = C1 /. eq2[[1]]
```

$$-\frac{g}{2}$$

13.6.2 Approximation: trial function for simple harmonics

We consider the Lagrangian for the simple harmonics,

$$L = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos\theta) .$$

For $\theta \approx 0$,

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ...$$

$$L = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(\frac{\theta^2}{2} - \frac{\theta^4}{24}) .$$

The Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} .$$

Since

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} . \qquad \text{and} \qquad \frac{\partial L}{\partial \theta} = -mgl(\theta - \frac{\theta^3}{6}) ,$$

we have

$$ml^2 \ddot{\theta} + mgl(\theta - \frac{\theta^3}{6}) = 0 ,$$

or

$$\ddot{\theta} + \omega_0{}^2 (\theta - \frac{\theta^3}{6}) = 0 ,$$

where

$$\omega_0 = \sqrt{\frac{g}{l}} \,.$$

When

$$x = \theta \,, \qquad ml^2 = 1 \,, \qquad \varepsilon = \frac{{\omega_0}^2}{6} \ (>0)$$

we get

$$\ddot{x} + {\omega_0}^2 x - \varepsilon x^3 = 0 \,.$$

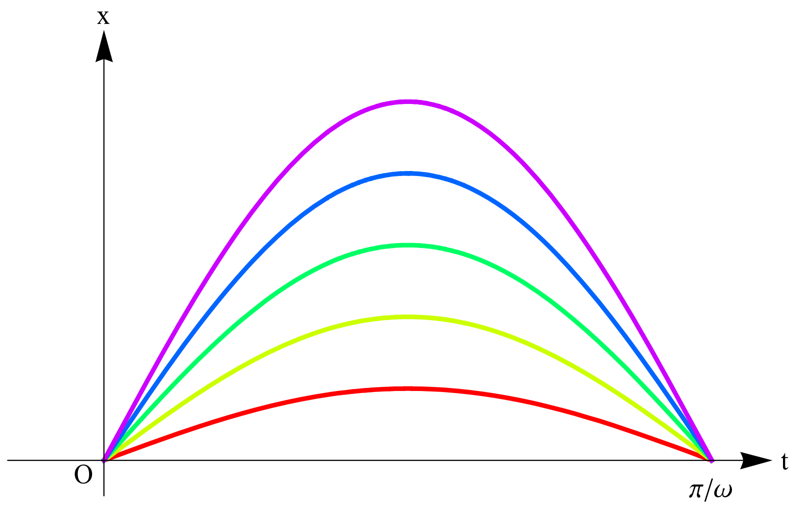
$$L = \frac{1}{2}(\dot{x}^2 - {\omega_0}^2 x^2) + \frac{\varepsilon}{4} x^4 \,.$$

We now consider the integral

$$I = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt \,.$$

When $\varepsilon = 0$,

$$x = A_0 \sin(\omega_0 t)$$



For small $\varepsilon\,(\approx 0)$, we assume that

$$x = A\sin(\omega t) \qquad \text{(trial function)}$$

$$\dot{x} = A\omega\cos(\omega t)$$

Here A and ω are unknown parameters. We choose

$$t_1 = 0, \qquad t_2 = \frac{2\pi}{\omega}.$$

$$x(t_1) = 0, \qquad x(t_2) = 0 \qquad \text{(fixed).}$$

$$\int_0^{2\pi/\omega} dt \sin^2 \omega t = \frac{\pi}{\omega}, \qquad \int_0^{2\pi/\omega} dt \cos^2 \omega t = \frac{\pi}{\omega}$$

$$\int_0^{2\pi/\omega} dt \sin^4 \omega t = \frac{3\pi}{4\omega}$$

Using these results, we have

$$I = \frac{1}{2}A^2\frac{\pi}{\omega}(\omega^2 - {\omega_0}^2) + \frac{\varepsilon}{4}\frac{3}{4}\frac{\pi}{\omega}A^4$$

I is a function of A.

$$\begin{aligned}\frac{\partial I}{\partial A} &= A\frac{\pi}{\omega}(\omega^2 - {\omega_0}^2) + \frac{3\varepsilon}{16}\frac{\pi}{\omega}4A^3 \\ &= \frac{3\varepsilon}{4}\frac{\pi}{\omega}A[A^2 - \frac{4}{3\varepsilon}({\omega_0}^2 - \omega^2)]\end{aligned}.$$

When $\frac{\partial I}{\partial A}$=0, we have

$$\varepsilon\frac{3}{4}A^2 = {\omega_0}^2 - \omega^2,$$

or

$$\omega = \omega_0(1 - -\frac{1}{8}A^2)^{1/2} \approx \omega_0(1 - -\frac{1}{16}A^2),$$

since

$$\varepsilon = \frac{{\omega_0}^2}{6}.$$

Then the period is obtained as

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0}(1 + \frac{1}{16}A^2).$$

13.7 VariationalD (Mathematica program)

We suppose that the functional is given by $f(y, y', x)$. Using the VariationalD [Mathematica], one can calculate

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right),$$

where the variation of the integral J is defined as

$$\delta J = \int_1^2 \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y dx .$$

VariationalD

`VariationalD[f, u[x], x]`
returns the variational derivative of the integral $\int f\, dx$ with respect to $u[x]$, where the integrand f is a function of u, its derivatives, and x.

`VariationalD[f, u[x, y, ...], {x, y, ...}]`
returns the variational derivative of the multiple integral $\int f\, dx\, dy \ldots$ with respect to $u[x, y, \ldots]$, where f is a function of u, its derivatives and the coordinates x, y, ….

`VariationalD[f, {u[x, y, ...], v[x, y, ...], ...}, {x, y, ...}]`
gives a list of variational derivatives with respect to u, v, ….

13.8 EulerEquations (Mathematica program)

Using the EulerEquations [Mathematica], one can derive the Euler (Lagrange, in physics) equation given by

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 . \quad (1)$$

EulerEquations

`EulerEquations[f, u[x], x]` returns the Euler–Lagrange differential equation obeyed by *u*[*x*] derived from the functional *f*, where *f* depends on the function *u*[*x*] and its derivatives as well as the independent variable *x*.
`EulerEquations[f, u[x, y, ...], {x, y, ...}]` returns the Euler–Lagrange differential equation obeyed by *u*[*x*, *y*, ...].
`EulerEquations[f, {u[x, y, ...], v[x, y, ...], ...}, {x, y, ...}]` returns a list of Euler–Lagrange differential equations obeyed by *u*[*x*, *y*, ...], *v*[*x*, *y*, ...], ...

13.9 FirstIntegral (Mathematica program)

Here we note that the Euler (Lagrange) equation can be rewritten as

$$\frac{\partial f}{\partial x}-\frac{d}{dx}(f-y'\frac{\partial f}{\partial y'})=0\,, \qquad (2)$$

since

$$\frac{\partial f}{\partial x}-\frac{d}{dx}(f-y'\frac{\partial f}{\partial y'})=\frac{\partial f}{\partial x}-\frac{df}{dx}+\frac{d}{dx}(y'\frac{\partial f}{\partial y'})$$

or

$$\frac{\partial f}{\partial x}-\frac{d}{dx}(f-y'\frac{\partial f}{\partial y'})=\frac{\partial f}{\partial x}-(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'+\frac{\partial f}{\partial y'}y'')+y''\frac{\partial f}{\partial y'}+y'\frac{d}{dx}(\frac{\partial f}{\partial y'})$$

$$=-\frac{\partial f}{\partial y}y'+y'\frac{d}{dx}(\frac{\partial f}{\partial y'})=y'[-\frac{\partial f}{\partial y}+\frac{d}{dx}(\frac{\partial f}{\partial y'})]=0$$

(a) The case when *f* is independent of *x*.

Since $\frac{\partial f}{\partial x}=0$ in Eq.(2), we have

$$\frac{d}{dx}(f - y'\frac{\partial f}{\partial y'}) = 0,$$

or

$$f - y'\frac{\partial f}{\partial y'} = constant. \quad (3)$$

Thus, when f is independent of x, FirstIntegrals[f, y(x), x] leads to the calculation of $f - y'\frac{\partial f}{\partial y'}$.

This corresponds to the Hamiltonian (or energy function) in the physics.

(b) The case when f is independent of y.

Since $\frac{\partial f}{\partial y} = 0$ in Eq.(1), we have

$$\frac{\partial f}{\partial y'} = constant. \quad (4)$$

When f is independent of y, FirstIntegrals[f, y(x), x] leads to the calculation of $\frac{\partial f}{\partial y'}$

((Note))

When you use FirstIntegrals[f, y(x), x] in your Mathematica program, yo do not have to check in advance whether f is independent of y or f is independent of x. The Mathematica will check for you automatically. In this sense, the FirstIntegrals are a very convenient program.

FirstIntegrals

`FirstIntegrals[f, x[t], t]`
returns a list of first integrals corresponding to the coordinate x[t] and independent variable t of the integrand f.

`FirstIntegrals[f, {x[t], y[t], ...}, t]`
returns a list of first integrals corresponding to the coordinates x, y, ... and independent variable t.

((**Mathematica**))

Here is an example of the simple harmonics. The Lagrangian is given by

$$L(x,\dot{x},t)=\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2$$

```
Needs["VariationalMethods`"];

L = 1/2 m x'[t]^2 - 1/2 k x[t]^2;

EulerEquations[L, x[t], t]

-k x[t] - m x''[t] == 0

FirstIntegrals[L, x[t], t]

{FirstIntegral[t] → 1/2 (k x[t]^2 + m x'[t]^2)}

VariationalD[L, x[t], t]

-k x[t] - m x''[t]
```

13.10 Shortest distance between two points in a plane

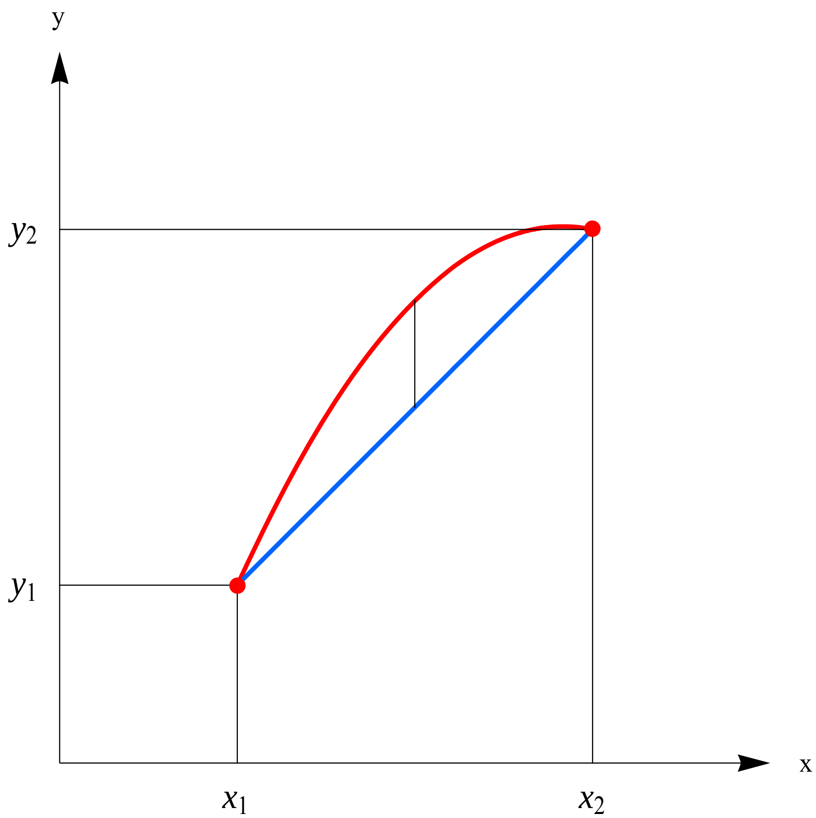


Fig. Varied paths of the function of y(x) in the one dimensional extremum problem.

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + y'^2}$$

The total length of any curve going between points 1 and 2 is

$$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} dx f(y, y', x),$$

with

$$f(y, y', x) = \sqrt{1 + y'^2}.$$

We calculate the Euler equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Since

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} = \text{const.}$$

we have

$$y' = a \;\; (= \text{constant}).$$

or

$$y = ax + b,$$

which is the equation of the straight line. The constants a and b are determined by the condition that the curve passes through the two end points (x_1, y_1) and (x_2, y_2).

In general, curves that give the shortest distance between two points on a given surface are called the geodesics of the surface.

((**Mathematica**))

```
Clear["Global`*"]

