Chapter 13S Symmetric top

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Lagrange's equation
Euler's angles
First integral
Symmetric top
Precession
Nutation

13S1 Angular velocity

This part is already discussed in Chapter 1S.

We derive the angular velocity in the new coordinate (x', y', z'). The rotation matrix is given by

$$\Re(t) = \Re_z(-\psi(t))\Re_x(-\theta(t))\Re_z(-\phi(t))$$

where the Euler angles are dependent on time t.

$$\mathbf{\Omega}_\phi = \Re(t)\begin{pmatrix} 0 \\ 0 \\ \dot{\phi}(t) \end{pmatrix}_{x,y,z} = \begin{pmatrix} \sin\theta(t)\sin\psi(t)\dot{\phi}(t) \\ \sin\theta(t)\cos\psi(t)\dot{\phi}(t) \\ \cos\theta(t)\dot{\phi}(t) \end{pmatrix}_{x',y',z'}$$

where $\dot{\phi}(t)$ is directed along the z axis.

$$\mathbf{\Omega}_\theta = \Re_x(-\theta(t))\Re_z(-\phi(t)\begin{pmatrix} \dot{\theta}(t) \\ 0 \\ 0 \end{pmatrix}_{\xi,\eta,\varsigma} = \begin{pmatrix} \cos\psi(t)\dot{\theta}(t) \\ -\sin\psi(t)\dot{\theta}(t) \\ 0 \end{pmatrix}_{x',y',z'}$$

where $\dot{\theta}(t)$ is directed along the ξ axis.

$$\boldsymbol{\Omega}_{\psi} = \Re_z(-\psi(t))\begin{pmatrix} 0 \\ 0 \\ \dot{\psi}(t) \end{pmatrix}_{\xi',\eta',\varsigma'} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}(t) \end{pmatrix}_{x',y,z'}$$

where $\dot{\psi}(t)$ is directed along the ζ' axis. Then we have the angular velocity in the new coordiante (x', y', z') as

$$\boldsymbol{\Omega}_{x'y'z'} = \boldsymbol{\Omega}_{\phi} + \boldsymbol{\Omega}_{\theta} + \boldsymbol{\Omega}_{\psi}$$
$$= \begin{pmatrix} \Omega_{x'} \\ \Omega_{y'} \\ \Omega_{z'} \end{pmatrix} = \begin{pmatrix} \cos\psi(t)\dot{\theta}(t) + \sin\theta(t)\sin\psi(t)\dot{\phi}(t) \\ -\sin\psi(t)\dot{\theta}(t) + \sin\theta(t)\cos\psi(t)\dot{\phi}(t) \\ \cos\theta(t)\dot{\phi}(t) + \dot{\psi}(t) \end{pmatrix}_{x',y',z'}$$

The angular velocity in the original (x, y, z) coordinate is obtained as

$$\boldsymbol{\Omega}_{x,y,z} = \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix}$$
$$= \Re^{-1}(t)\boldsymbol{\Omega}_{x'y'z'} = \Re^{T}(t)\boldsymbol{\Omega}_{x'y'z'}$$
$$= \begin{pmatrix} \cos\phi(t)\dot{\theta}(t) + \sin\theta(t)\sin\phi(t)\dot{\psi}(t) \\ \sin\phi(t)\dot{\theta}(t) - \sin\theta(t)\cos\phi(t)\dot{\psi}(t) \\ \dot{\phi}(t) + \cos\theta(t)\dot{\psi}(t) \end{pmatrix}_{x,y,z}$$

13S.2 Kinetic energy

The kinetic energy is given by

$$T = T_1 + T_3$$

with

$$T_1 = \frac{1}{2}I_1(\Omega_{1x'}{}^2 + \Omega_{1y'}{}^2) = \frac{1}{2}I_1(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

and

$$T_3 = \frac{1}{2} I_3 \Omega_{1z'}{}^2 = \frac{1}{2} I_3 (\cos\theta\dot{\phi} + \dot{\psi})^2 .$$

What is the potential energy? The center of mass of the symmetrical top is l from the bottom. In other words, we have

$$\mathbf{r}' = \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix}_{x',y',z'}$$

This vector r' is transformed to the position vector r in the original (x, y, z) coordinate system. Using the rotation matrix

$$\Re = \Re_z(-\psi)\Re_x(-\theta)\Re_z(-\phi) =$$

$$= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\theta\sin\phi\cos\psi - \cos\phi\sin\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{pmatrix}$$

we have

$$\mathbf{r} = \Re^{-1} \begin{pmatrix} 0 \\ 0 \\ l \end{pmatrix} = \begin{pmatrix} l\sin\theta\sin\phi \\ -l\cos\phi\sin\theta \\ l\cos\theta \end{pmatrix}_{x,y,z}$$

Then the potential energy V is given by

$$V = mgz = mgl\cos\theta .$$

13S.3 Lagrange's equation

The Lagrangian L is obtained as

$$L = T - V = \frac{1}{2} I_1(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{2} I_3(\cos\theta\dot{\phi} + \dot{\psi})^2 - mgl\cos\theta$$

((**The Lagrange's equation**))

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}:$$

$$\sin\theta[mgl + (I_1 - I_3)\dot{\phi}^2 \cos\theta] = I_3\dot{\phi}\dot{\psi}\sin\theta + I_1\ddot{\theta}.$$

((**First integral**)):

We note that ϕ and ψ are cyclic. In other words, L is independent of ϕ and ψ. So the corresponding angular momenta are constant

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_3 \cos^2\theta + I_1 \sin^2\theta)\dot{\phi} + I_3\dot{\psi}\cos\theta \qquad \text{for } \phi.$$

$$P_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = I_3\Omega_{z'} \qquad \text{for } \psi$$

Energy conservation:

$$E = mgl\cos\theta + \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\phi}^2\cos^2\theta + 2\dot{\theta}\dot{\psi}\cos\theta) + \frac{1}{2}I_3\dot{\psi}^2$$

13S.3 Solution

Here we put

$$P_\psi = I_3\omega_3 = I_1 a\,, \qquad P_\phi = I_1 b$$

where a, b, and ω_3 are constants. From the first integral, we have

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$

$$\dot{\psi} = a\frac{I_1}{I_3} - \frac{(b - a\cos\theta)\cos\theta}{\sin^2\theta}$$

From the Lagrange's equation and the first integral, we get

$$I_1\ddot{\theta} = I_1(a^2+b^2)\frac{\cos\theta}{\sin^3\theta} - I_1 ab\frac{3+\cos(2\theta)}{2\sin^3\theta} + mgl\sin\theta$$

13.S4 Effective potential energy V_{eff}

The energy conservation law can be rewritten as

$$\begin{aligned} E_1 &= mgl\cos\theta + \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) \\ &= mgl\cos\theta + \frac{1}{2}I_1[\dot{\theta}^2 + (\frac{b-a\cos\theta}{\sin\theta})^2] \end{aligned},$$

where

$$E_1 = E - \frac{a^2 I_1{}^2}{2I_3} = E - \frac{1}{2}I_3\omega_3{}^2,$$

$$\alpha = \frac{2E}{I_1} - \frac{a^2 I_1}{I_3} = \frac{2(E - \frac{a^2 I_1{}^2}{2I_3})}{I_1} = \frac{2E_1}{I_1}, \qquad \beta = \frac{2mgl}{I_1}.$$

$$a = \frac{P_\psi}{I_1} = \frac{I_3}{I_1}\omega_3, \qquad b = \frac{P_\phi}{I_1}$$

Here E is the total energy (constant) and E_1 is also constant. Then we get

$$\begin{aligned} E_1 &= \frac{I_1}{2}\dot{\theta}^2 + \frac{I_1}{2}(\frac{b-a\cos\theta}{\sin\theta})^2 + mgl\cos\theta \\ &= \frac{I_1}{2}\dot{\theta}^2 + V_{eff}(\theta) \end{aligned}$$

where $V_{\mathrm{eff}}(\theta)$ is an effective potential, given by

$$V_{eff}(\theta) = \frac{I_1}{2}(\frac{b-a\cos\theta}{\sin\theta})^2 + mgl\cos\theta.$$

The above equation is similar to that for the motion of a particle in a central-force field. The figure shown below indicates that for the value of E_1 the motion is limited by two extreme values of θ_1 and θ_2, which correspond to the turning points of the central-force problem. For $E_1 = E_0$, θ is limited to the single value θ_0. The motion is a steady precession at a fixed angle of inclination (θ_0). The condition that $V_{eff}(\theta)$ has a local minimum at $\theta = \theta_0$, is obtained from

$$\frac{\partial V_{eff}}{\partial \theta}|_{\theta=\theta_0} = 0,$$

or

$$(-b + a\cos\theta_0)(-a + b\cos\theta_0) - \frac{mg}{I_1} L \sin^4\theta_0 = 0$$

or

$$(-b + a\cos\theta_0)(-a + b\cos\theta_0) - \frac{\beta}{2}\sin^4\theta_0 = 0$$

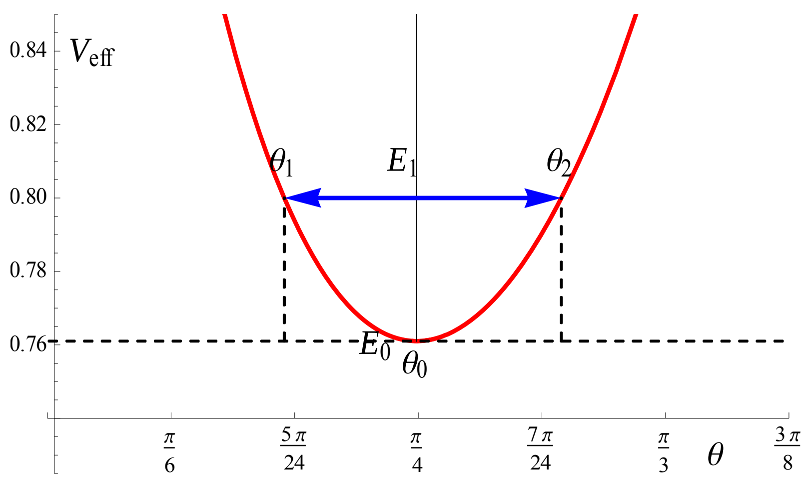


Fig. The plot of V_{eff} as a function of θ. $I_1 = 1$. $m = 1$. $g = 1$. $L = 1$. $a = 2.5$. $b = 2$. Veff has a local minimum at $\theta = \theta_0$.

13.S5 Summary

(1) Energy conservation law:

$$\dot{\theta}^2 \sin^2\theta = (\frac{2E}{I_1} - \frac{a^2 I_1}{I_3} - \frac{2mgl}{I_1}\cos\theta)\sin^2\theta - (b - a\cos\theta)^2$$

or

$$\dot{\theta}^2 \sin^2\theta = (\alpha - \beta\cos\theta)\sin^2\theta - (b - a\cos\theta)^2 \qquad (1)$$

When $u = \cos\theta$, the energy conservation law can be rewritten as

$$\dot{u}^2 = f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2 .$$

The formal solution of the above equation is obtained as

$$t - t_0 = \int_{u_0}^{u} \frac{du}{\sqrt{f(u)}} = \int_{u_0}^{u} \frac{du}{\sqrt{(1-u^2)(\alpha - \beta u) - (b - au)}}$$

(2) Equations of motion:

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} = \frac{b - au}{1-u^2}, \qquad (2)$$

$$\dot{\psi} = a\frac{I_1}{I_3} - \frac{(b - a\cos\theta)\cos\theta}{\sin^2\theta} = a\frac{I_1}{I_3} - \frac{(b-au)u}{1-u^2}, \qquad (3)$$

$$\ddot{\theta} = (a^2 + b^2)\frac{\cos\theta}{\sin^3\theta} - ab\frac{3+\cos(2\theta)}{2\sin^3\theta} + \frac{\beta}{2}\sin\theta . \qquad (4)$$

The equations (2), (3), and (4) will be solved numerically by using the Mathematica. We do not use Eq.(1) for the solution of the problem here.