<< "VariationalMethods`"
```

F = $\sqrt{1+y'[x]^2}$

$\sqrt{1+y'[x]^2}$

```
eq1 = VariationalD[F, y[x], x]
```

$-\frac{y''[x]}{\left(1+y'[x]^2\right)^{3/2}}$

```
eq2 = EulerEquations[F, y[x], x]
```

$-\frac{y''[x]}{\left(1+y'[x]^2\right)^{3/2}} == 0$

```
eq3 = FirstIntegrals[F, y[x], x]
```

$\left\{\text{FirstIntegral}[y] \to -\frac{y'[x]}{\sqrt{1+y'[x]^2}},\ \text{FirstIntegral}[x] \to -\frac{1}{\sqrt{1+y'[x]^2}}\right\}$

eq4 = $\left(\frac{y'[x]}{\sqrt{1+y'[x]^2}}\right)^2 == c^2$

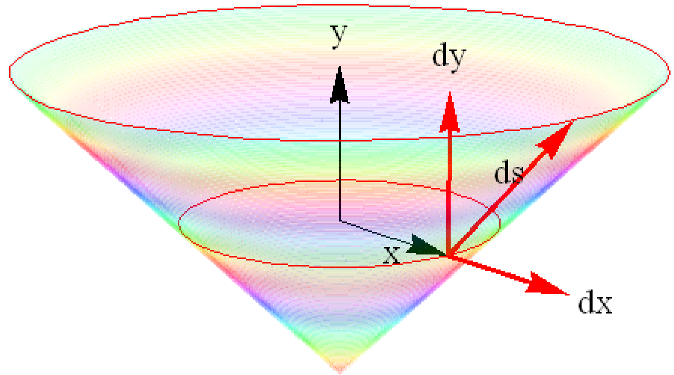
$\frac{y'[x]^2}{1+y'[x]^2} == c^2$

```
eq5 = Solve[eq4, y'[x]]
```

$\left\{\left\{y'[x] \to -\frac{c}{\sqrt{1-c^2}}\right\},\ \left\{y'[x] \to \frac{c}{\sqrt{1-c^2}}\right\}\right\}$

Since y'[x] is constant, y[x] is a straight line.

13.11 Minimum surface of revolution



Suppose we form a surface of revolution by taking some curve passing between two fixed end points, and revolving it about the *y* axis. The problem is to find the curve for which the surface area is a minimum.

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1 + y'^2}$$

The area of a strip of the surface is

$$2\pi x ds = 2\pi x\sqrt{1 + y'^2}\,dx$$

$$A = \int_{x_1}^{x_2} f(y, y', x)dx$$

with

$$f = f(y, y', x) = 2\pi x\sqrt{1 + y'^2}$$

Since

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial y'} = 2\pi x \frac{y'}{\sqrt{1+y'^2}} = \text{const}$$

we have

$$\frac{d}{dx}\left(\frac{xy'}{\sqrt{1+y'^2}}\right) = 0, \qquad \frac{xy'}{\sqrt{1+y'^2}} = a$$

or

$$y' = \frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}}$$

Then we get

$$y = a\int \frac{dx}{\sqrt{x^2 - a^2}} = a \operatorname{arccos} h\left(\frac{x}{a}\right) + b,$$

or

$$x = a\cosh\left(\frac{y-b}{a}\right),$$

which is the equation of a catenary.

((**Mathematica**))

```
Clear["Global`*"]

<< "VariationalMethods`"
```

F = $x\sqrt{1+y'[x]^2}$

$x\sqrt{1+y'[x]^2}$

eq1 = VariationalD[F, y[x], x]

$$\frac{-y'[x]-y'[x]^3-x\,y''[x]}{\left(1+y'[x]^2\right)^{3/2}}$$

eq2 = EulerEquations[F, y[x], x]

$$\frac{-y'[x]-y'[x]^3-x\,y''[x]}{\left(1+y'[x]^2\right)^{3/2}}==0$$

eq3 = FirstIntegrals[F, y[x], x]

$$\left\{\text{FirstIntegral}[y]\to-\frac{x\,y'[x]}{\sqrt{1+y'[x]^2}}\right\}$$

$$\text{eq4}=\left(\frac{x\,y'[x]}{\sqrt{1+y'[x]^2}}\right)^2==a^2$$

$$\frac{x^2\,y'[x]^2}{1+y'[x]^2}==a^2$$

eq5 = Solve[eq4, y'[x]]

$$\left\{\left\{y'[x]\to-\frac{a}{\sqrt{-a^2+x^2}}\right\},\left\{y'[x]\to\frac{a}{\sqrt{-a^2+x^2}}\right\}\right\}$$

eq51 = eq5[[2]] /. Rule → Equal

```
eq6 = eq51[[1]]
```

$$y'[x] == \frac{a}{\sqrt{-a^2 + x^2}}$$

```
eq7 = DSolve[{eq6, y[a] == b}, y[x], x] // Simplify
```

$$\{\{y[x] \to b - a\,\text{Log}[a] + a\,\text{Log}[x + \sqrt{-a^2 + x^2}]\}\}$$

```
eq8 = Y == y[x] /. eq7[[1]]
```

$$Y == b - a\,\text{Log}[a] + a\,\text{Log}[x + \sqrt{-a^2 + x^2}]$$

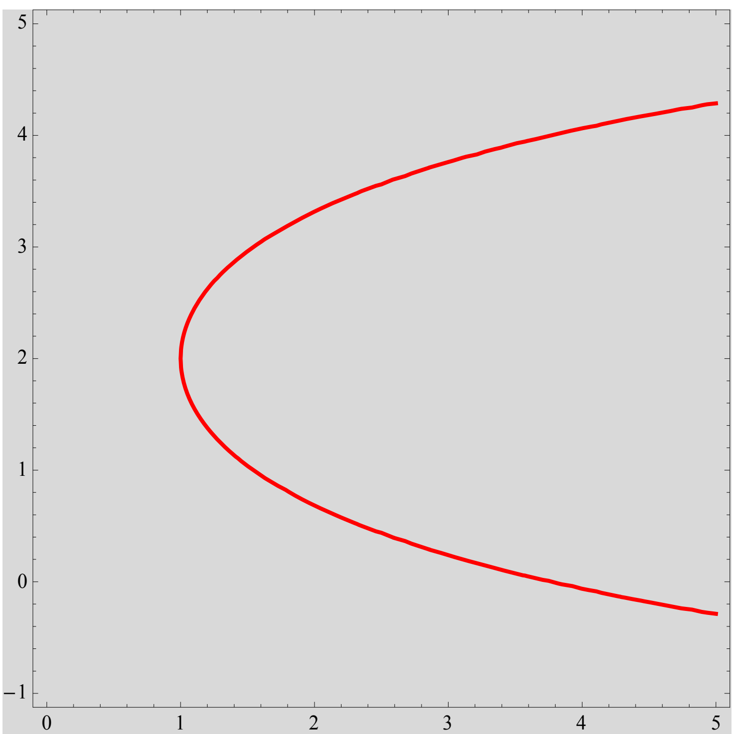
```
eq9 = Solve[eq8, x] // Expand
```

$$\{\{x \to \frac{1}{2}\,a\,e^{\frac{b}{a} - \frac{Y}{a}} + \frac{1}{2}\,a\,e^{-\frac{b}{a} + \frac{Y}{a}}\}\}$$

```
eq10 = X == x /. eq9[[1]] /. {a → 1, b → 2}
```

$$X == \frac{e^{2-Y}}{2} + \frac{e^{-2+Y}}{2}$$

```
ContourPlot[Evaluate[eq10], {X, 0, 5}, {Y, -1, 5}, ContourStyle → {Red, Thick},
 Background → LightGray]
```



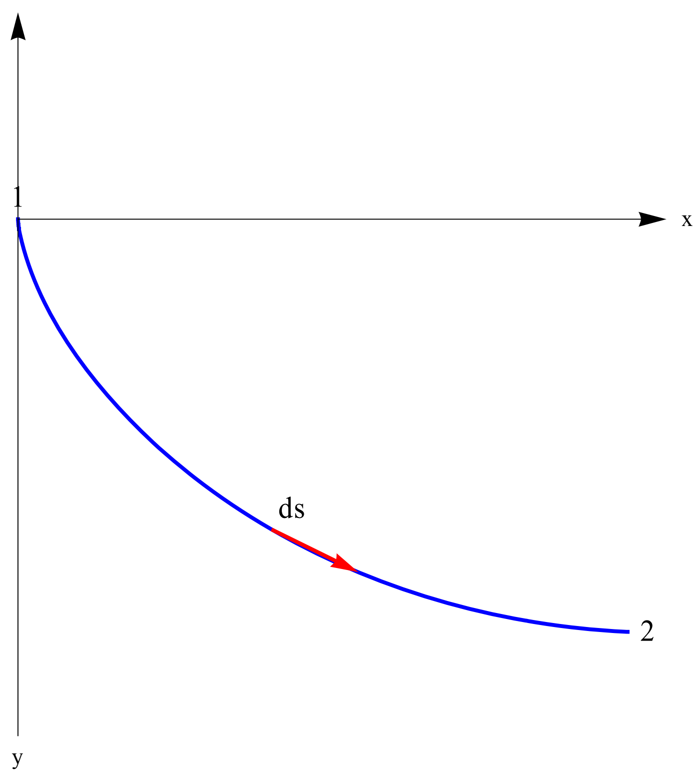
13.12 The brachistochrone problem

The brachistochrone problem was one of the earliest problems posed in the calculus of variations. Newton was challenged to solve the problem in 1696, and did so the very next day. In fact, the solution, which is a segment of a cycloid, was found by Leibniz, L'Hospital, Newton, and the two Bernoullis. Johann Bernoulli solved the problem using the analogous one of considering the path of light refracted by transparent layers of varying density. Actually, Johann Bernoulli had originally found an incorrect proof that the curve is a cycloid, and challenged his brother Jacob to find the required curve. When Jacob correctly did so, Johann tried to substitute the proof for his own.

BRACHIS = SHORT

CHRONOS = TIME

The well-known problem is to find the curve joining two points, along which a particle falling from rest under the influence of gravity travels from the higher to lower point in the least time.



$$ds = \sqrt{1 + y'^2}\,dx$$

$$t_{12} = \int_1^2 \frac{ds}{v}$$

If y is measured down from the initial point of release,

$$\frac{1}{2}mv^2 - mgy = 0,$$

or

$$v = \sqrt{2gy}.$$

Then the time t_{12} is given by

$$t_{12} = \int_1^2 \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx = \frac{1}{\sqrt{2g}} \int_1^2 \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx,$$

and $f(y, y', x)$ is defined as

$$f(y, y', x) = \frac{\sqrt{1+y'^2}}{\sqrt{y}}.$$

Euler-Lagrange equation;

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}.$$

Since

$$\frac{\partial f}{\partial y} = \sqrt{1+y'^2}\,(-\frac{1}{2}y^{-3/2})\,, \qquad \frac{\partial f}{\partial y'} = \frac{1}{\sqrt{y}}\frac{y'}{\sqrt{1+y'^2}} = \frac{y'}{\sqrt{y(1+y'^2)}}\,,$$

we have

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= \frac{d}{dx}\frac{y'}{\sqrt{y(1+y'^2)}} \\
&= \frac{y''\sqrt{y(1+y'^2)} - y'\dfrac{\{y'(1+y'^2)+y2y'y''\}}{2\sqrt{y(1+y'^2)}}}{y(1+y'^2)}\,. \\
&= \frac{2y''y(1+y'^2) - y'^2\left(1+y'^2+2yy''\right)}{2[y(1+y'^2)]^{3/2}} \\
&= \frac{2y''y - y'^2\left(1+y'^2\right)}{2[y(1+y'^2)]^{3/2}}
\end{aligned}$$

Then we get

$$-\frac{1}{2y^{3/2}}\sqrt{1+y'^2} - \frac{2y''y - y'^2\left(1+y'^2\right)}{2[y(1+y'^2)]^{3/2}} = 0$$

$$-\frac{1}{y^{3/2}}\sqrt{1+y'^2}\,y^{3/2}(1+y'^2)^{3/2} - [2y''y - y'^2\left(1+y'^2\right)] = 0$$

$$-(1+y'^2)^2 - 2y''y + y'^2\left(1+y'^2\right)] = 0$$

$$-(1+y'^2) - 2y''y = 0$$

Thus we find

$$\frac{y''}{1+y'^2} + \frac{1}{2y} = 0$$

or

$$\frac{y''}{1+y'^2}y' + \frac{1}{2y}y' = 0$$

or

$$\frac{1}{2}\ln(1+y'^2) + \frac{1}{2}\ln y = const\,.$$

((**Mathematica**))

Brachistrochrone prblem