13.S6 Characteristic motion

(a) Roots of $f(u) = 0$

We are interested in the roots of

$$f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2 = 0$$

Since β>0, the solution must go to positive infinity for $u \to \infty$, and to negative infinity for u $\to -\infty$. At the physical limits ($u = \pm 1$),

$$f(u = \pm 1) = -(b - au)^2 \leq 0$$

So these conditions constrain the functional form of the soluition f(u) = 0 to the three roots

$$-1 \leq u_1 \leq u_2 \leq 1 \leq u_3$$

The physical motion is bounded to the range $u_1 \leq u \leq u_2$.

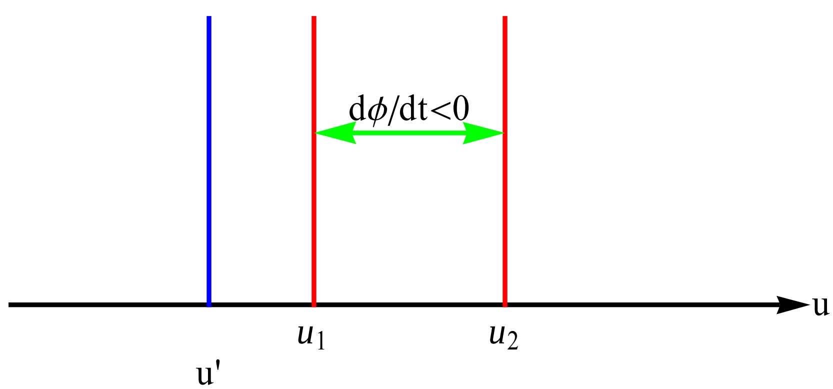
(b) Precession with nutation

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} = \frac{b - au}{1-u^2}$$

The precession $\phi(t)$ reverses direction when $\dot{\phi} = 0$. This corresponds to the turning point at

$$u' = \frac{b}{a}.$$

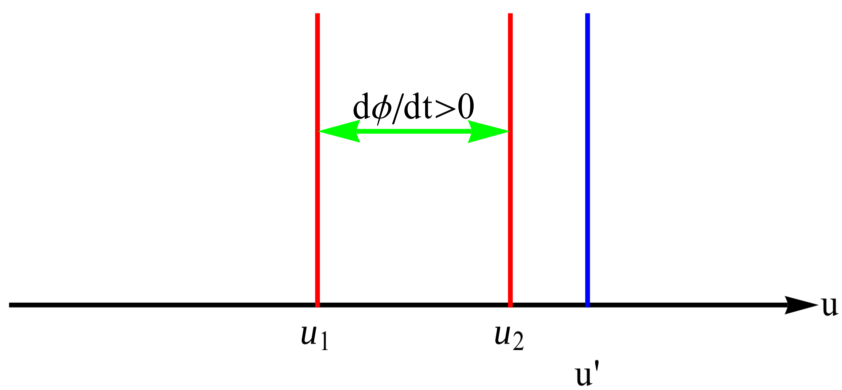
(i) $u' < u_1 < u_2$



$\dot{\phi} < 0$

ϕ monotonically increases with time. The turning point is not in the allowed region ($u_1<u<u_2$).

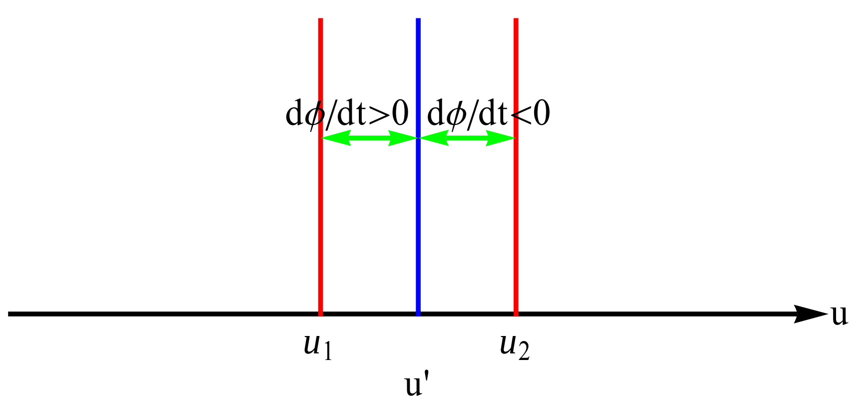
(ii) $u_1< u_2<u'$



$\dot{\phi} > 0$

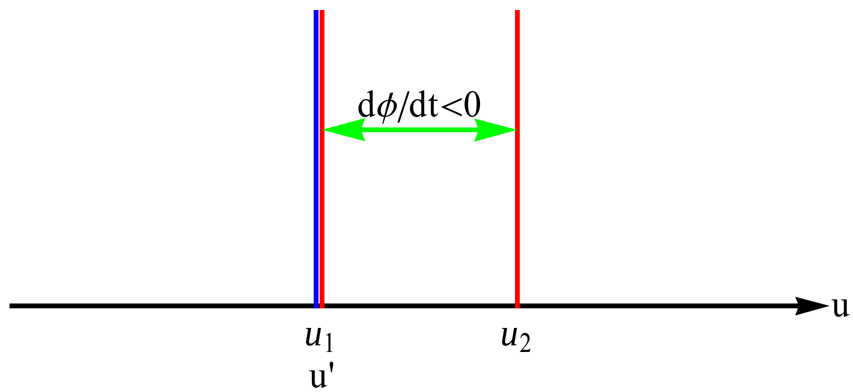
ϕ monotonically increases with time. The turning point is not in the allowed region ($u_1<u<u_2$).

(iii) $u_1<u'<u_2$



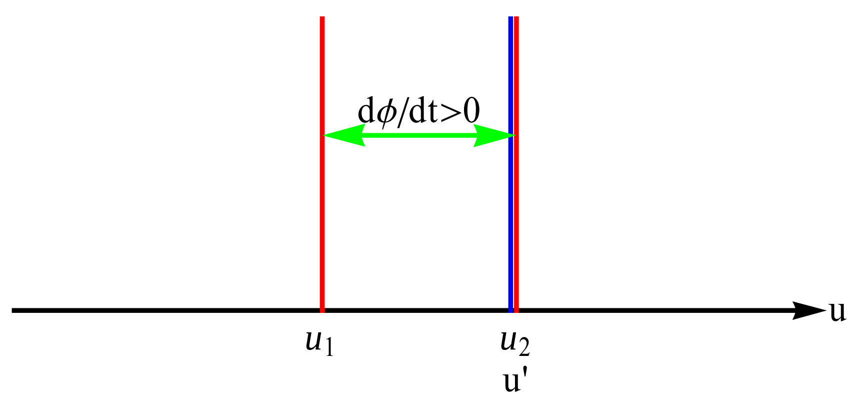
ϕ reverses the direction. $\dot{\phi} > 0$ for $u < u'$. $\dot{\phi} < 0$ for $u > u'$.

(iv) $u' = u_1 < u_2$



If the turning point is at $u' = u_1$, one get a cusp.

(v) $u_1 < u_2 = u'$



If the turning point is at $u' = u_2$, one get a cusp.

13.S6 Initial conditions

Suppose that the top is set spinning about its symmetry axis and released with zero intial precession and nutation;

$$\theta(t=0) = \theta_0 \qquad (u = u_2 = \cos\theta_0)$$

$\dot{\theta}(t=0)=0$, $\dot{\phi}(t=0)=0$, $\dot{\psi}(t=0)=\omega_3$ (initial conditions)

From the energy conservation law, we have

$$E_1 = mgl\cos\theta_0 = E - \frac{1}{2}I_3\omega_3^{\;2}$$

or

$$E = mgl\cos\theta_0 + \frac{1}{2}I_3\omega_3^{\;2}$$

From the two relations

$$\sin^2\theta\dot{\theta}^2 = \dot{u}^2 = f(u) = (1-u^2)(\alpha-\beta u) - (b-au)^2,$$

and

$$\dot{\phi} = \frac{b-au}{1-u^2},$$

we have

$$u(0) = \cos\theta_0 = u_2 = u' = \frac{b}{a},$$

which leads to the cusp motion ($u_1<u_2 = u'$). Since u_2 is one of the roots in $f(u) = 0$

$$f(u_2) = (1-u_2^{\;2})(\alpha-\beta u_2) - (b-au_2)^2 = (1-u_2^{\;2})(\alpha-\beta u_2) = 0$$

or

$$u_2 = u' = \frac{\alpha}{\beta} = \frac{b}{a}$$

Note that

$$\frac{\alpha}{\beta} = \frac{E_1}{mgl} = \cos\theta_0 = u_2 = u' = u_0$$

13.S7 Fast top

In the above case, the total energy E is given by

$$E = mgl\cos\theta_0 + \frac{1}{2}I_3\omega_3^{\ 2}.$$

Here we assume that the initial kinetic energy of rotation about the z axis is assumed large compared to the maximum change in the potential energy.

$$\frac{1}{2}I_3\omega_3^{\ 2} >> 2mgl$$

What is the other root u_1?

$$\begin{aligned} f(u) &= (\alpha - \beta u)(1-u^2) - (b-au)^2 \\ &= \beta(\frac{\alpha}{\beta} - u)(1-u^2) - a^2(\frac{b}{a} - u)^2 \\ &= \beta(u_0 - u)[(1-u^2) - a^2(u_0 - u)^2 \\ &= \beta(u_0 - u)[(1-u^2) - \frac{a^2}{\beta}(u_0 - u)] \end{aligned}$$

Then u_1 is the root of the quadratic equation

$$(1-u_1^{\ 2}) - \frac{a^2}{\beta}(u_0 - u_1) = 0$$

We put

$$u_0 - u_1 = x.$$

Then we have

$$x^2 + px - q = 0$$

with

$$p = -2u_0 + \frac{a^2}{\beta} \approx \frac{a^2}{\beta} > 0, \qquad q = 1 - u_0{}^2 = \sin^2\theta_0 > 0$$

We note that

$$\frac{a^2}{\beta} = \frac{I_3}{I_1}\frac{\frac{1}{2}I_3\omega_3{}^2}{mgl} >> 2.$$

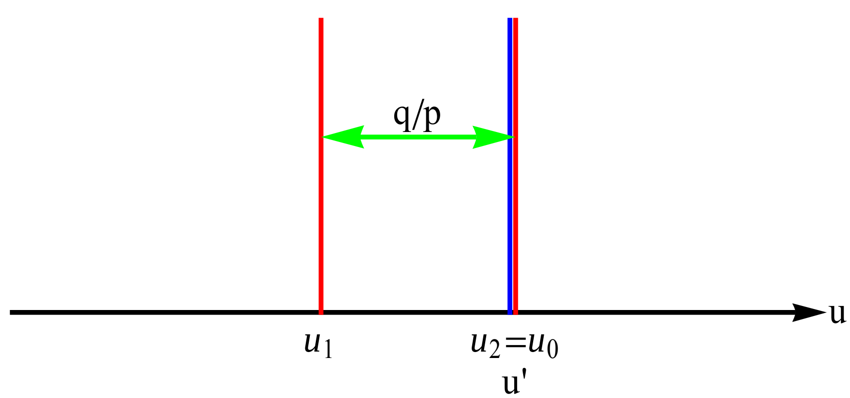
The solution of the quadratic equation is given by

$$x_1 = \frac{q}{p} = \frac{\beta \sin^2\theta_0}{a^2} \quad \text{or} \qquad x_3 = -p - \frac{q}{p},$$

where $p^2 >> 4q$. Since $0<x_1<1$ and $x_3<-1$, we have

$$u_1 = u_0 - \frac{q}{p} = u_0 - \frac{I_1}{I_3}\frac{2mgl}{I_3\omega_3{}^2}\sin^2\theta_0$$

The extent of the nutation, as measured by $u_0 - u_1$, goes down as $1/\omega_3{}^2$. The faster the top is spun, the less is the nutation.