```
<< "VariationalMethods`"
```

$$F = \sqrt{\frac{1 + y'[x]^2}{y[x]}};$$

```
eq1 = VariationalD[F, y[x], x]
```

$$\frac{-1 - y'[x]^2 - 2\,y[x]\,y''[x]}{2\,y[x]^3\left(\frac{1+y'[x]^2}{y[x]}\right)^{3/2}}$$

```
eq2 = FirstIntegrals[F, y[x], x]
```

$$\left\{\text{FirstIntegral}[x] \to -\frac{1}{y[x]\sqrt{\frac{1+y'[x]^2}{y[x]}}}\right\}$$

```
eq3 = FirstIntegral[x] /. eq2[[1]]
```

$$-\frac{1}{y[x]\sqrt{\frac{1+y'[x]^2}{y[x]}}}$$

eq4 = eq3^2 == $\frac{1}{2\,a}$ // Simplify

$$\frac{1}{y[x]+y[x]\,y'[x]^2}==\frac{1}{2\,a}$$

eq5 = Solve[eq4, y'[x]]

$$\left\{\left\{y'[x]\to-\frac{i\sqrt{-2\,a+y[x]}}{\sqrt{y[x]}}\right\},\left\{y'[x]\to\frac{i\sqrt{-2\,a+y[x]}}{\sqrt{y[x]}}\right\}\right\}$$

eq6 = y'[x] - (y'[x] /. eq5[[2]]) == 0

$$-\frac{i\sqrt{-2\,a+y[x]}}{\sqrt{y[x]}}+y'[x]==0$$

eq7 = $-\frac{\sqrt{2\,a-y[x]}}{\sqrt{y[x]}}$ + y'[x] == 0;

This equation can be solved as

$$\frac{dx}{dy}=\sqrt{\frac{y}{2\,a-y}}$$

or

$$x=\int_0^y\sqrt{\frac{u}{2\,a-u}}\,du$$

```
eq7 = Simplify[∫_0^y Sqrt[u/(2 a - u)] du, {0 < y < 2 a, a > 0}]
```

$$\frac{\sqrt{y}\,(-2a+y)+2a\sqrt{2a-y}\,\mathrm{ArcTan}\left[\frac{1}{\sqrt{-1+\frac{2a}{y}}}\right]}{\sqrt{2a-y}}$$

```
rule1 = {y → 2 a Sin[θ]^2};

Y = y /. rule1
```

$2\,a\,\mathrm{Sin}[\theta]^2$

```
eq8 = FullSimplify[eq7 /. rule1, a > 0 && 0 < θ < π/2]
```

$2\,a\,(\theta - \mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta])$

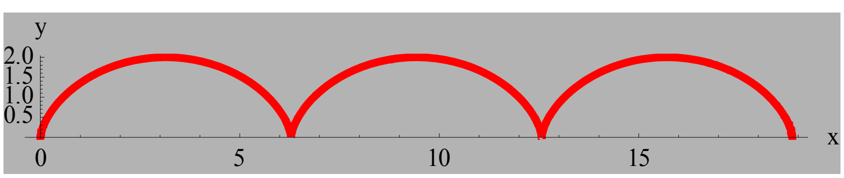
```
X1 = eq8 /. {a → 1, θ → ϕ / 2} // Simplify
```

$\phi - \mathrm{Sin}[\phi]$

```
Y1 = Y /. {a → 1, θ → ϕ / 2}
```

$2\,\mathrm{Sin}\left[\frac{\phi}{2}\right]^2$

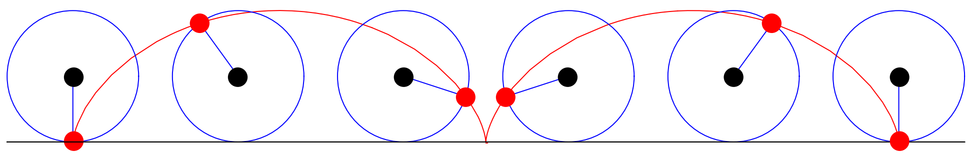
```
ParametricPlot[{X1, Y1}, {ϕ, 0, 6 π},
 PlotStyle → {Hue[0], Thickness[0.01]},
 Background → GrayLevel[0.7],
 AspectRatio → Automatic, AxesLabel → {"x", "y"}]
```

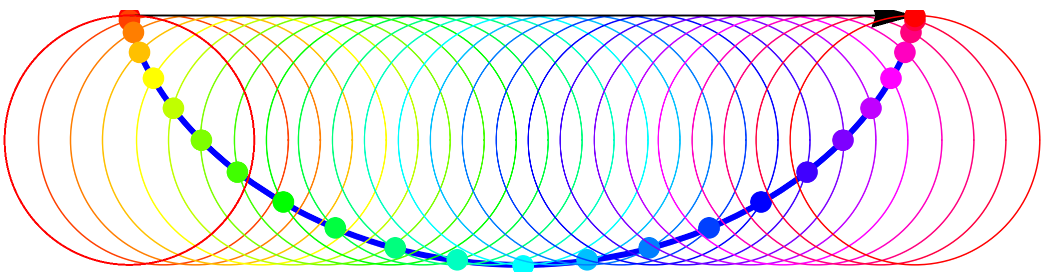


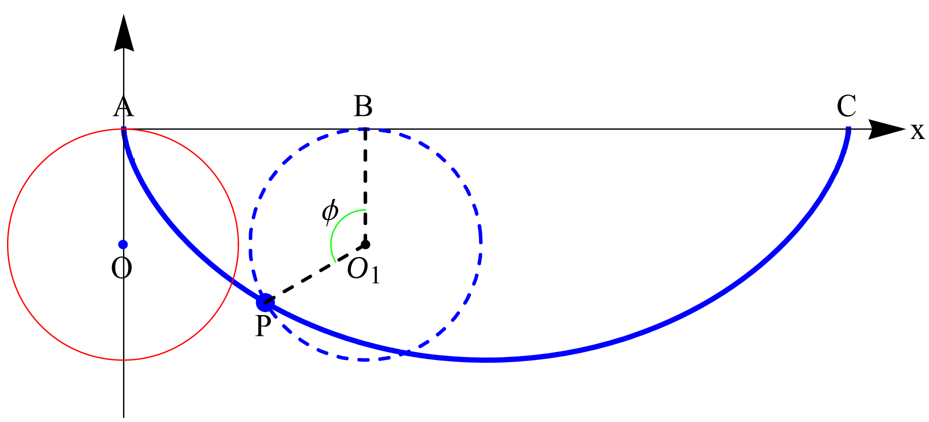
13.13 Cycloid

The cycloid is the locus of a point on the rim of a circle of radius rolling along a straight line. It was studied and named by Galileo in 1599. Galileo attempted to find the area by weighing pieces of metal cut into the shape of the cycloid. Torricelli, Fermat, and Descartes all found the area. The cycloid was also studied by Roberval in 1634, Wren in 1658, Huygens in 1673, and Johann Bernoulli in 1696. Roberval and Wren found the arc length (MacTutor Archive). Gear teeth were also made out of cycloids, as first proposed by Desargues in the 1630s (Cundy and Rollett 1989).

((**Mathematica**))







$$\overrightarrow{AB} = (R\phi, 0)$$

$$\overrightarrow{AP} = (R(\phi - \sin\phi), R(1 - \cos\phi))$$

$$Arc(PB) = R\phi$$

13.14 The branchistochrone problem with initial velocity

When the particle is projected with a kinetic energy $\frac{1}{2}mv_0{}^2$, we have

$$\frac{1}{2}mv^2 - mgy = \frac{1}{2}mv_0{}^2 = mgz$$

where

$$z = \frac{v_0{}^2}{2g}$$

then we have

$$v = \sqrt{2g(y+z)}$$

$$t_{12} = \frac{1}{\sqrt{2g}}\int_{t_1}^{t_2}\frac{\sqrt{1+y'^2}}{\sqrt{y+z}}dx = \frac{1}{\sqrt{2g}}\int_{t_1}^{t_2}\frac{\sqrt{1+Y'^2}}{\sqrt{Y}}dx$$

where

$$Y = y + z$$

Euler-Lagrange's equation

$$Y(1+Y'^2) = 2a$$

or

$$\frac{dY}{dx} = (\frac{2a}{Y} - 1)^{1/2}$$

For $x = 0$, $Y = z$,

$$x = \int_z^Y \sqrt{\frac{u}{2a - u}} du$$
$$= a[\arccos(1 - \frac{Y}{a}) - \sin[\arccos(1 - \frac{Y}{a})] - a[\arccos(1 - \frac{z}{a}) - \sin[\arccos(1 - \frac{z}{a})]$$

$$\arccos(1 - \frac{Y}{a}) = \arccos(1 - \frac{y+z}{a}) = \theta$$

$$\arccos(1 - \frac{z}{a}) = \theta_0$$

$$\cos\theta_0 = 1 - \frac{z}{a}, \qquad \cos\theta = 1 - \frac{y+z}{a},$$

$$y = 0, \qquad z = \frac{v_0^{\ 2}}{2g} = a(1 - \cos\theta_0)$$

$$x = a[(\theta - \sin\theta) - (\theta_0 - \sin\theta_0)]$$
$$y = a[(1 - \cos\theta) - (1 - \cos\theta_0)]$$

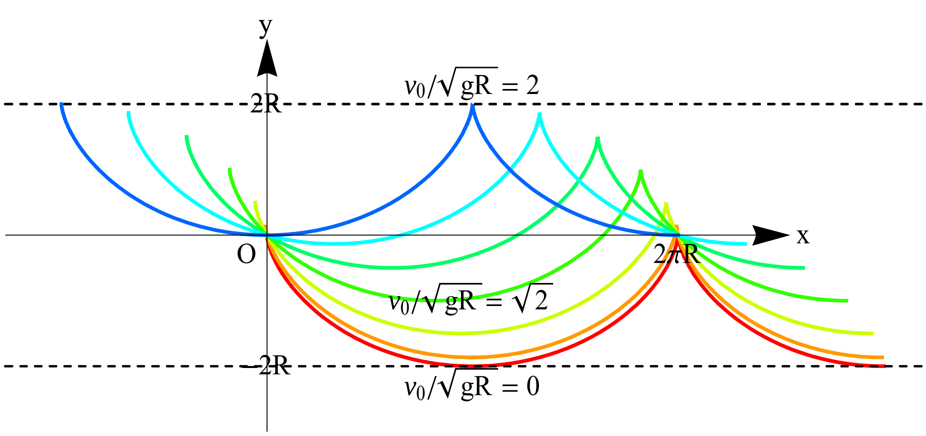


Fig. Cycloid motion with the initial velocity at y = 0 which is changed as a parameter.

$v_0 = \sqrt{2gR(1-\cos\theta_0)}$, where $\theta_0 = 0$, $\pi/6$, $\pi/3$, $\pi/2$, $2\pi/3$, $5\pi/6$, and π.

---

http://www.sewanee.edu/physics/TAAPT/TAAPTTALK.html?x=47&y=53

http://mathworld.wolfram.com/BrachistochroneProblem.html

http://curvebank.calstatela.edu/brach/brach.htm

13.15 Simple pendulum

We consider a simple harmonics

$$L = L(\theta, \dot{\theta}, t) = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$$

L is independent of t.

Lagrange equation

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \text{ ,}$$

with

$$\omega_0^{\ 2} = \frac{g}{l}$$

First Integral: L is independent of t.

$$\dot{\theta}^2 + 2\omega_0^{\ 2}(1-\cos\theta) = const$$

Initial condition: $\dot{\theta}(0) = v(0)$ and $\theta(0) = 0$

$$\frac{1}{2}\dot{\theta}^2 + \omega_0{}^2(1-\cos\theta) = \frac{v(0)^2}{2}$$

$U(\theta) = \omega_0{}^2(1-\cos\theta)$ is a potential and $\frac{v(0)^2}{2}$ is the total energy. When $v(0) = 2\omega_0$ is the critical angular velocity. For

For $v(0) < 2\omega_0$, a sinusoidal oscillation is observed.
For $v(0) < 2\omega_0$, a continuous rotation occurs. In other words, θ monotonically increases with increasing *t*.

See the lecture note on the physics of simple pendulum in much more detail.
http://www2.binghamton.edu/physics/docs/physics-of-simple-pendulum-9-15-08.pdf

((**Mathematica**))

```
Clear["Global`*"]