13.S.7 Angular frequency of fast top

$$\begin{aligned}\dot{u}^2 &= f(u) = \beta(u_0 - u)[(1-u^2) - \frac{a^2}{\beta}(u_0 - u)] \\ &\approx \beta(u_0 - u)[\sin^2\theta_0 - \frac{a^2}{\beta}(u_0 - u)] \\ &= (u_0 - u)a^2 x_1 - a^2(u_0 - u)^2 \\ &= a^2 x(x_1 - x)\end{aligned}$$

or

$$\dot{x}^2 = a^2 x(x_1 - x)$$

where $x = u_0 - u$ and the initial condition is $x(0) = 0$. The solution for $x(t)$ is given by

$$x = x_1 \sin^2\frac{at}{2} = \frac{x_1}{2}(1 - \cos at)$$

The angular frequency of nutation is

$$a = \frac{P_\psi}{I_1} = \frac{I_3}{I_1}\omega_3$$

((**Mathematica**))

```
eq1 = x'[t]^2 == a^2 x[t] (x1 - x[t]);
eq2 =
 DSolve[{eq1, x[0] == 0}, x[t], t] // Simplify[#, a > 0] & //
  FullSimplify;
x[t_] = x[t] /. eq2[[1]]
```

$$x1\,\mathrm{Sin}\left[\frac{a\,t}{2}\right]^2$$

13S.8 Precession of fast top

$$\dot{\phi} = \frac{b-au}{1-u^2} = \frac{a(\frac{b}{a}-u)}{\sin^2\theta} \approx \frac{a(u_0-u)}{\sin^2\theta_0} \approx \frac{ax}{\sin^2\theta_0} = \frac{ax_1}{2\sin^2\theta_0}(1-\cos at)$$

or

$$\dot{\phi} = \frac{\beta}{2a}(1-\cos at).$$

The average angular frequency for the precession is

$$<\dot{\phi}> = \frac{\beta}{2a} = \frac{mgl}{I_3\omega_3}.$$

13S.9 True regular precession

What is the condition for the regular precession without any nutation? In this case, the angle θ remains constant; $\ddot{\theta} = \dot{\theta} = 0$, and $\theta = \theta_0$. This condition is equivalent to the condition that $f(u)$ has double roots; $u_1 = u_2$.

We return to the Lagrange's equation,

$$\sin\theta[mgl + (I_1 - I_3)\dot{\phi}^2\cos\theta] = I_3\dot{\phi}\dot{\psi}\sin\theta + I_1\ddot{\theta}$$

When $\ddot{\theta} = 0$, we have

$$(I_1 - I_3)\dot{\phi}^2\cos\theta_0 - I_3\dot{\phi}\dot{\psi} + mgl = 0, \qquad (1)$$

or

$$mgl = \dot{\phi}[I_3\dot{\psi} - (I_1 - I_3)\dot{\phi}\cos\theta_0]. \qquad (2)$$

Equation (1) is a quadratic equation for $\dot{\phi}$. The discriminant should be positive,

$$D = (I_3\dot{\psi})^2 - 4mgl\cos\theta_0(I_1 - I_3) \geq 0$$

It is evident that form Eq.(2), $\dot{\phi} = 0$ is not a solution. From Eq.(1), there are two roots for $\dot{\phi}$, fast precession (large $\dot{\phi}$) and slow precession (small $\dot{\phi}$).

For the slow precession (small $\dot{\phi}$), we have

$$mgl = \dot{\phi}[I_3\dot{\psi} - (I_1 - I_3)\dot{\phi}\cos\theta_0] \approx I_3\dot{\phi}\dot{\psi}$$

or

$$\dot{\phi} \approx \frac{mgl}{I_3\dot{\psi}} \approx \frac{mgl}{I_3\omega_3} = \frac{\beta}{2a}. \qquad \text{(slow)}.$$

For the fast precession (large $\dot{\phi}$), we have

$$I_3\dot{\psi} \approx (I_1 - I_3)\dot{\phi}\cos\theta_0 \approx I_1\dot{\phi}\cos\theta_0$$

or

$$\dot{\phi} \approx \frac{I_3\omega_3}{I_1\cos\theta_0} \qquad \text{(fast)}$$

13S.10 **$u = 1$.**

A top is set spinning with its figure axis initially vertical; $\theta = 0$ and $\dot{\theta} = 0$ at $t = 0$.

$$\sin^2\theta\dot{\phi} = b - a\cos\theta = 0$$

When $\theta = 0$ at $t = 0$, we have $u_2 = 1$,

$$a = b.$$

The energy E_1 is given by

$$E_1 = E - \frac{1}{2}I_3\omega_3^2 = mgl$$

$$\alpha = \frac{2E_1}{I_1}, \qquad \beta = \frac{2mgl}{I_1}.$$

Then we have

$$\alpha = \beta.$$

and

$$\begin{aligned}\dot{u}^2 &= f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2 \\ &= (1-u^2)\beta(1-u) - a^2(1-u)^2 \\ &= (1-u)^2[\beta(1+u) - a^2]\end{aligned}$$

Then we have

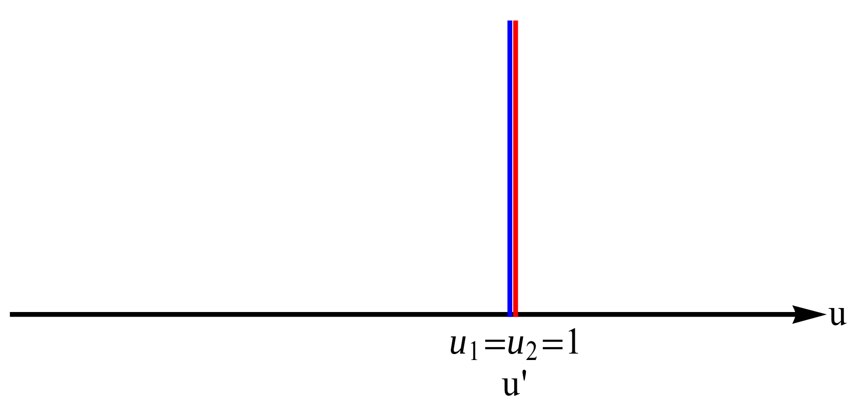
$$u_1 = u_2 = 1$$

The third root is

$$u_3 = \frac{a^2}{\beta} - 1.$$

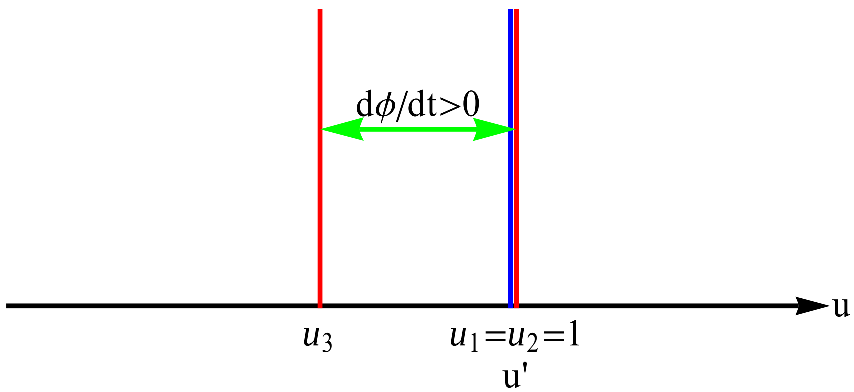
(i) For $\frac{a^2}{\beta} > 2$, we have $u_3>1$.

The top continues to spin about the vertical.



(ii) For $\frac{a^2}{\beta} < 2$, we have u_3<1.

The top will nutate between $\theta = 0$ and $\theta = \theta_3$.



The critical angular velocity, ω_{3c}, above which only vertical motion is possible, is given by

$$\omega_{3c} = \frac{4mglI1}{I_3^{\ 2}}$$

13.S 11 Differential equations for symmetric top

We solve the following differential equations

$$\ddot{\theta} = (a^2 + b^2)\frac{\cos\theta}{\sin^3\theta} - ab\frac{3 + \cos(2\theta)}{2\sin^3\theta} + \frac{\beta}{2}\sin\theta. \qquad (1)$$

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} = \frac{b - au}{1 - u^2}, \qquad (2)$$

$$\dot{\psi} = a\frac{I_1}{I_3} - \frac{(b - a\cos\theta)\cos\theta}{\sin^2\theta} = a\frac{I_1}{I_3} - \frac{(b - au)u}{1 - u^2}, \qquad (3)$$

with the initial conditions given by

$\theta'(0) = 0$. $\phi(0) = 0°$. $\psi(0) = 0°$.

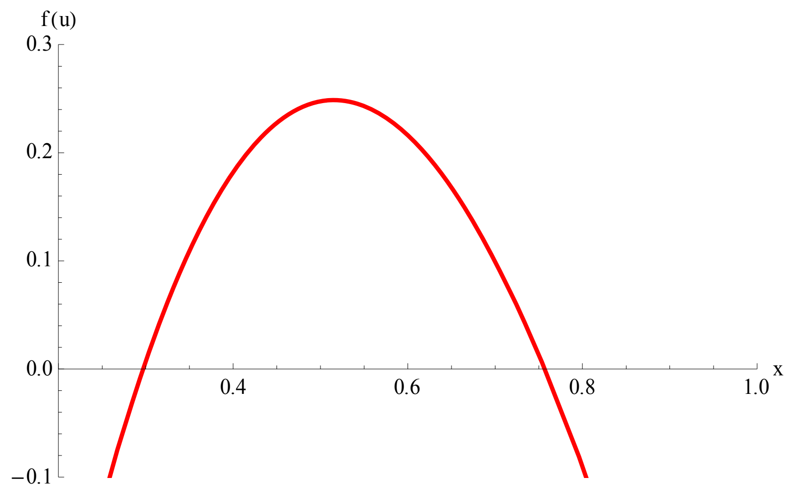
$\theta(0)$ is changed as a parameter.

13S.12 Numerical simulation -1

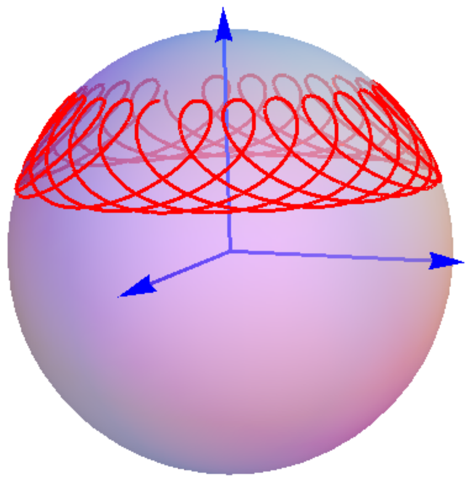
$\alpha = 1.6.$ $\beta = 2.0.$ $a = 2.5.$ $b = 1.7.$

Note that since b/a = 1.7/2.5=0.68, there is an angle θ satisfying $u' = \cos\theta = 0.68$ (θ= 47.16º). The values of u where $f(u) = 0$, are given by

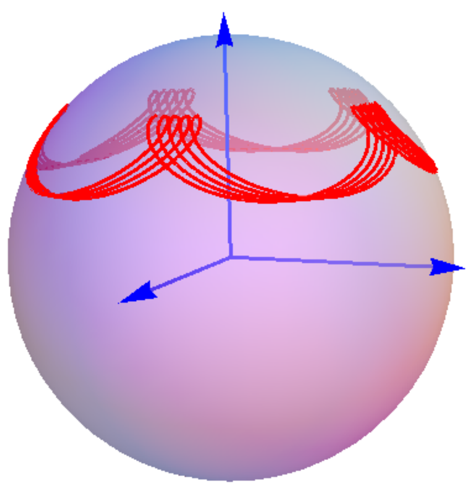
$u_1 = 0.29068$ ($\theta_1 = 73.10°$) u_2=0.757826 ($\theta_2 = 40.7278°$)



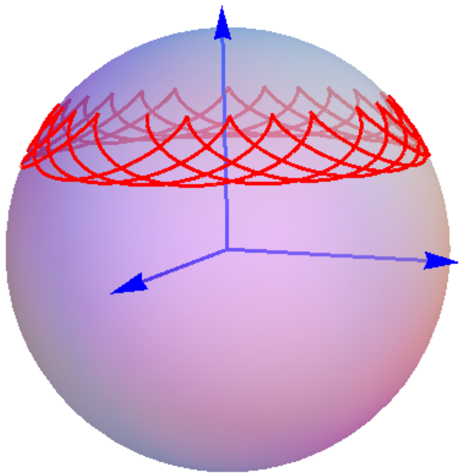
$\theta(0)=42°$



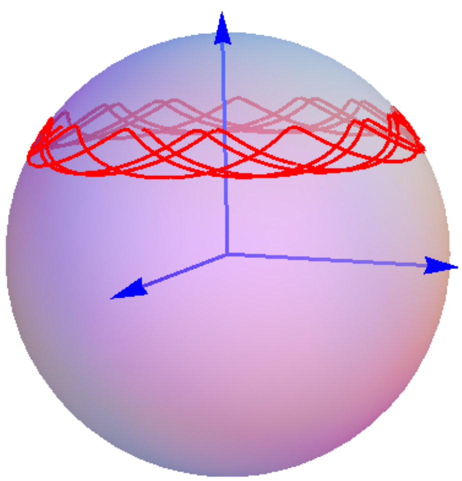
$\theta(0)=45°$



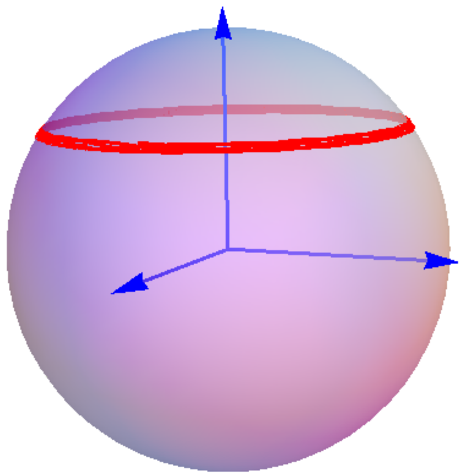
$\theta(0)=47.16°$



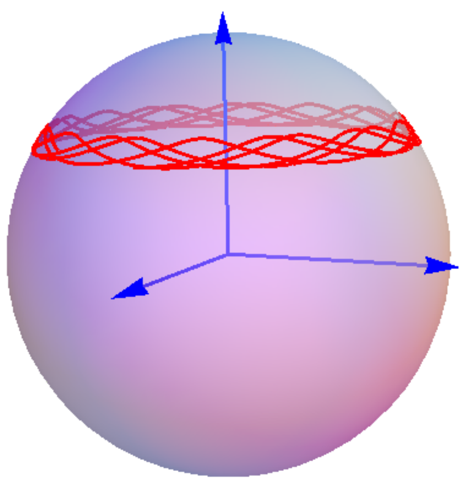
$\theta(0)=50°$



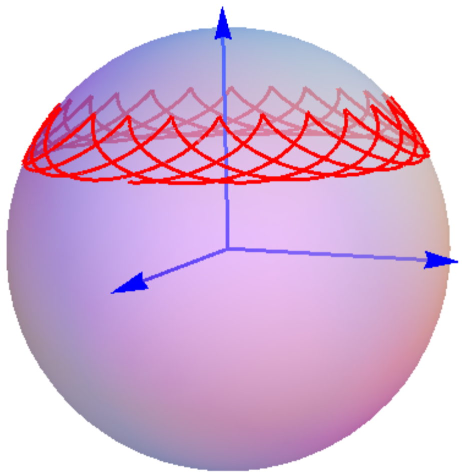
$\theta(0)=55°$



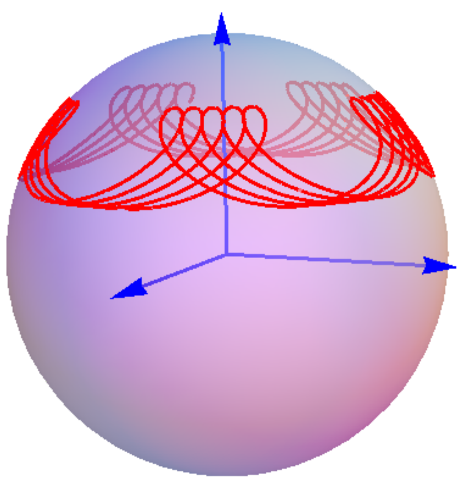
$\theta(0)=60°$

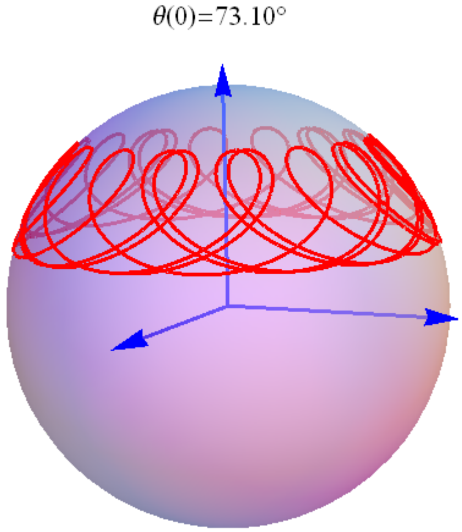


$\theta(0)=65°$



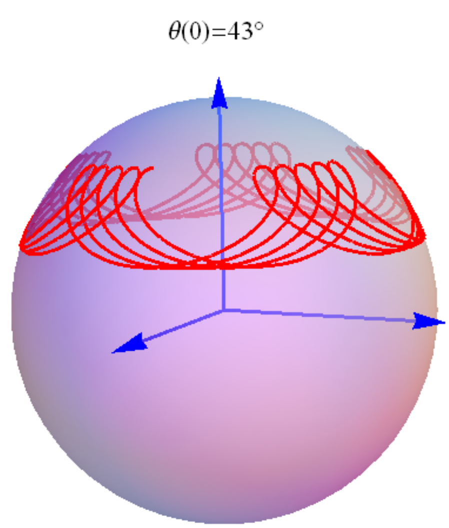
$\theta(0)=70°$





((**Mathematica**))

```
Clear["Global`*"]

Firsteq = θ''[t] == (a^2 + b^2) Cot[θ[t]] Csc[θ[t]]^2 - 1/2 a b (3 + Cos[2 θ[t]]) Csc[θ[t]]^3 + β/2 Sin[θ[t]];

Secondeq = ϕ'[t] == Csc[θ[t]] (-a Cot[θ[t]] + b Csc[θ[t]]);

Thirdeq = ψ'[t] == (a I1)/I3 + a Cot[θ[t]]^2 - b Cot[θ[t]] Csc[θ[t]];

rule1 = {I1 → 2, I3 → 1};


α = 1.6; β = 2; a = 2.5; b = 1.7;


def1 = {Firsteq, Secondeq, Thirdeq} /. rule1 ;
Initial = {θ[0] == 43 °, θ'[0] == 0, ϕ[0] == 0 °, ψ[0] == 0 °};

def2 = Join[def1, Initial]; eq1 = NDSolve[def2, {θ[t], ϕ[t], ψ[t]}, {t, 0, 70} ];

θ[t_] = θ[t] /. eq1[[1]]; ϕ[t_] = ϕ[t] /. eq1[[1]]; ψ[t_] = ϕ[t] /. eq1[[1]];

p1 = ParametricPlot3D[{Sin[θ[t]] Cos[ϕ[t]], Sin[θ[t]] Sin[ϕ[t]], Cos[θ[t]]},
  {t, 0, 70}, PlotStyle → {{Red, Thick}}, Boxed → False, Axes → False];
p2 = Graphics3D[{Opacity[0.5], Sphere[{0, 0, 0}, 1]}];
p3 = Graphics3D[{Blue, Thick, Arrow[{{0, 0, 0}, {1.1, 0, 0}}], Arrow[{{0, 0, 0}, {0, 1.1, 0}}],
   Arrow[{{0, 0, 0}, {0, 0, 1.1}}], Text[Style["θ(0)=43°", Black, 15], {0, 0, 1.3}]}];
Show[p1, p2, p3, PlotRange → All]
```

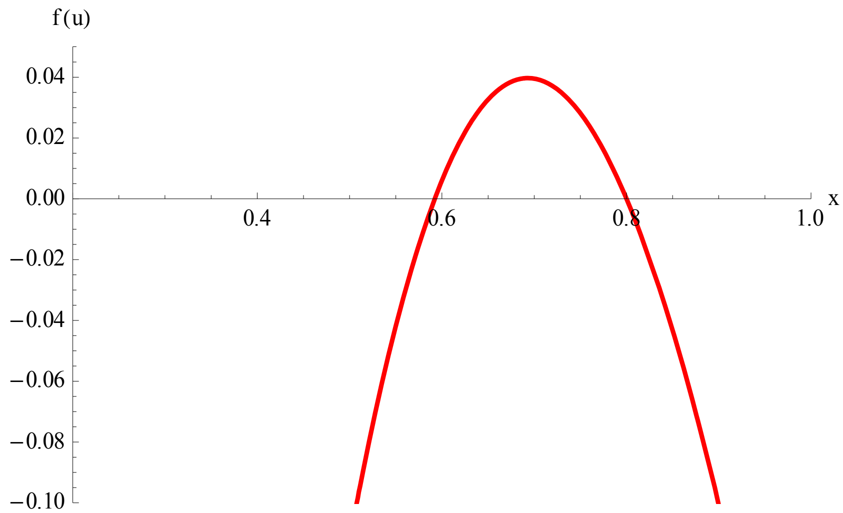


13.S13 Numerical simulation-2

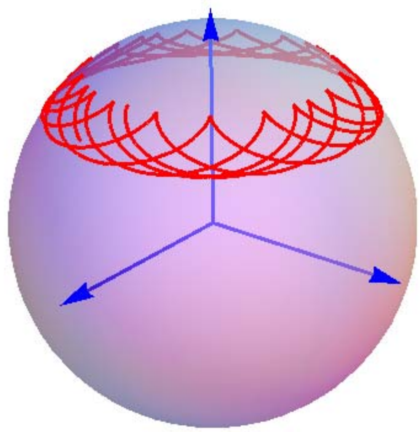
$\alpha = 1.6.$ $\beta = 2.0.$ $a = 2.5.$ $b = 2.$