<< "VariationalMethods`"
```

$$L = \frac{1}{2} m \ell^2 \theta'[t]^2 - m g \ell (1 - Cos[\theta[t]])$$

$$-g m \ell (1 - Cos[\theta[t]]) + \frac{1}{2} m \ell^2 \theta'[t]^2$$

```
eq1 = VariationalD[L, θ[t], t]
```

$$-m \ell (g Sin[\theta[t]] + \ell \theta''[t])$$

```
eq2 = EulerEquations[L, θ[t], t]
```

$$-m \ell (g Sin[\theta[t]] + \ell \theta''[t]) == 0$$

```
eq3 = FirstIntegrals[L, θ[t], t] // Simplify
```

$$\left\{FirstIntegral[t] \to \frac{1}{2} m \ell \left(-2 g (-1 + Cos[\theta[t]]) + \ell \theta'[t]^2\right)\right\}$$

```
eq4 = Solve[eq2, θ''[t]]
```

$$\left\{\left\{\theta''[t] \to -\frac{g Sin[\theta[t]]}{\ell}\right\}\right\}$$

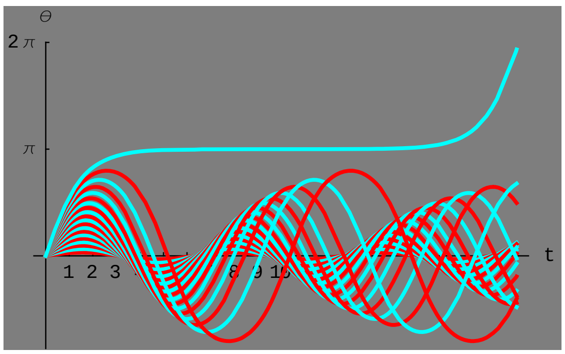
```
eq5 = θ''[t] - (θ''[t] /. eq4[[1]]) == 0
```

$$\frac{g Sin[\theta[t]]}{\ell} + \theta''[t] == 0$$

$$eq6 = eq5 /. \{g \to \ell \omega 0^2\} // Simplify$$

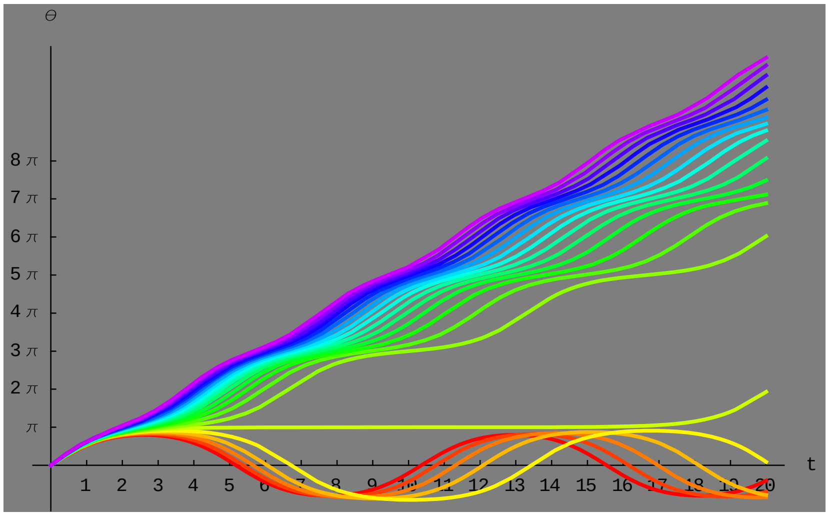
$$\omega 0^2 Sin[\theta[t]] + \theta''[t] == 0$$

```
phase[{θ0_, v0_}, ω0_, tmax_, opts__] :=
 Module[{numso1, numgraph},
  numso1 =
   NDSolve[{ ω0^2 Sin[θ[t]] + v'[t] == 0, v[t] == θ'[t], θ[0] == θ0, v[0] == v0},
     {θ[t], v[t]}, {t, 0, tmax}] // Flatten;
  numgraph = Plot[Evaluate[θ[t] /. numso1], {t, 0, tmax}, opts,
    DisplayFunction → Identity]]

phlist =
 phase[{0, #}, 1, 20,  PlotStyle → Hue[5 (# - 0.1)], AxesLabel → {"t", "θ"},
    Prolog → AbsoluteThickness[2], Background → GrayLevel[0.5],
    PlotRange → All, Ticks → { Range[0, 10],  π Range[-3, 3]},
    DisplayFunction → Identity] & /@ Range[0.1, 2.0, 0.1];
Show[phlist, DisplayFunction → $DisplayFunction]
```



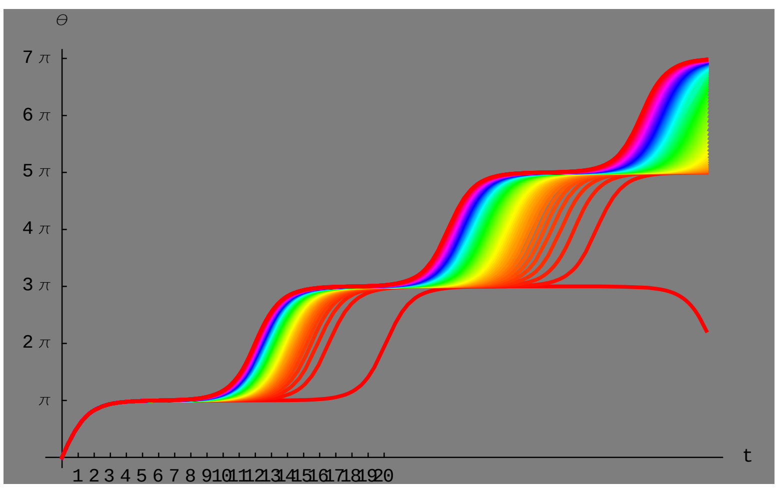
- Graphics -

```
phlist =
 phase[{0, #}, 1, 20,  PlotStyle → Hue[2 (# - 1.9)], AxesLabel → {"t", "θ"},
    Prolog → AbsoluteThickness[2], Background → GrayLevel[0.5],
    PlotRange → All, Ticks → { Range[0, 20],  π Range[0, 8]},
    DisplayFunction → Identity] & /@ Range[1.9, 2.3, 0.02];
Show[phlist, DisplayFunction → $DisplayFunction]
```




```
- Graphics -
```

```
phlist =
 phase[{0, #}, 1, 40,  PlotStyle → Hue[10 000 (# - 2)], AxesLabel → {"t", "θ"},
    Prolog → AbsoluteThickness[2], Background → GrayLevel[0.5],
    PlotRange → All, Ticks → { Range[0, 20],  π Range[0, 8]},
    DisplayFunction → Identity] & /@ Range[2, 2.0001, 0.000001];
Show[phlist, DisplayFunction → $DisplayFunction]
```



13.16 Ginzburg-Landau equation for superconductivity

We introduce the order parameter $\psi(\mathbf{r})$ with the property that

$$\psi^*(\mathbf{r})\psi(\mathbf{r}) = n_s(\mathbf{r}).$$

which is the local concentration of superconducting electrons. We first set up a form of the free energy density $F_s(\boldsymbol{r})$,

$$F_s(\mathbf{r}) = F_N + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi\right|^2 + \frac{\mathbf{B}^2}{8\pi},$$

where β is positive and the sign of α is dependent on temperature. We must minimize the free energy with respect to the order parameter $\psi(\boldsymbol{r})$ and the vector potential $\boldsymbol{A}(\boldsymbol{r})$. We set

$$\Im = \int F_s(\mathbf{r}) dr ,$$

where the integral is extending over the volume of the system. If we vary

$$\psi(\mathbf{r}) \to \psi(\mathbf{r}) + \delta\psi(\mathbf{r}), \qquad \mathbf{A}(\mathbf{r}) \to \mathbf{A}(\mathbf{r}) + \delta\mathbf{A}(\mathbf{r}),$$

we obtain the variation in the free energy such that

$$\Im + \delta\Im .$$

By setting $\delta\Im = 0$, we obtain the GL equation

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A}\right)^2\psi = 0,$$

and the current density

$$\mathbf{J}_s = \frac{q^*\hbar}{2m^* i}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{q^{*2}|\psi|^2}{m^* c}\mathbf{A},$$

or

$$\mathbf{J}_s = \frac{q^*}{2m^*}[\psi^*(\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi + \psi(-\frac{\hbar}{i}\nabla - \frac{q^*}{c}\mathbf{A})\psi^*]$$

At a free surface of the system we must choose the gauge to satisfy the boundary condition that no current flows out of the superconductor into the vacuum.

$$\mathbf{n}\cdot\mathbf{J}_s = 0$$

((**Mathematica**))

Derivation of Ginzburg Landau equation

```
Clear["Global`*"]

<< "VariationalMethods`"

Needs["VectorAnalysis`"]

SetCoordinates[Cartesian[x, y, z]];

A = {A1[x, y, z], A2[x, y, z], A3[x, y, z]};
```

$$\text{eq1} = \alpha\,(\psi[x,y,z]\,\psi c[x,y,z]) + \frac{1}{2}\beta\left(\psi[x,y,z]^2\,\psi c[x,y,z]^2\right) + \frac{1}{2\,m}\left(\left(\frac{\hbar}{i}\,\text{Grad}[\psi[x,y,z]] - \frac{q}{c}\,A\,\psi[x,y,z]\right).\left(-\frac{\hbar}{i}\,\text{Grad}[\psi c[x,y,z]] - \frac{q}{c}\,A\,\psi c[x,y,z]\right)\right) \text{ // Expand;}$$

```
eq2 = VariationalD[eq1, ψc[x, y, z], {x, y, z}] // Expand
```

$$\alpha\,\psi[x,y,z] + \frac{q^2\,A1[x,y,z]^2\,\psi[x,y,z]}{2\,c^2\,m} + \frac{q^2\,A2[x,y,z]^2\,\psi[x,y,z]}{2\,c^2\,m} + \frac{q^2\,A3[x,y,z]^2\,\psi[x,y,z]}{2\,c^2\,m} + \beta\,\psi[x,y,z]^2\,\psi c[x,y,z] + \frac{i\,q\,\hbar\,\psi[x,y,z]\,A3^{(0,0,1)}[x,y,z]}{2\,c\,m} + \frac{i\,q\,\hbar\,A3[x,y,z]\,\psi^{(0,0,1)}[x,y,z]}{c\,m} - \frac{\hbar^2\,\psi^{(0,0,2)}[x,y,z]}{2\,m} + \frac{i\,q\,\hbar\,\psi[x,y,z]\,A2^{(0,1,0)}[x,y,z]}{2\,c\,m} + \frac{i\,q\,\hbar\,A2[x,y,z]\,\psi^{(0,1,0)}[x,y,z]}{c\,m} - \frac{\hbar^2\,\psi^{(0,2,0)}[x,y,z]}{2\,m} + \frac{i\,q\,\hbar\,\psi[x,y,z]\,A1^{(1,0,0)}[x,y,z]}{2\,c\,m} + \frac{i\,q\,\hbar\,A1[x,y,z]\,\psi^{(1,0,0)}[x,y,z]}{c\,m} - \frac{\hbar^2\,\psi^{(2,0,0)}[x,y,z]}{2\,m}$$

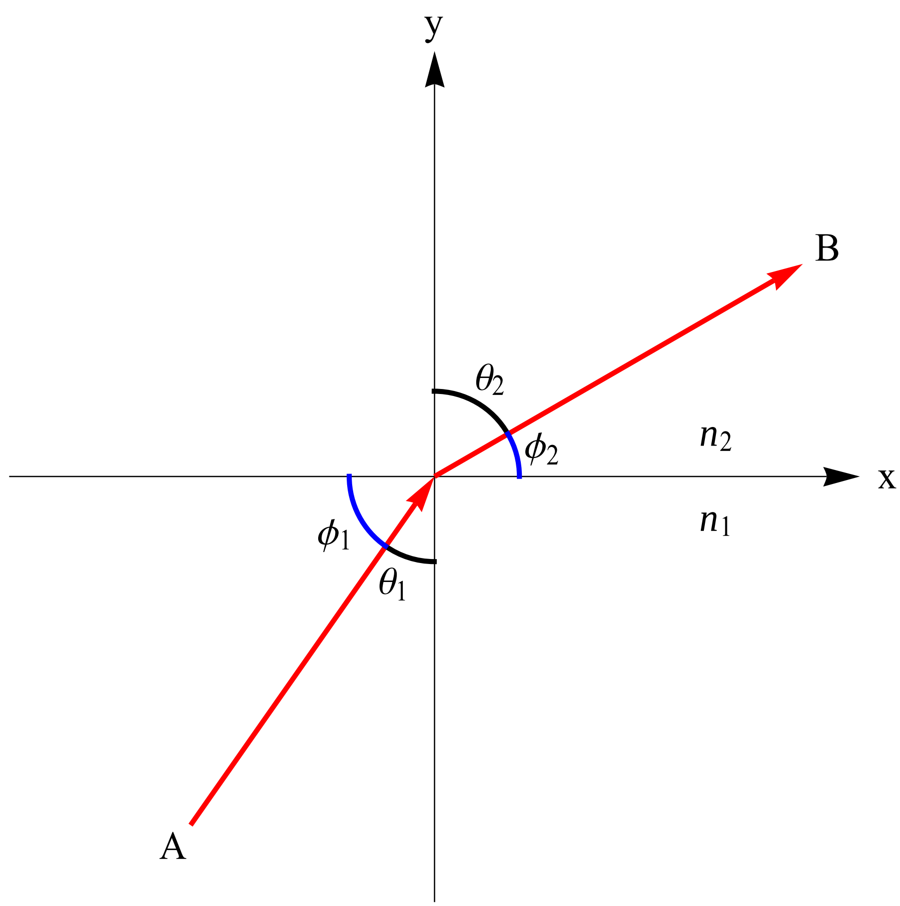
We need to calculate the following

```
OP1 := (ℏ/ⅈ D[#, x] - q/c A1[x, y, z] # &)

OP2 := (ℏ/ⅈ D[#, y] - q/c A2[x, y, z] # &)

OP3 := (ℏ/ⅈ D[#, z] - q/c A3[x, y, z] # &)

eq3 =
  α ψ[x, y, z] + β ψ[x, y, z]^2 ψc[x, y, z] +
    1/(2 m) (OP1[OP1[ψ[x, y, z]]] + OP2[OP2[ψ[x, y, z]]] +
       OP3[OP3[ψ[x, y, z]]]) // Expand
```

$$\alpha\,\psi[x,y,z] + \frac{q^2\,A1[x,y,z]^2\,\psi[x,y,z]}{2\,c^2\,m} + \frac{q^2\,A2[x,y,z]^2\,\psi[x,y,z]}{2\,c^2\,m} + \frac{q^2\,A3[x,y,z]^2\,\psi[x,y,z]}{2\,c^2\,m} + \beta\,\psi[x,y,z]^2\,\psi c[x,y,z] + \frac{i\,q\,\hbar\,\psi[x,y,z]\,A3^{(0,0,1)}[x,y,z]}{2\,c\,m} + \frac{i\,q\,\hbar\,A3[x,y,z]\,\psi^{(0,0,1)}[x,y,z]}{c\,m} - \frac{\hbar^2\,\psi^{(0,0,2)}[x,y,z]}{2\,m} + \frac{i\,q\,\hbar\,\psi[x,y,z]\,A2^{(0,1,0)}[x,y,z]}{2\,c\,m} + \frac{i\,q\,\hbar\,A2[x,y,z]\,\psi^{(0,1,0)}[x,y,z]}{c\,m} - \frac{\hbar^2\,\psi^{(0,2,0)}[x,y,z]}{2\,m} + \frac{i\,q\,\hbar\,\psi[x,y,z]\,A1^{(1,0,0)}[x,y,z]}{2\,c\,m} + \frac{i\,q\,\hbar\,A1[x,y,z]\,\psi^{(1,0,0)}[x,y,z]}{c\,m} - \frac{\hbar^2\,\psi^{(2,0,0)}[x,y,z]}{2\,m}$$