Note that since $b/a = 2/2.5<1$, there is an angle θ satisfying $\cos\theta = 2/2.5 = 0.8$ ($\theta = 36.87°$). The values of u where $f(u) = 0$, are given by

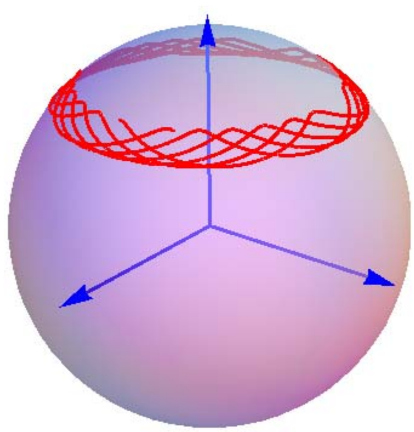
$$u_1 = 0.592 \ (\theta_1 = 53.70°) \qquad u_2 = 0.8 \ (\theta_2 = 36.87°)$$



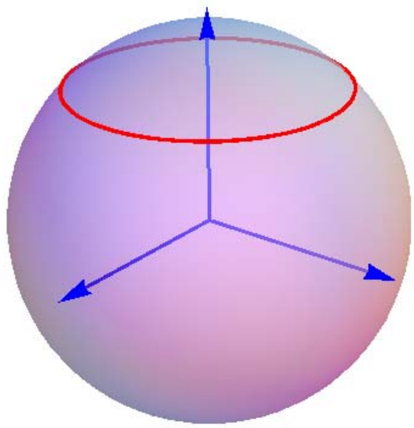
$\theta(0)=36.869°$



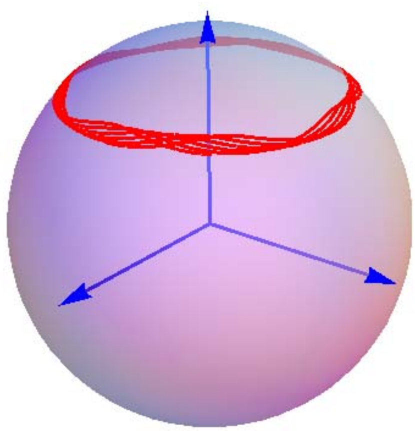
$\theta(0)=40°$



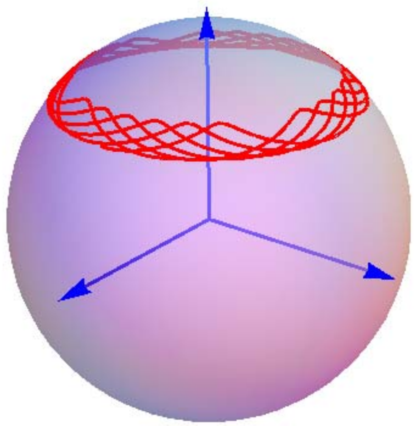
$\theta(0)=45°$



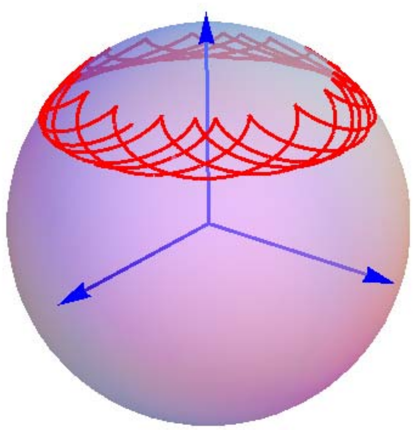
$\theta(0)=47.5°$



$\theta(0)=50°$



$\theta(0)=53.70°$



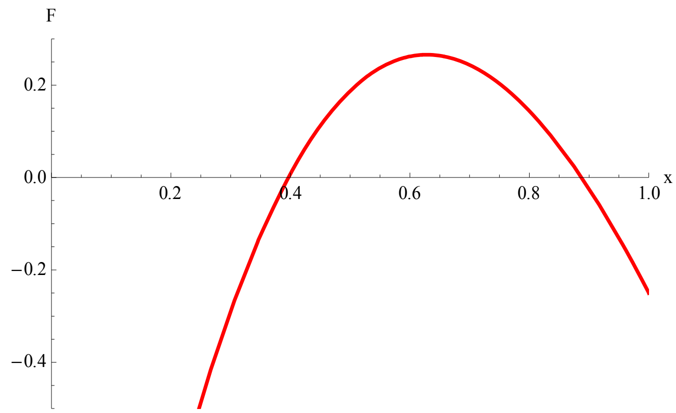
13S.12 Numerical simulations - 3

Parameters:

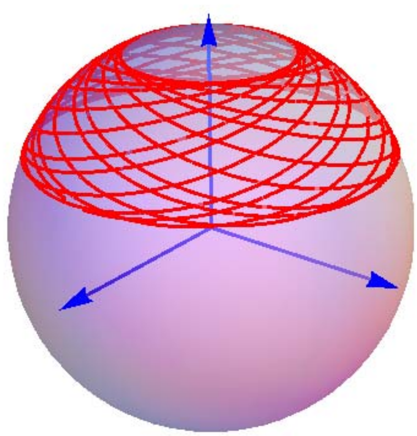
$\alpha = 2.8.$ $\beta = 2.0.$ $a = 1.75.$ $b = 2.$

Note that since $b/a = 2/1.75>1$, there is no angle θ satisfying $\cos\theta = b/a$. The values of u where $f(u) = 0$, are

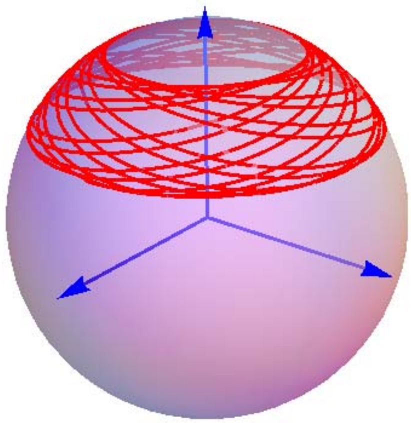
$u_1 = 0.408$ (65.92°) u_2=0.914 (23.94°)