```
eq2 - eq3 // Simplify

0
```

13.17 Fermat theorem (optics)



Fermat's principle in geometrical optics states that a ray of light travelling in a region of variable refractive index follows a path such that the total optical path length (physical path length) is stationary.

We can derive Snell's law of refraction at an interface.

$$ds = dx\sqrt{1+[y'(x)]^2}\,.$$

Suppose that the index of refraction n depends only on y. The total time T is

$$T = \int_A^B \frac{1}{c} n(y)\sqrt{1+y'^2}\,dx\,.$$

Since the integrand does not contain the independent variable *x* explicitly, we use the first integral

$$f = n(y)\sqrt{1+y'^2},$$

$$f - y'\frac{\partial f}{\partial y'} = 0,$$

or

$$n(y)\sqrt{1+y'^2} - y'n(y)\frac{y'}{\sqrt{1+y'^2}} = n(y)\frac{1}{\sqrt{1+y'^2}} = k,$$

where *k* is constant. *y'* is the tangent of the angle ϕ between the instantaneous direction of the ray and the *x* axis.

Since $y' = \tan\phi$,

$$n(y)\frac{1}{\sqrt{1+y'^2}} = \frac{n(y)}{\sqrt{1+\tan^2\phi}}k = n(y)\cos\phi,$$

or

$$n_1\cos\phi_1 = n_2\cos\phi_2,$$

or

Snell's law

$$n_1 \cos(\frac{\pi}{2} - \theta_1) = n_2 \cos(\frac{\pi}{2} - \theta_2)$$
.
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

+

13.18 Extension of Hamiltonian's principle to nonholonomic systems

It is possible to extend Hamiltonian's principle, at least in a formal sense, to cover certain types of nonholonomic systems. With nonholonomic systems the generalized co-ordinates are not independent of each other, and it is not possible to reduce them further by means of Eqs. of constraint of the form;

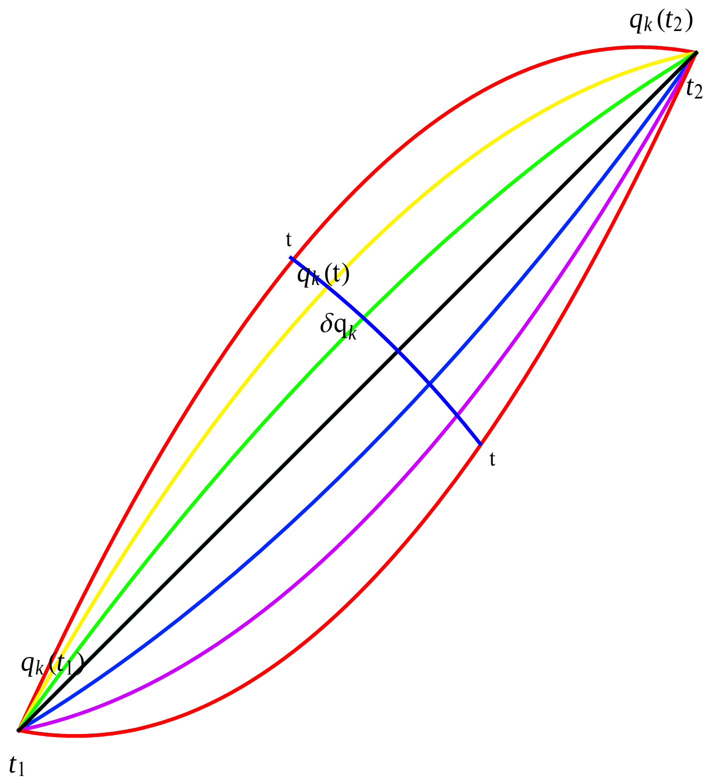
$$f(q_1, q_2, ..., q_n, t) = 0 .$$

It appears that a resonably straightforward treatment of nonholonomic systems by a variational principle is possible, only when Eqs. of constraint can be put in the form

$$\sum_{k=1}^{n} a_{lk} dq_k + a_{lt} dt = 0 ,$$

$(l = 1, 2, ..., m.)$

a linear relation connecting the differentials of q's. Note that a_{lk} and a_{lt} may be functions of q's and t.



The constraint Eqs. valid for the virtual displacement are

$$\sum_{k=1}^{n} a_{lk}\delta q_k = 0\,, \tag{1}$$

where t = const. We can use Eq.(1) to reduce the number of virtual displacements ti independent ones.

13.19 Method of Lagrange undetermined multipliers

If Eq.(1) holds, then we have

$$\lambda_l \sum_{k=1}^{n} a_{lk}\delta q_k = 0\,. \tag{2}$$

(l = 1, 2, ..., m).

where λ_l are undetermined quantities;

$$\lambda_l = \lambda_l(q_1, q_2, ..., q_n, t)$$

In addition, we have Hamilton's principle given by

$$\delta\int_{t1}^{t2} Ldt = 0,$$

is assumed for the nonholonomic system.

$$\delta I = \int_{t1}^{t_2} dt \sum_{k=1}^{n} (\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k})\delta q_k = 0 \qquad (3)$$

From Eq.(2)

$$\int_{t_1}^{t_2} dt \sum_{k,l} \lambda_l a_{lk} \delta q_k = 0, \qquad (4)$$

The sum of Eq.(3) and (4) then yields the relation

$$\int_{t1}^{t_2} dt \sum_{k=1}^{n} (\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_{l=1}^{m} \lambda_l a_{lk})\delta q_k = 0$$

The δq_k 's are still not independent. They are connected by the *m* relations.

$$\sum_{k=1}^{n} a_{lk}\delta q_k = 0 \qquad (l = 1, 2, ..., m).$$

or

$$a_{11}\delta q_1 + a_{12}\delta q_2 + ... + a_{1n}\delta q_n = 0$$
$$a_{21}\delta q_1 + a_{22}\delta q_2 + ... + a_{2n}\delta q_n = 0$$
$$..$$
$$a_{m1}\delta q_1 + a_{m2}\delta q_2 + ... + a_{mn}\delta q_n = 0$$

$$|\, \delta q_1, \delta q_2, ..., \delta q_{n-m}, |\, \delta q_{n-m+1}, \delta q_{m+2}, ..., \delta q_n \,|$$

The first (n - m) of these may be chosen independently.

$$\int_{t1}^{t_2} dt \sum_{k=1}^{n-m} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_{l=1}^{m} \lambda_l a_{lk}\right)\delta q_k$$
$$+ \int_{t1}^{t_2} dt \sum_{k=n-m+1}^{n} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_{l=1}^{m} \lambda_l a_{lk}\right)\delta q_k = 0$$

Suppose that we now choose the λ_l's to be such that

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_{l=1}^{m} \lambda_l a_{lk} = 0$$

for $k = n - m+1, n - m + 2, ..., n$. With the λ_l determined above, we can write as

$$\int_{t1}^{t_2} dt \sum_{k=1}^{n-m} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_{l=1}^{m} \lambda_l a_{lk}\right)\delta q_k = 0$$

where δq_k is independent. Here it follows that

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_{l=1}^{m} \lambda_l a_{lk} = 0, \qquad \text{with } k = 1, 2, ..., n - m.$$

Finally, we have

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \sum_{l=1}^{m} \lambda_l a_{lk} = 0$$

for $k = 1, 2, ..., n$.

((Note))

We have now $(n + m)$ unknown parameters

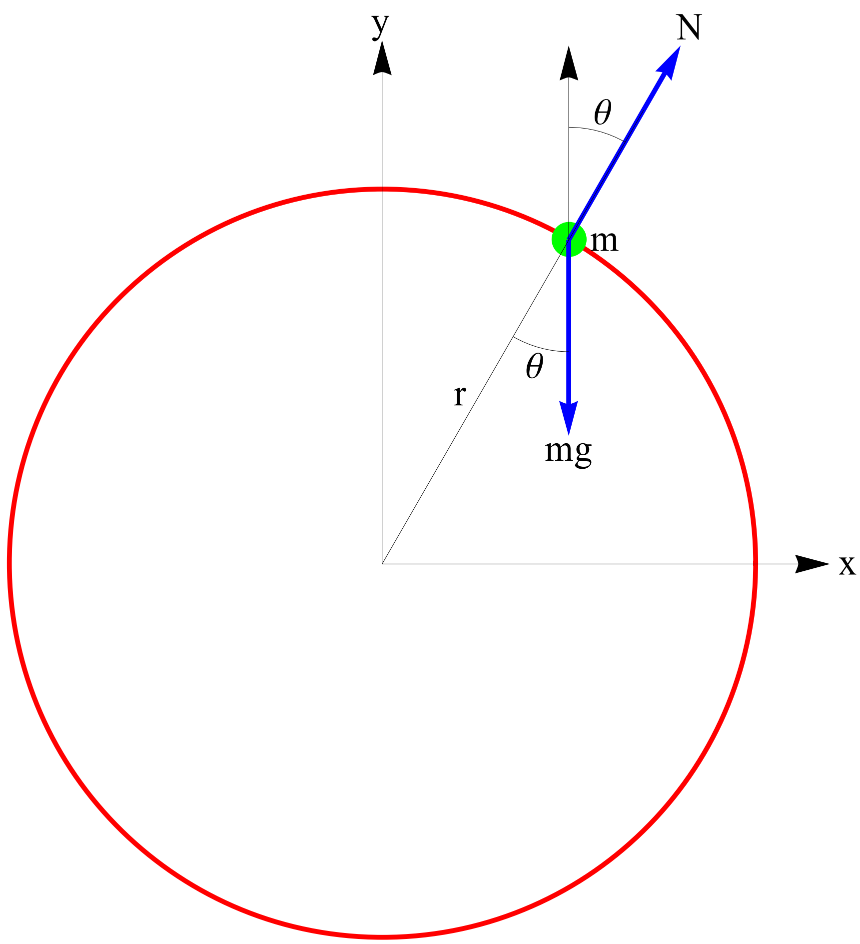
$(q_1, q_2, ..., q_n)$,

$(\lambda_1, \lambda_2, ..., \lambda_m)$,

The additional equations needed are exactly the equations of constraint liking up the q_k's

$$\sum_{k=1}^{n} a_{lk}\dot{q}_k + a_{lt} = 0 \qquad (l = 1, 2, ..., m)$$

13.20



$$x^2 + y^2 = r^2 \qquad\qquad xdx + ydy = 0$$

The Lagrangian L;

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

λ: Lagrange multiplier

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \lambda x \qquad\qquad M\ddot{x} = \lambda x$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) - \frac{\partial L}{\partial y} = \lambda y \qquad\qquad M\ddot{y} + mg = \lambda y$$

((Note))

$$\sum F_y = N\cos\theta - mg = N\frac{y}{r} - mg = m\ddot{y}$$

$$\sum F_x = N\sin\theta = N\frac{x}{r} = m\ddot{x}$$

$$\lambda = \frac{N}{r}$$

with

$$x^2 + y^2 = r^2 \qquad\qquad x\dot{x} + y\dot{y} = 0,$$

$$x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = 0$$

$$m\ddot{x}\dot{x} = \frac{N}{r}x\dot{x}.$$

$$m\ddot{y}\dot{y} = \frac{N}{r}y\dot{y} - mg\dot{y}.$$

$$m(\ddot{x}\dot{x} + \ddot{y}\dot{y}) = \frac{N}{r}(x\dot{x} + y\dot{y}) - mg\dot{y}$$

or

$$\frac{d}{dt}[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy] = 0$$

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy = mgr \qquad \text{(energy conservation)}$$

What is the normal force *N*?

$$m\ddot{x}x = \frac{N}{r}x^2 .$$

$$m\ddot{y}y = \frac{N}{r}y^2 - mgy .$$

or

$$m(\ddot{x}x + \ddot{y}y) = \frac{N}{r}(x^2 + y^2) - mgy$$

$$-m(\dot{x}^2 + \dot{y}^2) = \frac{N}{r}(x^2 + y^2) - mgy$$

Since $x^2 + y^2 = r^2$,

$$-2mg(r - y) = \frac{N}{r}(x^2 + y^2) - mgy = Nr - mgy$$

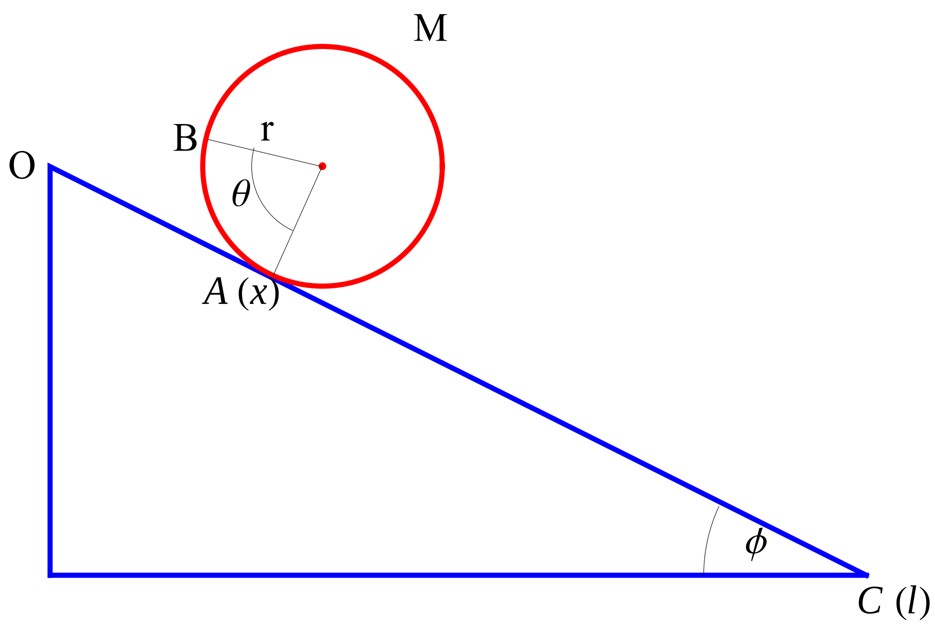
or

$$N = mg(3\frac{y}{r} - 2)$$

When $N = 0$,

$$\frac{y}{r} = \frac{2}{3} = \cos\theta .$$

13.21 Example: Rolling of hollow cylinder on the incline



Equation of rolling constraint,

$$OA = x = r\theta , \quad \text{or} \qquad -dx + rd\theta = 0 .$$

Lagrangian L is given by

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 - Mg(l - x)\sin\phi ,$$

where I is the moment of inertia and is given by $I = Mr^2$ for the hollow cylinder.

λ: Lagrange multiplier

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = -\lambda \qquad M\ddot{x} - Mg\sin\phi + \lambda = 0$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = \lambda r \qquad Mr^2\ddot{\theta} - \lambda r = 0$$

with the equation of constraint

$$\dot{x} = r\dot{\theta} .$$

These equations constitute three equations for three unknown θ, x, and λ.

$$\ddot{x} = \frac{g\sin\phi}{2} ,$$

$$\ddot{\theta} = \frac{g\sin\phi}{2r} ,$$

$$\lambda = M\ddot{x} = \frac{Mg\sin\phi}{2} . \qquad \text{(friction force of constraint)}$$

13.22 Example: Constraint

A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R. The only external force is that of a gravity. If the smaller cylinder starts rolling from rest on top of the larger cylinder, use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder.

Lagrangian:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}mr^2\dot{\phi}^2 - mgy .$$

The first term is the kinetic energy for the translation of the center of mass. The second term is the rotational energy around the center of mass, where the moment of inertia for the hoop is

$$I = mr^2 .$$

Equation of constraint:

$$x^2 + y^2 = (R+r)^2 .$$

or

$$xdx + ydy = 0 .$$

$$\tan\theta = \frac{x}{y},$$

or

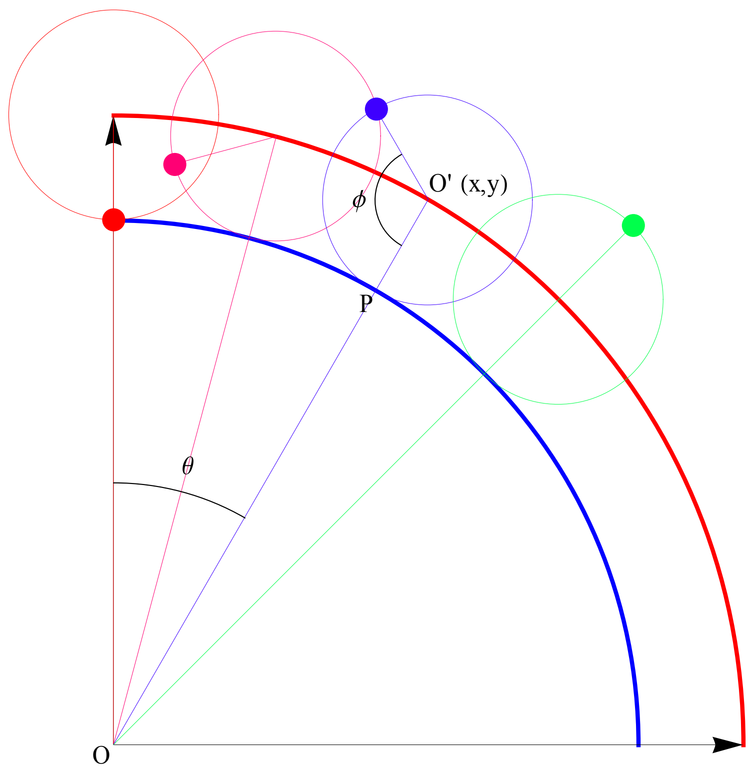
$$\sec^2\theta d\theta = \frac{ydx - xdy}{y^2} = \frac{(x^2+y^2)}{y^2} d\theta ,$$

or

$$ydx - xdy = (x^2+y^2)d\theta = (R+r)^2 d\theta .$$

Then we have

$$d\theta = \frac{ydx - xdy}{(R+r)^2} .$$



The condition for the rolling without slip:

$$R\theta = r\phi, \qquad \text{or} \qquad rd\phi - Rd\theta = 0$$

or

$$rd\phi - \frac{R}{(R+r)^2}(ydx - xdy) = 0$$

Thus the constraint of equation is

$$rd\phi - \frac{R}{(R+r)^2}(ydx - xdy) = 0 \qquad (1)$$

and

$$xdx + ydy = 0 \qquad (2)$$

We introduce Lagrange's undetermined multipliers,

$$m\ddot{x} = \lambda_0[\frac{-Ry}{(R+r)^2}] + \lambda_1 x = N_x, \qquad \text{for } \delta x \qquad (3)$$

$$m\ddot{y} + mg = \lambda_0[\frac{Rx}{(R+r)^2}] + \lambda_1 y = N_y \,, \qquad \text{for } \delta y \qquad (4)$$

$$mr^2\ddot{\phi} = \lambda_0 r = N_\phi \,, \qquad \text{for } \delta\phi \qquad (5)$$

where N_ϕ is the torque of the hoop around its center. Note that

$$N_\phi = \frac{r}{R+r}(xN_y - yN_x) \,.$$

$$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P} \,, \qquad \overrightarrow{OP} = \frac{R}{R+r}\overrightarrow{OO'}$$

$$\overrightarrow{O'P} = -\frac{r}{R+r}\overrightarrow{OO'}$$

Then we have

$$\begin{aligned}\overrightarrow{O'P} \times (N_x\mathbf{e}_x + N_y\mathbf{e}_y) &= -\mathbf{e}_z N_\phi \\ &= -\frac{r}{R+r}(x\mathbf{e}_x + y\mathbf{e}_y) \times (N_x\mathbf{e}_x + N_y\mathbf{e}_y) \\ &= -\frac{r}{R+r}(xN_y - yN_x)\mathbf{e}_z\end{aligned}$$

Eq.(3) $\times \dot{x}$ + Eq.(4) $\times \dot{y}$ + Eq.(5) $\times \dot{\phi}$

$$\begin{aligned} m\ddot{x}\dot{x} + m\ddot{y}\dot{y} + mg\dot{y} + mr^2\ddot{\phi}\dot{\phi} &= \lambda_0[r\dot{\phi} + \frac{R(x\dot{y} - \dot{x}y)}{(R+r)^2}] + \lambda_1(x\dot{x} + \lambda_1 y\dot{y}) \\ &= 0 \end{aligned}$$

or

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy + \frac{1}{2}mr^2\dot{\phi}^2 = mg(R+r) \,, \qquad (6)$$

or

$$\dot{x}^2 + \dot{y}^2 + 2gy + r^2\dot{\phi}^2 = 2g(R+r)$$

From Eqs.(1) and (2),

$$r\dot{\phi} = \frac{R}{(R+r)^2}(y\dot{x} - x\dot{y})$$
,

or

$$\begin{aligned} r^2\dot{\phi}^2 &= \frac{R^2}{(R+r)^4}(y\dot{x} - x\dot{y})^2 \\ &= \frac{R^2}{(R+r)^4}(y^2\dot{x}^2 - 2x\dot{x}y\dot{y} + x^2\dot{y}^2) \\ &= \frac{R^2}{(R+r)^4}[(y^2\dot{x}^2 - 2x\dot{x}y\dot{y} + x^2\dot{y}^2) + (x^2\dot{x}^2 + 2x\dot{x}y\dot{y} + y^2\dot{y}^2)] \\ &= \frac{R^2}{(R+r)^4}(\dot{x}^2 + \dot{y}^2)(x^2 + y^2) \\ &= \frac{R^2}{(R+r)^2}(\dot{x}^2 + \dot{y}^2) \end{aligned}$$

since

$$x\dot{x} + y\dot{y} = 0 .$$

Thus we get

$$r^2\dot{\phi}^2 = \frac{R^2}{(R+r)^2}(\dot{x}^2 + \dot{y}^2) . \qquad (7)$$

The energy conservation law can be rewritten as

$$(\dot{x}^2 + \dot{y}^2)[1 + \frac{R^2}{(R+r)^2}] + 2gy = 2g(R+r) . \qquad (9)$$

At the point where the hoop falls off the cylinder, the component of ***N*** along the normal direction is equal to zero.

$$N_y \cos\theta + N_x \sin\theta = \frac{1}{R+r}(N_y y + N_x x) = 0 ,$$

or

$$N_y y + N_x x = 0 .$$

Eq.(3) x *x* + Eq.(4) x *y:*

$$m(x\ddot{x}+y\ddot{y}+gy)=\lambda_0[\frac{Rxy}{(R+r)^2}]+\lambda_0[\frac{-Rxy}{(R+r)^2}]+\lambda_1(x^2+y^2)$$
$$=\lambda_1(x^2+y^2)=\lambda_1(R+r)^2$$
$$=(N_x x+N_y y)$$

We note that

$$0=\frac{d}{dt}(x\dot{x}+y\dot{y})=\ddot{x}x+\dot{x}^2+\ddot{y}y+\dot{y}^2 .$$

Then we have

$$m(-\dot{x}^2-\dot{y}^2+gy)=(N_x x+N_y y)=\lambda_1(R+r)^2 . \qquad (10)$$

From Eqs.(9) and (10),

$$\dot{x}^2+\dot{y}^2=2g\frac{(R+r)-y}{1+\frac{R^2}{(R+r)^2}}$$

$$(N_x x+N_y y)=\lambda_1(R+r)^2$$
$$=m(-\dot{x}^2-\dot{y}^2+gy)$$
$$=mg[-\frac{2(R+r)-2y}{1+\frac{R^2}{(R+r)^2}}+y]$$

Then we have

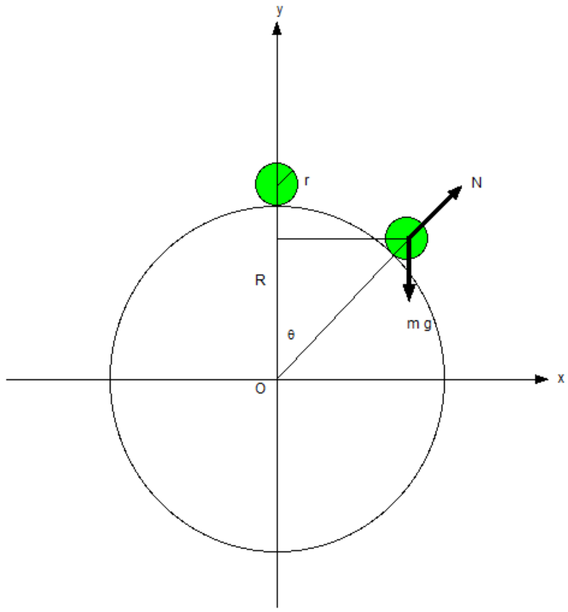
$$y=\frac{2(R+r)}{3+\frac{R^2}{(R+r)^2}} .$$

from the condition

$$N_x x+N_y y=0 .$$

((Another method))

A cylinder of radius *r* and mass *m* on top of a fixed sphere of radius *R*. The first sphere is slightly displaced so that it rolls (without slipping) down the second sphere. What is the angle q at which the first sphere loses a contact with the second sphere.

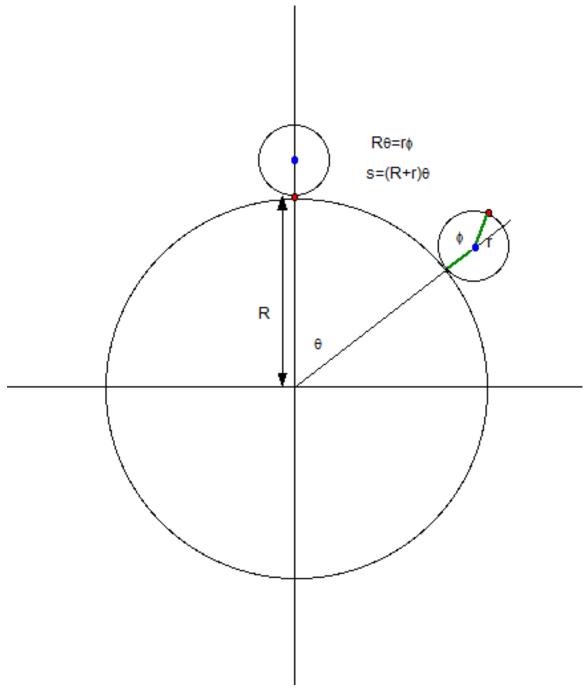


Energy-conservation law

$$mg(R+r) = mg(R+r)\cos\theta + \frac{1}{2}mv_{cm}{}^2 + \frac{1}{2}mr^2\omega^2$$

The condition for no slipping:

$$v_{cm} = \frac{ds}{dt} = (R+r)\dot{\theta} = (R+r)\frac{r}{R}\dot{\phi} = \frac{r(R+r)}{R}\omega$$
$$R\dot{\theta} = r\dot{\phi}$$



where ϕ is the rotation angle of the ball

$$g(R+r)(1-\cos\theta)=[\frac{(R+r)^2}{R^2}+1]\frac{1}{2}r^2\omega^2 \qquad (1)$$

Newton's second law (centripetal acceleration)

$$mg\cos\theta - N = m\frac{{v_{cm}}^2}{R+r}=m\frac{1}{R+r}\frac{r^2(R+r)^2}{R^2}\omega^2 = mr^2\omega^2\frac{R+r}{R^2}$$

When $N = 0$,

$$g\cos\theta = r^2\omega^2\frac{R+r}{R^2} \qquad (2)$$

From Eqs.(1) and (2), we have

$$(R+r)^2(1-\cos\theta)=\frac{1}{2}[(R+r)^2+R^2]\cos\theta$$

or

$$\cos\theta=\frac{(R+r)^2}{\frac{3(R+r)^2}{2}+\frac{R^2}{2}}=\frac{2}{3+\frac{R^2}{(R+r)^2}}$$

or

$$y = (R + r)\cos\theta = \frac{2(R + r)}{3 + \frac{R^2}{(R + r)^2}}$$

13.23 Energy function

Consider a general Lagrangian

$$L = L(q_1, q_2, q_3, \ldots\ldots, q_n, \dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots\ldots, \dot{q}_n, t)$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t}$$

From the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}.$$

Then

$$\begin{aligned}\frac{dL}{dt} &= \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial t} \\ &= \sum_j \frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j\right] + \frac{\partial L}{\partial t}\end{aligned}$$

or

$$\frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L\right) + \frac{\partial L}{\partial t} = 0$$

We define the energy function given by

$$h(q,\dot{q},t) = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

where

$$\frac{dh}{dt} + \frac{\partial L}{\partial t} = 0 .$$

If the Lagrangian L is not an explicit function of t, then

$$\frac{dh}{dt} = 0 \rightarrow h \text{ is conserved}$$

For a conserved system, $V = V(q_1, q_2, ...q_n)$.

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

$$\begin{aligned} h(q,\dot{q},t) &= \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - L \\ &= 2T - (T - V) = T + V = E \end{aligned}$$

where E corresponds to the total energy of the system.

13.24 Hamiltonian *H*

The Hamiltonian is described by

$$H = H(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n, t) = \sum_j p_j \dot{q}_j - L ,$$

which then has to be rewritten by eliminating all the generalized velocities in favor of the generalized momenta

Normally, the Hamiltonian for each problem should be constructed via the Lagrangian formulation.

(1) Choose a set of generalized co-ordinate qi, and construct

$$L(q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t) .$$

(2) Define the conjugate momenta as a function of $q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t$.

$$p_i = p_i(q, \dot{q}) = p_i(q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ..., \dot{q}_n) = \frac{\partial L(q, \dot{q})}{\partial \dot{q}_i} .$$

(3) Use the energy function

$$h = h((q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t) = \sum_i p_i \dot{q}_i - L((q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t) .$$

and construct $h((q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)$.

(4) Obtain $\dot{q}_i$ as a function of $(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n, t)$.

(5) Construct $H(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n, t)$.

Note that The Hamiltonian *H* is constructed in the same manner as the energy function. But they are functions of different variables.

13.25 Hamilton's equation

$$dL = \sum_i (\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i) + \frac{\partial L}{\partial t} dt = \sum_i (\dot{p}_i dq_i + p_i d\dot{q}_i) + \frac{\partial L}{\partial t} dt$$

where

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad \dot{p}_i = \frac{\partial L}{\partial q_i}.$$

We define the Hamiltonian *H* as

$$H = H(q, p, t) = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t). \qquad \text{(Legendre transformation)}$$

$$\begin{aligned} dH &= \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - dL \\ &= \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - \frac{\partial L}{\partial t} dt - \sum_i (\dot{p}_i dq_i + p_i d\dot{q}_i) \\ &= \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt \end{aligned}$$

which means that *p* and *q* are independent variables.

Consider a function of *q*, *p*, and *t* only. Then we have

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i\right) + \frac{\partial H}{\partial t} dt$$

which is compared with

$$dH = \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt$$

Then we have canonical equations of Hamiltonian

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

13.26 Example

(1) Simple harmonics

$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$$

$$\frac{\partial L}{\partial \dot{q}} = p = m\dot{q}$$

$$\begin{aligned} h(q,\dot{q},t) &= p\dot{q} - L \\ &= m\dot{q}\dot{q} - (\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2) \\ &= \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2 \end{aligned}$$

Note that

$$\frac{dh}{dt} + \frac{\partial L}{\partial t} = 0. \qquad \text{and} \qquad \frac{\partial L}{\partial t} = 0$$

or

$$h = \text{const.}$$

Construction of the Hamiltonian:

$$\dot{q} = \frac{p}{m}$$

$$H(q,p,t) = \frac{1}{2}m\frac{p^2}{m^2} + \frac{1}{2}kq^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

where

$$k = m\omega^2$$

(2) Particle (mass m and charge q) in the presence of electromagnetic field

$$L = \frac{1}{2} m\mathbf{v}^2 - q\phi + q\mathbf{A}\cdot\mathbf{v}$$

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}$$

or

$$m\mathbf{v} = \mathbf{p} - q\mathbf{A}$$

$$\begin{aligned} H &= \mathbf{p}\cdot\mathbf{v} - L \\ &= (m\mathbf{v} + q\mathbf{A})\cdot\mathbf{v} - (\frac{1}{2} m\mathbf{v}^2 - q\phi + q\mathbf{A}\cdot\mathbf{v}) \end{aligned}$$

or

$$\begin{aligned} H &= \frac{1}{2} m\mathbf{v}^2 + q\phi \\ &= \frac{1}{2} m \frac{1}{m^2}(\mathbf{p} - q\mathbf{A})^2 + q\phi \\ &= \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi \end{aligned}$$

13.26 Derivation of Lorentz force from the Lagrangian

The Lagrangian for a particle with charge q in an electromagnetic field described by scalar potential ϕ and vector potential $\boldsymbol{A}$ is

$$L = \frac{1}{2} m\mathbf{v}^2 - q\phi + q\mathbf{A}\cdot\mathbf{v}$$

Find the equation of motion of the charged particle.

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

where

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad B = \nabla\times\mathbf{A}\,.$$

((**Mathematica**))

Derivation of Lorentz force