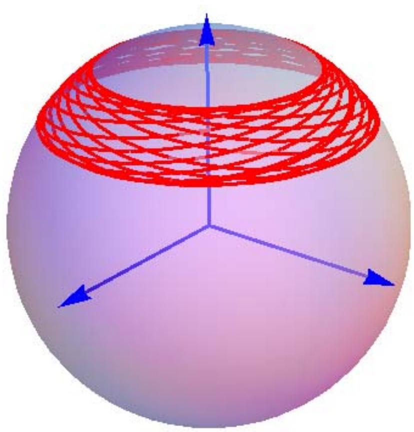
$\theta(0)$=65°



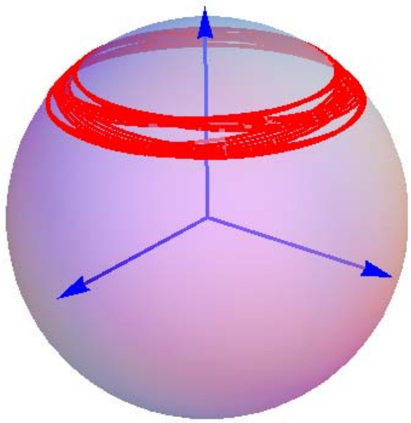
$\theta(0)=60°$



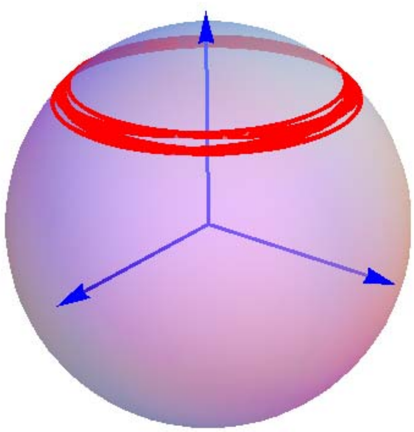
$\theta(0)=55°$



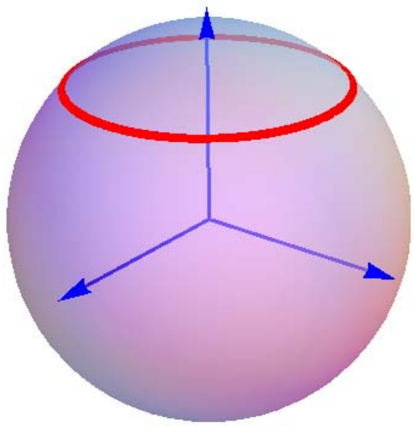
$\theta(0)=50°$



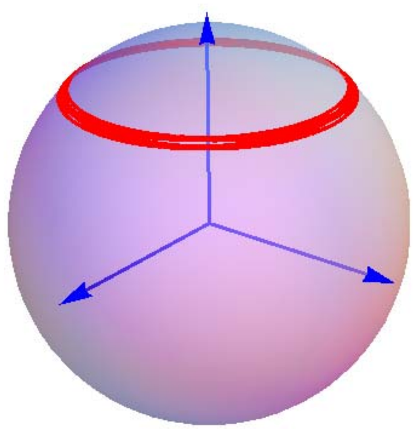
$\theta(0)=47.5°$



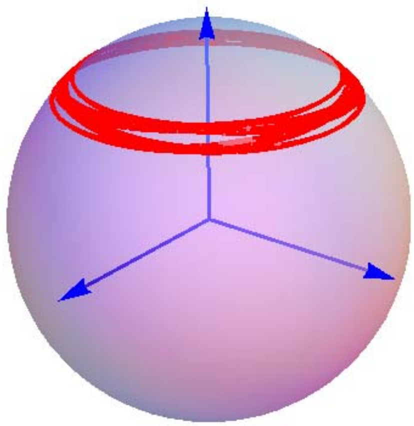
$\theta(0)=45°$



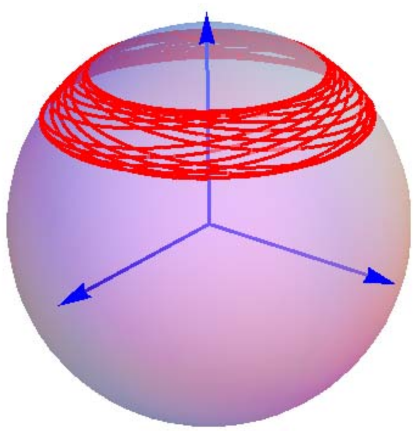
$\theta(0)=42.5°$



$\theta(0)=40°$



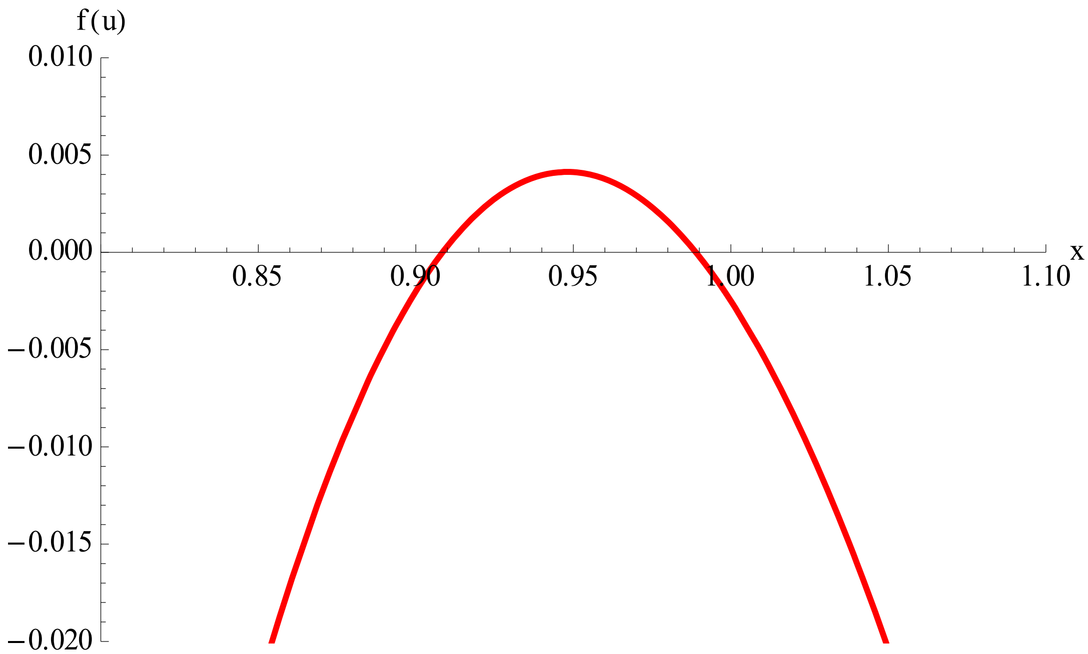
$\theta(0)=35°$



13S.13 Numerical calculations Example-4

Parameters:

$\alpha = 2.0.$ $\beta = 2.0.$ $a = 2.5.$ $b = 2.45$



The values of u where $f(u) = 0$, are

$u_1 = 0.908475$ ($\theta_1 = 24.705°$). $u_2 = 0.988875$ ($\theta_2 = 8.554°$).

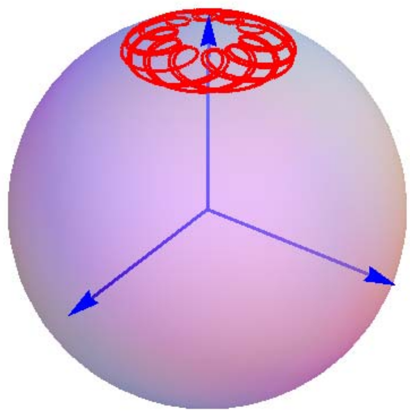
Note that there is an angle θ satisfying $\cos\theta = 2.45/2.5 = 0.98$ ($\theta = 11.478°$).

Initial conditions:

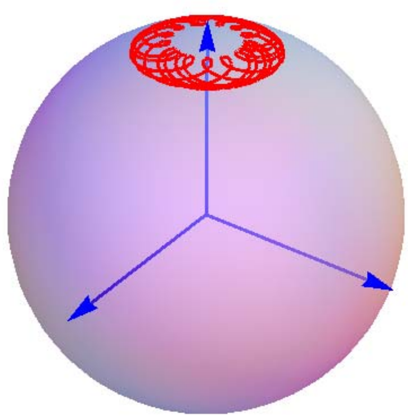
$\theta'(0) = 0.$ $\phi(0) = 0°.$ $\psi(0) = 0°.$

$\theta(0)$ is changed as a parameter.

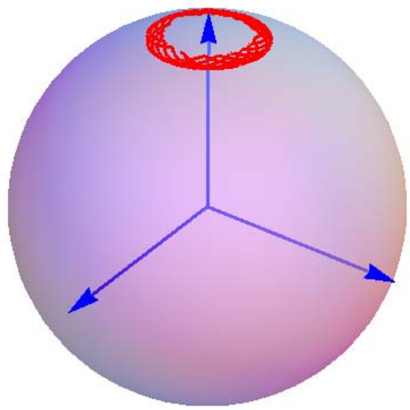
$\theta(0)=8.7°$



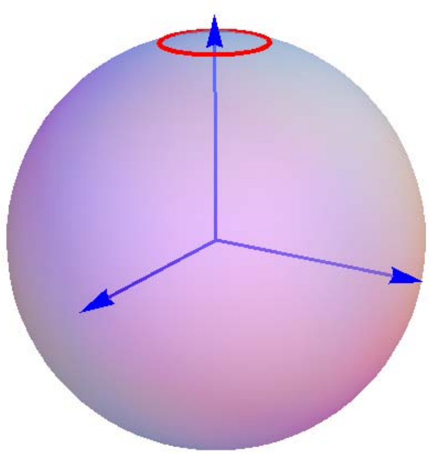
$\theta(0)=10°$

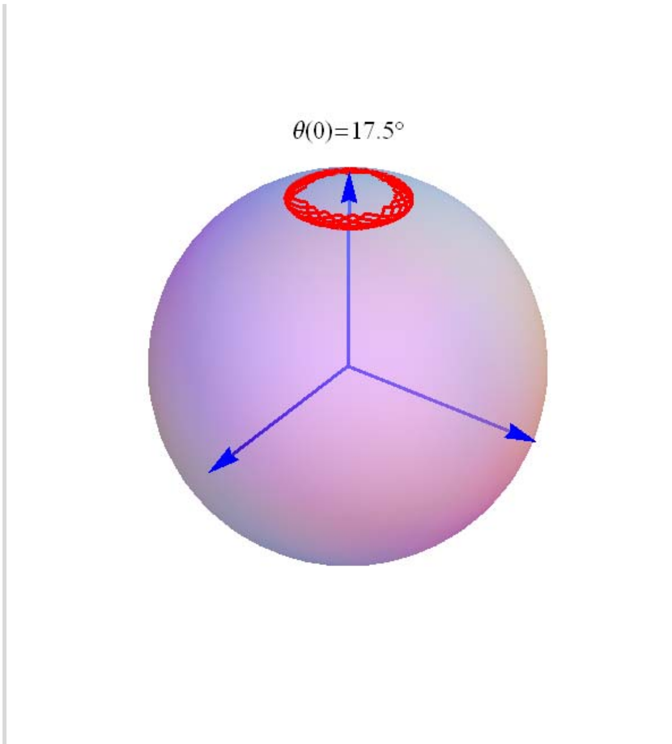


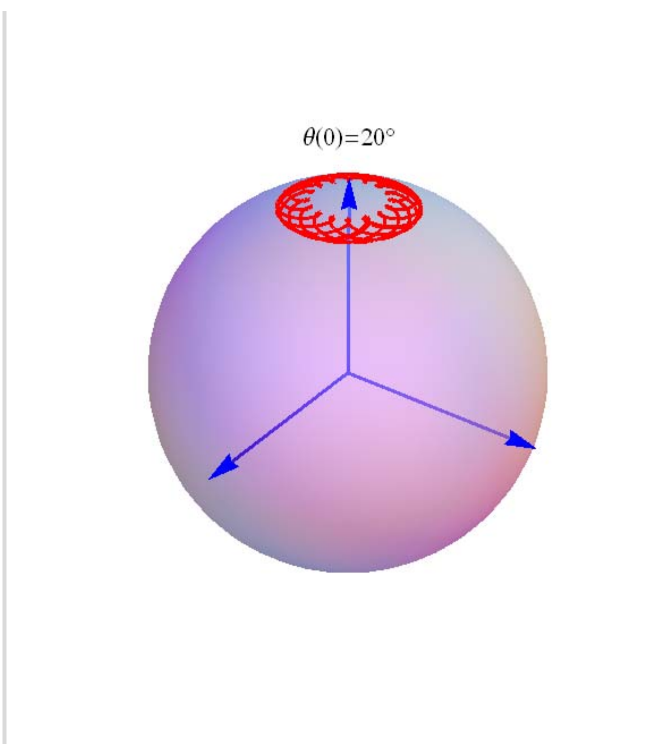
$\theta(0)=12.5°$

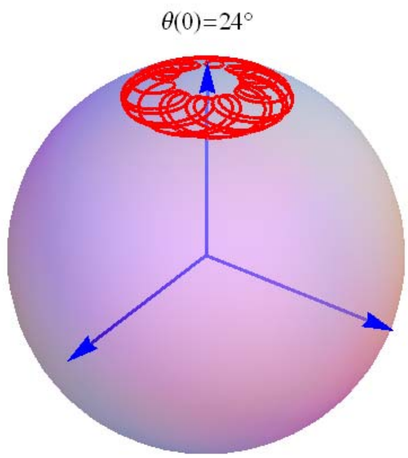


$\theta(0)=15°$




$\theta(0)$=17.5°

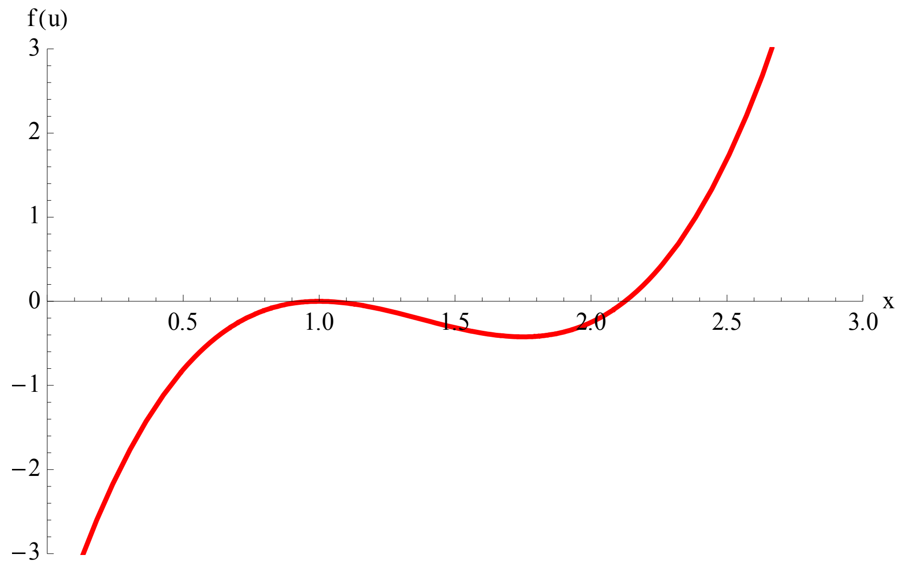

$\theta(0)$=20°



13S.13 Numerical calculations Example-5

Parameters:

$\alpha = 2.0.$ $\beta = 2.0.$ $a = 2.5.$ $b = 2.5$



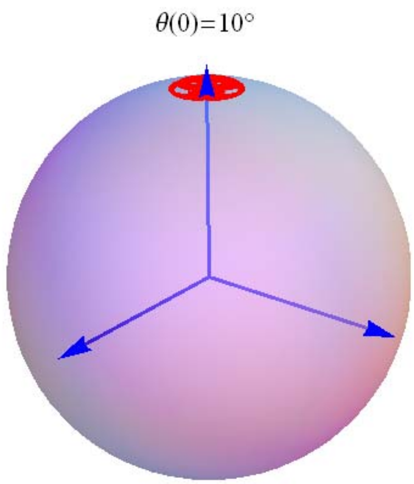
The values of u where $f(u) = 0$, are

$$u_2 = u_1 = 1$$

Initial conditions:

$$\theta'(0) = 0. \qquad \phi(0) = 0^{\circ}. \qquad \psi(0) = 0^{\circ}.$$

$\theta(0)$ is changed as a parameter.