```
<< "VariationalMethods`"

Needs["VectorAnalysis`"]

SetCoordinates[Cartesian[x, y, z]];

r = {x[t], y[t], z[t]};

A = {A1[x[t], y[t], z[t]], A2[x[t], y[t], z[t]], A3[x[t], y[t], z[t]]};

v = {x'[t], y'[t], z'[t]};
```

$$eq1 = \frac{m}{2}\, v.v - q\ \phi[x[t], y[t], z[t]] + q\ A.v\ //\ Expand;$$

```
eq2 = VariationalD[eq1, {x[t], y[t], z[t]}, t] // Simplify
```

$$\{-m\, x''[t] - q\, (y'[t]\, (A1^{(0,1,0)}[x[t], y[t], z[t]] - A2^{(1,0,0)}[x[t], y[t], z[t]]) + z'[t]\, (A1^{(0,0,1)}[x[t], y[t], z[t]] - A3^{(1,0,0)}[x[t], y[t], z[t]]) + \phi^{(1,0,0)}[x[t], y[t], z[t]]),$$
$$-m\, y''[t] - q\, (z'[t]\, (A2^{(0,0,1)}[x[t], y[t], z[t]] - A3^{(0,1,0)}[x[t], y[t], z[t]]) + \phi^{(0,1,0)}[x[t], y[t], z[t]] + x'[t]\, (-A1^{(0,1,0)}[x[t], y[t], z[t]] + A2^{(1,0,0)}[x[t], y[t], z[t]])),$$
$$-m\, z''[t] - q\, (\phi^{(0,0,1)}[x[t], y[t], z[t]] + y'[t]\, (-A2^{(0,0,1)}[x[t], y[t], z[t]] + A3^{(0,1,0)}[x[t], y[t], z[t]]) + x'[t]\, (-A1^{(0,0,1)}[x[t], y[t], z[t]] + A3^{(1,0,0)}[x[t], y[t], z[t]]))\}$$

```
eq3 = FirstIntegrals[eq1, {x[t], y[t], z[t]}, t] // Simplify
```

$$\{FirstIntegral[t] \to \frac{1}{2}\, (2\, q\, \phi[x[t], y[t], z[t]] + m\, (x'[t]^2 + y'[t]^2 + z'[t]^2))\}$$

```
rule1 = {x → x[t], y → y[t], z → z[t]};

AA = {A1[x, y, z], A2[x, y, z], A3[x, y, z]};

B = Curl[AA] /. rule1;

Cross[v, B] // Simplify;

E1 = -Grad[ϕ[x, y, z]] /. rule1 // Simplify;

eq4 = -m D[r, {t, 2}] + q E1 + q Cross[v, B] // Simplify
```

$$\{-m\, x''[t] - q\, (y'[t]\, (A1^{(0,1,0)}[x[t], y[t], z[t]] - A2^{(1,0,0)}[x[t], y[t], z[t]]) + z'[t]\, (A1^{(0,0,1)}[x[t], y[t], z[t]] - A3^{(1,0,0)}[x[t], y[t], z[t]]) + \phi^{(1,0,0)}[x[t], y[t], z[t]]),$$
$$-m\, y''[t] - q\, (z'[t]\, (A2^{(0,0,1)}[x[t], y[t], z[t]] - A3^{(0,1,0)}[x[t], y[t], z[t]]) + \phi^{(0,1,0)}[x[t], y[t], z[t]] + x'[t]\, (-A1^{(0,1,0)}[x[t], y[t], z[t]] + A2^{(1,0,0)}[x[t], y[t], z[t]])),$$
$$-m\, z''[t] - q\, (\phi^{(0,0,1)}[x[t], y[t], z[t]] + y'[t]\, (-A2^{(0,0,1)}[x[t], y[t], z[t]] + A3^{(0,1,0)}[x[t], y[t], z[t]]) + x'[t]\, (-A1^{(0,0,1)}[x[t], y[t], z[t]] + A3^{(1,0,0)}[x[t], y[t], z[t]]))\}$$

```
eq2 - eq4 // Simplify

{0, 0, 0}
```

13.27 Relativistic-covariant Lagrangian formalism

A. Lagrangian *L* (simple case)

Proper time

$$(dx_\mu')^2 = a_{\mu\lambda}a_{\mu\sigma}dx_\lambda dx_\sigma = \delta_{\lambda\sigma}dx_\lambda dx_\sigma = (dx_\mu)^2$$

We define the proper time as

$$(ds)^2 = c^2(dt)^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2 = c^2(dt')^2 - (dx_1')^2 - (dx_2')^2 - (dx_3')^2$$

$$(ds)^2 = c^2(dt)^2\{1 - \frac{1}{c^2}[(\frac{dx_1}{dt})^2 + (\frac{dx_2}{dt})^2 + (\frac{dx_3}{dt})^2]\} = c^2(dt)^2(1 - \frac{\mathbf{u}^2}{c^2})$$

or

$$d\tau = \frac{ds}{c} = dt\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}$$

where τ is a proper time and $\boldsymbol{u}$ is the velocity of the particle in the frame *S*. The integral $\int_a^b ds$ taken between a given pair of world points has its maximum value if it is taken along the straight line joining two points.

$$S = -\alpha\int_a^b ds = -\alpha c\int_{t_a}^{t_b} dt\sqrt{1 - \frac{\mathbf{u}^2}{c^2}} = \int_{t_a}^{t_b} Ldt\ ,$$

where

$$L = -\alpha c\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}\ .$$

Nonrelativistic case:

$$L = -\alpha c(1 - \frac{\mathbf{u}^2}{c^2})^{1/2} = -\alpha c(1 - \frac{\mathbf{u}^2}{2c^2}) = \frac{\alpha}{2c}\mathbf{u}^2 - \alpha c\ .$$

In the classical mechanics,

$$\frac{\alpha}{2c} = \frac{m}{2} \qquad \text{or} \qquad \alpha = mc\,.$$

Therefore the Lagrangian L is given by

$$L = -mc^2 (1 - \frac{\mathbf{u}^2}{c^2})^{1/2}\,.$$

The momentum $\boldsymbol{p}$ is defined by

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}} = \frac{m\mathbf{u}}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} = m\mathbf{u}\gamma(\mathbf{u}) = m\frac{d\mathbf{r}}{d\tau} = m\frac{d\mathbf{r}}{dt}\frac{dt}{d\tau}\,.$$

((**Note**))

This momentum coincides with the components of four-vector momentum p_μ defined by

$$p_1 = m\frac{dx_1}{d\tau}$$
$$p_2 = m\frac{dx_2}{d\tau}$$
$$p_3 = m\frac{dx_3}{d\tau}$$
$$p_4 = m\frac{dx_4}{d\tau}$$

B. Hamiltonian

The Hamiltonian H is defined by

$$H = \mathbf{p}\cdot\mathbf{u} - L = \gamma(\mathbf{u})m\mathbf{u}^2 + mc^2\frac{1}{\gamma(\mathbf{u})} = \frac{\gamma(\mathbf{u})^2 m\mathbf{u}^2 + mc^2}{\gamma(\mathbf{u})} = \gamma(\mathbf{u})mc^2 = \frac{mc^2}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} = E\,,$$

or

$$E = \frac{mc^2}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}}\,.$$

We have

$$\frac{E^2}{c^2} = \frac{m^2c^2}{1-\frac{\mathbf{u}^2}{c^2}} = \frac{m^2c^2(1-\frac{\mathbf{u}^2}{c^2}) + m^2\mathbf{u}^2}{1-\frac{\mathbf{u}^2}{c^2}} = m^2c^2 + \mathbf{p}^2 .$$

C. Lagrangian form in the presence of an electromagnetic field

The action function for a charge in an electromagnetic field

$$S = \int_a^b (-mcds + qA_\mu dx_\mu) ,$$

where the second term is invariant under the Lorentz transformation.

$$A_\mu = (\mathbf{A}, i\frac{1}{c}\phi) , \qquad \text{and} \qquad dx_\mu = (dx_1, dx_2, dx_3, icdt) .$$

Then we have

$$S = \int_a^b (-mcds + qA_\mu dx_\mu) = \int_a^b [-mc^2\sqrt{1-\frac{\mathbf{u}^2}{c^2}} + q(\mathbf{A}\cdot\mathbf{u} - \phi)]dt .$$

The integrand in the Lagrangian function of a charge (q) in the electromagnetic field,

$$L = -mc^2\sqrt{1-\frac{\mathbf{u}^2}{c^2}} + q(\mathbf{A}\cdot\mathbf{u} - \phi) ,$$

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{u}} = \frac{m\mathbf{u}}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} + q\mathbf{A} ,$$

where

$$A_\mu = (\mathbf{A}, i\frac{1}{c}\phi) .$$

The Hamiltonian H is given by

$$H = \mathbf{p}\cdot\mathbf{u} - L = \frac{m\mathbf{u}^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} + e\mathbf{A}\cdot\mathbf{u} - (-mc^2\sqrt{1-\frac{\mathbf{u}^2}{c^2}} + q\mathbf{A}\cdot\mathbf{u} - q\phi),$$

or

$$H = \mathbf{p}\cdot\mathbf{u} - L = \frac{m\mathbf{u}^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} + e\mathbf{A}\cdot\mathbf{u} - (-mc^2\sqrt{1-\frac{\mathbf{u}^2}{c^2}} + q\mathbf{A}\cdot\mathbf{u} - q\phi),$$

$$H = \frac{mc^2}{\sqrt{1-\frac{\mathbf{u}^2}{c^2}}} + q\phi,$$

or

$$\left(\frac{H-q\phi}{c}\right)^2 = \frac{m^2c^2(1-\frac{\mathbf{u}^2}{c^2}) + m^2\mathbf{u}^2}{1-\frac{\mathbf{u}^2}{c^2}} = m^2c^2 + (\mathbf{p}-q\mathbf{A})^2.$$

D. Another expression for the Lagrangian

Here we use $d\tau$ instead of dt in the expression of Lagrangian.

$$ds = cd\tau$$

η_μ is a four-dimensional velocity defined by

$$\eta_\mu = \frac{dx_\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx_\mu}{dt} = (\gamma(\mathbf{u})u_1, \gamma(\mathbf{u})u_2, \gamma(\mathbf{u})u_3, ic\gamma(\mathbf{u})),$$

$$A_\mu\eta_\mu = A_1\eta_1 + A_2\eta_2 + A_3\eta_3 + A_4\eta_4 = \gamma(\mathbf{u})(\mathbf{u}\cdot\mathbf{A} - \phi).$$

since

$$A_\mu = (\mathbf{A}, i\frac{1}{c}\phi), \qquad \eta_4 = \frac{dt}{d\tau}\frac{dx_4}{dt} = ic\frac{dt}{d\tau},$$

$$S = \int_a^b (-mcds + qA_\mu dx_\mu) = \int_a^b (-mc^2 + qA_\mu\cdot\eta_\mu)d\tau,$$

$$L = -mc^2 + qA_\mu \eta_\mu .$$

E. Lagrangian and Hamiltonian

$$F_{\mu\nu}F_{\mu\nu} = 2(B_1{}^2 + B_2{}^2 + B_3{}^2) - \frac{2}{c^2}(E_1{}^2 + E_2{}^2 + E_3{}^2) .$$

This is invariant under the Lorentz transformation. We may try the Lagrangian density

$$L = -\frac{1}{4\mu_0}F_{\mu\nu}F_{\mu\nu} + J_\mu A_\mu .$$

By regarding each component of A_μ as an independent field, we find that
the Lagrange equation

$$\frac{\partial L}{\partial A_\mu} = \frac{\partial}{\partial x_\nu}[\frac{\partial L}{\partial(\frac{\partial A_\mu}{\partial x_\nu})}] ,$$

is equivalent to

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = \mu_0 J_\mu .$$

The Hamiltonian density H_{em} for the free Maxwell field can be evaluated as follows.

$$L_{em} = -\frac{1}{4\mu_0}F_{\mu\nu}F_{\mu\nu} ,$$

$$H_{em} = \frac{\partial L_{em}}{\partial\left(\frac{\partial A_\mu}{\partial x_4}\right)}\frac{\partial A_\mu}{\partial x_4} - L_{em} = -\frac{F_{4\mu}}{\mu_0}(F_{4\mu} + \frac{\partial A_4}{\partial x_\mu}) - \frac{1}{2\mu_0}(\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2) ,$$

or

$$H_{em} = \frac{1}{2}\varepsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2 - \varepsilon_0\mathbf{E}\cdot\nabla\phi ,$$

$$\int H_{em}d\mathbf{r} = \frac{1}{2}\int(\varepsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2)d\mathbf{r} - \int\varepsilon_0(\mathbf{E}\cdot\nabla\phi)d\mathbf{r} = \frac{1}{2}\int(\varepsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2)dr .$$

((**Note**))

$$\int (\mathbf{E}\cdot\nabla\phi)d\mathbf{r} = \int [\nabla\cdot(\mathbf{E}\phi) - \phi\nabla\cdot\mathbf{E}]d\mathbf{r} = \int \nabla\cdot(\mathbf{E}\phi)d\mathbf{r} = \int (\mathbf{E}\phi)\cdot d\mathbf{a} = 0\,,$$

where $\mathbf{E}\phi$ vanishes sufficiently rapidly at infinity.

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} = 0 \text{ (in this case).}$$