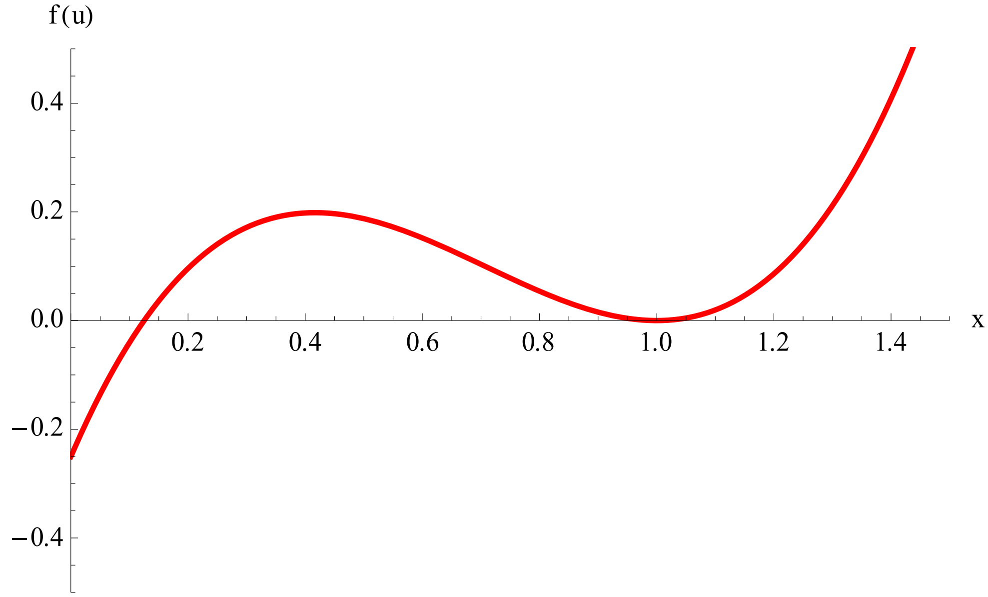


13S.14 Numerical calculations Example-6

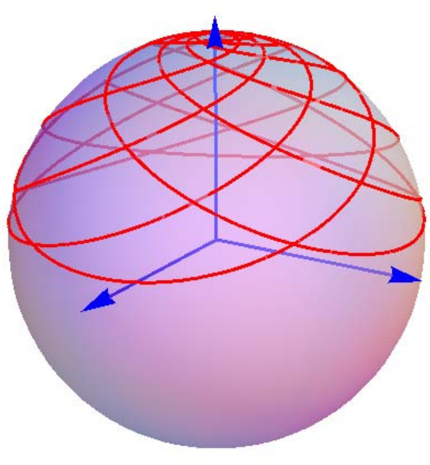
Parameters:

$\alpha = 2.0.$ $\beta = 2.0.$ $a = 1.5.$ $b = 1.5$

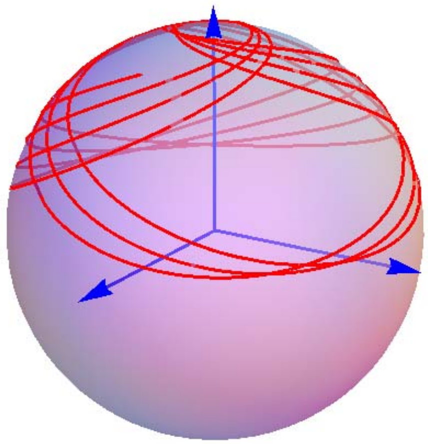
$u_1 = u_2 = 1$ (0°). $u_3 = 0.125$ (82.82°).



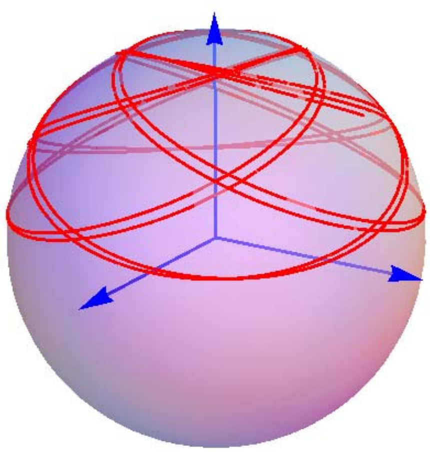
$\theta(0)=5°$



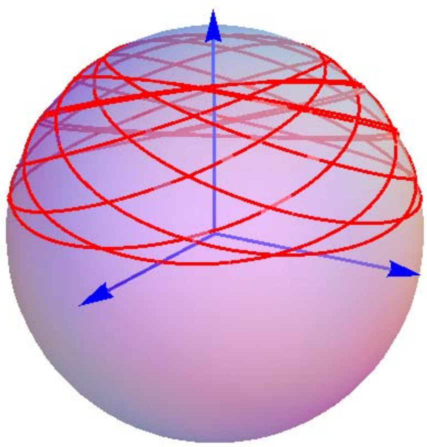
$\theta(0)=10°$



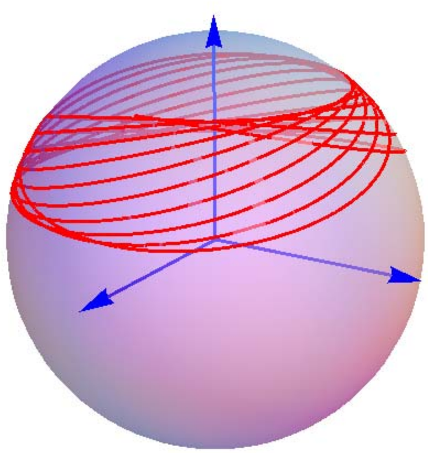
$\theta(0)=20°$



$\theta(0)=30°$



$\theta(0)=40°$



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H. Goldstein, C.P. Poole, and J.L.Safko, *Classical Mechanics*, 3rd edition (Addison Wesley, San Francisco, 2002).

J.M. Finn, *Classical Mechanics* (Infinity Science Press LLC, Hingham, Massachusetts, 2008).
P. Hamill, *Intermediate Dynamics* (Jones and Bartlett Publisher Sudbury, Massachusetts, 2010).
J.E. Hasbun, *Classical Mechanics with Matlab Applications* (Jones and Bartlett Publishers, Sundbury Massachusetts, 2009).
Jerry B. Marion, Classical Dynamic s of Particles and Systems, 2nd edition (Academic Press, New York, 1970).

APPENDIX

((**Mathematica**))

```
Clear["Global`*"];
```

The rotation with the angle ϕ around the z axis

```
D1 = RotationMatrix[-ϕ, {0, 0, 1}]; D1 // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\phi] & \text{Sin}[\phi] & 0 \\ -\text{Sin}[\phi] & \text{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The rotation with the angle θ around the ξ axis

```
C1 = RotationMatrix[-θ, {1, 0, 0}]; C1 // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\theta] & \text{Sin}[\theta] \\ 0 & -\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

The rotation with the angle ψ around the ζ' axis

```
B1 = RotationMatrix[-ψ, {0, 0, 1}]; B1 // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\psi] & \text{Sin}[\psi] & 0 \\ -\text{Sin}[\psi] & \text{Cos}[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The resultant rotation is described by a matrix A1 = B1 C1 D1

```
A1 = B1.C1.D1 // Simplify; A1 // MatrixForm
```

$$\begin{pmatrix} \mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi]+\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] \\ -\mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi]-\mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]-\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] \\ \mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] & -\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix}$$

```
D11 = D1 /. {ϕ → ϕ[t], θ → θ[t], ψ → ψ[t]}; C11 = C1 /. {ϕ → ϕ[t], θ → θ[t], ψ → ψ[t]};
B11 = B1 /. {ϕ → ϕ[t], θ → θ[t], ψ → ψ[t]};
A11 = B11.C11.D11;
```

```
Ωϕ1 = A11.{0, 0, ϕ'[t]}
```

{Sin[θ[t]] Sin[ψ[t]] ϕ′[t], Cos[ψ[t]] Sin[θ[t]] ϕ′[t], Cos[θ[t]] ϕ′[t]}

```
Ωθ1 = B11.C11.{θ'[t], 0, 0}
```

{Cos[ψ[t]] θ′[t], -Sin[ψ[t]] θ′[t], 0}

```
Ωψ1 = B11.{0, 0, ψ'[t]}
```

{0, 0, ψ′[t]}

```
Ω1 = Ωϕ1 + Ωθ1 + Ωψ1
```

{Cos[ψ[t]] θ′[t] + Sin[θ[t]] Sin[ψ[t]] ϕ′[t],
-Sin[ψ[t]] θ′[t] + Cos[ψ[t]] Sin[θ[t]] ϕ′[t], Cos[θ[t]] ϕ′[t] + ψ′[t]}

The angular velocity with respect to the body axes (x, y, z)

```
Ω = Inverse[A11].Ω1 // Simplify
```

{Cos[ϕ[t]] θ′[t] + Sin[θ[t]] Sin[ϕ[t]] ψ′[t],
Sin[ϕ[t]] θ′[t] - Cos[ϕ[t]] Sin[θ[t]] ψ′[t], ϕ′[t] + Cos[θ[t]] ψ′[t]}

Let the axis of symmetry be taken as the z axis fixed in the top. The moment of inertia is I3 about the z axis, and symmetry requires I1 = I2. The kinetic rotational energy is T.

$$\mathrm{T1} = \frac{1}{2}\,\mathrm{I1}\left((\Omega 1[[1]])^2 + (\Omega 1[[2]])^2\right)\ //\ \mathrm{Simplify}$$

$$\frac{1}{2}\,\mathrm{I1}\left(\theta'[t]^2 + \mathrm{Sin}[\theta[t]]^2\,\phi'[t]^2\right)$$

$$\mathrm{T3} = \frac{1}{2}\,\mathrm{I3}\,(\Omega 1[[3]])^2\ //\ \mathrm{Simplify}$$

$$\frac{1}{2}\,\mathrm{I3}\,(\mathrm{Cos}[\theta[t]]\,\phi'[t] + \psi'[t])^2$$

```
T = T1 + T3
```

$$\frac{1}{2}\,\mathrm{I1}\left(\theta'[t]^2 + \mathrm{Sin}[\theta[t]]^2\,\phi'[t]^2\right) + \frac{1}{2}\,\mathrm{I3}\,(\mathrm{Cos}[\theta[t]]\,\phi'[t] + \psi'[t])^2$$

The potential energy V is

```
Inverse[A1].{0, 0, l} // Simplify
{l Sin[θ] Sin[ϕ], -l Cos[ϕ] Sin[θ], l Cos[θ]}

V = m g l Cos[θ[t]];
```

The Lagrangian L is equal to L = T - V

```
L = T - V
```

$$-g\, l\, m \,\mathrm{Cos}[\theta[t]] + \frac{1}{2}\, I1 \left(\theta'[t]^2 + \mathrm{Sin}[\theta[t]]^2 \phi'[t]^2\right) + \frac{1}{2}\, I3 \left(\mathrm{Cos}[\theta[t]]\, \phi'[t] + \psi'[t]\right)^2$$

Here we use the variational method.

```
<< "VariationalMethods`"

eq11 = VariationalD[L, {ϕ[t], θ[t], ψ[t]}, t] // Simplify
```

$$\{\theta'[t]\,(-(I1 - I3)\,\mathrm{Sin}[2\,\theta[t]]\,\phi'[t] + I3\,\mathrm{Sin}[\theta[t]]\,\psi'[t]) - \left(I3\,\mathrm{Cos}[\theta[t]]^2 + I1\,\mathrm{Sin}[\theta[t]]^2\right)\phi''[t] - I3\,\mathrm{Cos}[\theta[t]]\,\psi''[t],$$
$$g\,l\,m\,\mathrm{Sin}[\theta[t]] + (I1 - I3)\,\mathrm{Cos}[\theta[t]]\,\mathrm{Sin}[\theta[t]]\,\phi'[t]^2 - I3\,\mathrm{Sin}[\theta[t]]\,\phi'[t]\,\psi'[t] - I1\,\theta''[t],$$
$$-I3\,(-\mathrm{Sin}[\theta[t]]\,\theta'[t]\,\phi'[t] + \mathrm{Cos}[\theta[t]]\,\phi''[t] + \psi''[t])\}$$

```
eq21 = EulerEquations[L, {ϕ[t], θ[t], ψ[t]}, t] // Simplify
```

$$\{\mathrm{Sin}[\theta[t]]\,\theta'[t]\,(2\,(I1 - I3)\,\mathrm{Cos}[\theta[t]]\,\phi'[t] - I3\,\psi'[t]) + \left(I3\,\mathrm{Cos}[\theta[t]]^2 + I1\,\mathrm{Sin}[\theta[t]]^2\right)\phi''[t] + I3\,\mathrm{Cos}[\theta[t]]\,\psi''[t] == 0,$$
$$\mathrm{Sin}[\theta[t]]\left(g\,l\,m + (I1 - I3)\,\mathrm{Cos}[\theta[t]]\,\phi'[t]^2\right) == I3\,\mathrm{Sin}[\theta[t]]\,\phi'[t]\,\psi'[t] + I1\,\theta''[t],$$
$$I3\,\mathrm{Sin}[\theta[t]]\,\theta'[t]\,\phi'[t] == I3\,(\mathrm{Cos}[\theta[t]]\,\phi''[t] + \psi''[t])\}$$

```
eq31 = FirstIntegrals[L, {ϕ[t], θ[t], ψ[t]}, t] // Simplify
```

$$\{\mathrm{FirstIntegral}[\phi] \to -\left(I3\,\mathrm{Cos}[\theta[t]]^2 + I1\,\mathrm{Sin}[\theta[t]]^2\right)\phi'[t] - I3\,\mathrm{Cos}[\theta[t]]\,\psi'[t],$$
$$\mathrm{FirstIntegral}[\psi] \to -I3\,(\mathrm{Cos}[\theta[t]]\,\phi'[t] + \psi'[t]),$$
$$\mathrm{FirstIntegral}[t] \to \frac{1}{2}\left(2\,g\,l\,m\,\mathrm{Cos}[\theta[t]] + I1\,\theta'[t]^2 + \left(I3\,\mathrm{Cos}[\theta[t]]^2 + I1\,\mathrm{Sin}[\theta[t]]^2\right)\phi'[t]^2 + 2\,I3\,\mathrm{Cos}[\theta[t]]\,\phi'[t]\,\psi'[t] + I3\,\psi'[t]^2\right)\}$$

Lagrangian L is a function of θ[t], θ'[t], ϕ'[t], ψ'[t] . In other words, ϕ[t], ψ[t], and t are the cyclic coordinates.

(1) $\partial L/\partial\phi'[t]=P\phi$ = constant. (2) $\partial L/\partial\psi'[t]=P\psi$ = constant. (3) Energy conservation.
(4) Lagrange equation.
a, b, and E1 are constants.

```
Pϕ = -(FirstIntegral[ϕ] /. eq31)
```

$\left(\text{I3}\,\text{Cos}[\theta[t]]^2+\text{I1}\,\text{Sin}[\theta[t]]^2\right)\phi'[t]+\text{I3}\,\text{Cos}[\theta[t]]\,\psi'[t]$

```
Pψ = -(FirstIntegral[ψ] /. eq31)
```

$\text{I3}\,(\text{Cos}[\theta[t]]\,\phi'[t]+\psi'[t])$

```
E1 = (FirstIntegral[t] /. eq31) // Expand
```

$$g\,l\,m\,\text{Cos}[\theta[t]]+\frac{1}{2}\text{I1}\,\theta'[t]^2+\frac{1}{2}\text{I3}\,\text{Cos}[\theta[t]]^2\,\phi'[t]^2+\frac{1}{2}\text{I1}\,\text{Sin}[\theta[t]]^2\,\phi'[t]^2+\text{I3}\,\text{Cos}[\theta[t]]\,\phi'[t]\,\psi'[t]+\frac{1}{2}\text{I3}\,\psi'[t]^2$$

Differential equations derived from the First Integrals
We put $P\psi == \text{I1}\ a$, $P\phi == \text{I1}\ b$. where a and b are constants.

```
s1 = Solve[{Pψ == I1 a, Pϕ == I1 b}, {ϕ'[t], ψ'[t]}] // Simplify
```

$$\left\{\left\{\phi'[t]\to\text{Csc}[\theta[t]]\,(-a\,\text{Cot}[\theta[t]]+b\,\text{Csc}[\theta[t]]),\ \psi'[t]\to\frac{a\,\text{I1}}{\text{I3}}+a\,\text{Cot}[\theta[t]]^2-b\,\text{Cot}[\theta[t]]\,\text{Csc}[\theta[t]]\right\}\right\}$$

```
Secondeq = s1[[1, 1]] /. Rule → Equal
```

$\phi'[t]==\text{Csc}[\theta[t]]\,(-a\,\text{Cot}[\theta[t]]+b\,\text{Csc}[\theta[t]])$

```
Thirdeq = s1[[1, 2]] /. Rule → Equal
```

$$\psi'[t]==\frac{a\,\text{I1}}{\text{I3}}+a\,\text{Cot}[\theta[t]]^2-b\,\text{Cot}[\theta[t]]\,\text{Csc}[\theta[t]]$$

Third differential equation from the Lagrange's (or Euler) eqution

```
seq11 = eq21[[2]] /. s1[[1]] // FullSimplify
```

$$\left(a^2+b^2\right)\text{I1}\,\text{Cot}[\theta[t]]\,\text{Csc}[\theta[t]]^2+g\,l\,m\,\text{Sin}[\theta[t]]==\text{I1}\left(a\,b\left(1+2\,\text{Cot}[\theta[t]]^2\right)\text{Csc}[\theta[t]]+\theta''[t]\right)$$

```
seq12 = Solve[seq11, θ''[t]] // Simplify
```

$$\left\{\left\{\theta''[t]\to\frac{\left(a^2+b^2\right)\text{I1}\,\text{Cot}[\theta[t]]\,\text{Csc}[\theta[t]]^2-\frac{1}{2}a\,b\,\text{I1}\,(3+\text{Cos}[2\,\theta[t]])\,\text{Csc}[\theta[t]]^3+g\,l\,m\,\text{Sin}[\theta[t]]}{\text{I1}}\right\}\right\}$$

```
seq13 = seq12 /. Rule → Equal
```

$$\left\{\left\{\theta''[t]==\frac{\left(a^2+b^2\right)\text{I1}\,\text{Cot}[\theta[t]]\,\text{Csc}[\theta[t]]^2-\frac{1}{2}a\,b\,\text{I1}\,(3+\text{Cos}[2\,\theta[t]])\,\text{Csc}[\theta[t]]^3+g\,l\,m\,\text{Sin}[\theta[t]]}{\text{I1}}\right\}\right\}$$

```
Firsteq = seq13[[1, 1]]
```

$$\theta''[t] == \frac{\left(a^2+b^2\right) \text{I1}\, \text{Cot}[\theta[t]]\, \text{Csc}[\theta[t]]^2 - \frac{1}{2} a\, b\, \text{I1}\, (3+\text{Cos}[2\,\theta[t]])\, \text{Csc}[\theta[t]]^3 + g\, l\, m\, \text{Sin}[\theta[t]]}{\text{I1}}$$

Energy conservation from the FirstIntegrals

Etot is the total energy and is constant.

```
s2 = (E1) /. s1[[1]] // Simplify
```

$$\frac{1}{2\,\text{I3}}\left(2\,g\,\text{I3}\,l\,m\,\text{Cos}[\theta[t]] + \text{I1}\left(a^2\,\text{I1} + a^2\,\text{I3}\,\text{Cot}[\theta[t]]^2 - 2\,a\,b\,\text{I3}\,\text{Cot}[\theta[t]]\,\text{Csc}[\theta[t]] + b^2\,\text{I3}\,\text{Csc}[\theta[t]]^2\right) + \text{I1}\,\text{I3}\,\theta'[t]^2\right)$$

```
Energy = s2 == Etot
```

$$\frac{1}{2\,\text{I3}}\left(2\,g\,\text{I3}\,l\,m\,\text{Cos}[\theta[t]] + \text{I1}\left(a^2\,\text{I1} + a^2\,\text{I3}\,\text{Cot}[\theta[t]]^2 - 2\,a\,b\,\text{I3}\,\text{Cot}[\theta[t]]\,\text{Csc}[\theta[t]] + b^2\,\text{I3}\,\text{Csc}[\theta[t]]^2\right) + \text{I1}\,\text{I3}\,\theta'[t]^2\right) == \text{Etot}$$

```
en11 = Solve[Energy /. θ'[t]^2 → x, x]
```

$$\left\{\left\{x \to \frac{1}{\text{I1}\,\text{I3}}\left(-a^2\,\text{I1}^2 + 2\,\text{Etot}\,\text{I3} - 2\,g\,\text{I3}\,l\,m\,\text{Cos}[\theta[t]] - a^2\,\text{I1}\,\text{I3}\,\text{Cot}[\theta[t]]^2 + 2\,a\,b\,\text{I1}\,\text{I3}\,\text{Cot}[\theta[t]]\,\text{Csc}[\theta[t]] - b^2\,\text{I1}\,\text{I3}\,\text{Csc}[\theta[t]]^2\right)\right\}\right\}$$

```
Energyeq = θ'[t]^2 == x /. en11[[1]] // Simplify
```

$$\frac{a^2\,\text{I1}}{\text{I3}} + \frac{2\,g\,l\,m\,\text{Cos}[\theta[t]]}{\text{I1}} + a^2\,\text{Cot}[\theta[t]]^2 + b^2\,\text{Csc}[\theta[t]]^2 + \theta'[t]^2 == \frac{2\,\text{Etot}}{\text{I1}} + 2\,a\,b\,\text{Cot}[\theta[t]]\,\text{Csc}[\theta[t]]$$