

Chapter 15
Three dimensional and four dimensional Green's function
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15.1 Green's function (summary)

$$L_1 y(\mathbf{r}_1) = -f(\mathbf{r}_1) \quad (\text{self adjoint})$$

The solution of this equation is given by

$$y(\mathbf{r}_1) = \int G(\mathbf{r}_1, \mathbf{r}_2) f(\mathbf{r}_2) d\mathbf{r}_2 + \varphi(\mathbf{r}_1),$$

where

$$L_1 = \nabla_1 \cdot [\mathbf{p}(\mathbf{r}_1) \cdot \nabla_1] + q(\mathbf{r}_1),$$

$$L_1 G(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2),$$

and

$$L_1 \varphi(\mathbf{r}_1) = 0.$$

(a) $\mathbf{p}(\mathbf{r}_1) = \mathbf{1}$ and $q(\mathbf{r}_1) = 0$

$$L_1 = \nabla_1^2$$

$$\nabla_1^2 G(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2)$$

The solution is

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|}.$$

(b) $\mathbf{p}(\mathbf{r}_1) = \mathbf{1}$ and $q(\mathbf{r}_1) = k^2$

$$L_1 = \nabla_1^2 + k^2,$$

$$(\nabla_1^2 + k^2)G(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2),$$

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{\exp(ik|\mathbf{r}_1 - \mathbf{r}_2|)}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|}.$$

The solution of $L_1 y(\mathbf{r}_1) = -f(\mathbf{r}_1)$ is

$$y(\mathbf{r}_1) = \int \frac{\exp(ik|\mathbf{r}_1 - \mathbf{r}_2|)}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} f(\mathbf{r}_2) d\mathbf{r}_2 + \varphi(\mathbf{r}_1).$$

with

$$(\nabla_1^2 + k^2)\varphi(r_1) = 0$$

$$(c) \quad \mathbf{p}(\mathbf{r}_1) = \mathbf{1} \text{ and } q(\mathbf{r}_1) = -k^2$$

$$L_1 = \nabla_1^2 - k^2,$$

$$(\nabla_1^2 - k^2)G(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2),$$

$$G(\mathbf{r}_1, \mathbf{r}_2) = \frac{\exp(-k|\mathbf{r}_1 - \mathbf{r}_2|)}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|}.$$

The solution of $L_1 y(\mathbf{r}_1) = -f(\mathbf{r}_1)$ is

$$y(\mathbf{r}_1) = \int \frac{\exp(-k|\mathbf{r}_1 - \mathbf{r}_2|)}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} f(\mathbf{r}_2) d\mathbf{r}_2 + \varphi(\mathbf{r}_1).$$

with

$$(\nabla_1^2 - k^2)\varphi(r_1) = 0$$

Table

Laplace
 ∇^2

Helmholtz
 $\nabla^2 + k^2$

Modified Helmholtz
 $\nabla^2 - k^2$

$$3D \quad \frac{1}{4\pi |\mathbf{r}_1 - \mathbf{r}_2|} \quad \frac{1}{4\pi} \frac{\exp(ik|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|} \quad \frac{1}{4\pi} \frac{\exp(-k|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

15.2 3D Green's function (Laplace)

We now consider the form of the Green's function (Laplace)

$$\nabla^2 G_0 = -\delta(\mathbf{r}).$$

The Fourier transform of $G(\mathbf{r})$ is defined as

$$G_0(\mathbf{q}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{r}).$$

The inverse Fourier transform is also defined as

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{q}),$$

where

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}}.$$

Then

$$\begin{aligned} \nabla^2 G_0(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} \nabla^2 e^{i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{q}) \\ &= \frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} (-q^2) e^{i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{q}), \\ &= -\frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} \end{aligned}$$

or

$$G_0(\mathbf{q}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{q^2}.$$

Then we have

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\mathbf{q} \frac{e^{i\mathbf{q} \cdot \mathbf{r}}}{q^2}$$

For convenience, we assume that the direction of $\mathbf{r}$ is the z axis. The angle between $\mathbf{r}$ and $\mathbf{q}$ is θ .

$$\mathbf{q} \cdot \mathbf{r} = qr \cos \theta$$

$$d\mathbf{q} = 2\pi q^2 dq \sin \theta d\theta$$

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^3} \int \int 2\pi q^2 dq \sin \theta d\theta e^{iqr \cos \theta} \frac{1}{q^2}$$

Note

$$\int \sin \theta d\theta e^{iqr \cos \theta} = -\frac{i}{qr} [e^{iqr} - e^{-iqr}],$$

we have

$$\begin{aligned} G_0(\mathbf{r}) &= \frac{1}{(2\pi)^3} \int q^2 dq \left(-\frac{2\pi i}{qr} \right) \frac{e^{iqr} - e^{-iqr}}{q^2} \\ &= \frac{-i}{(2\pi)^2 r} \int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q} \end{aligned}$$

We calculate $I = \int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q}$ by using the Cauchy's theorem.

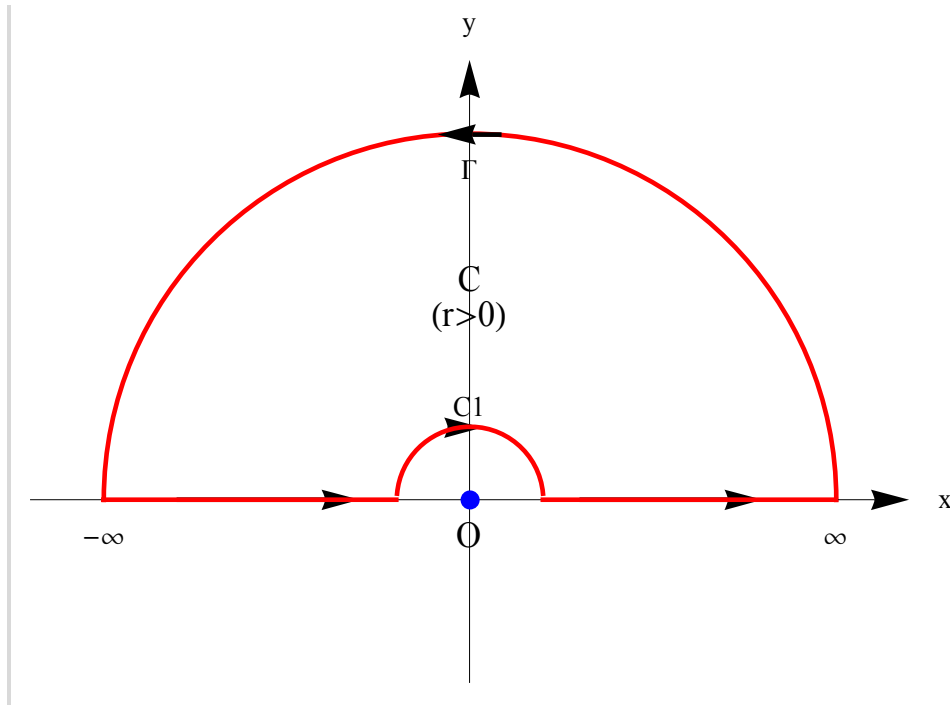


Fig. Upper half-plane contour for $r > 0$. The semi circle C_1 (clock wise).

$$P \int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q} + \int_{C_1} dz \frac{e^{izr}}{z} + \int_{\Gamma} dz \frac{e^{izr}}{z} = \oint_C dz \frac{e^{izr}}{z} = 0$$

or

$$P \int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q} = - \int_{C_1} dz \frac{e^{izr}}{z} = \pi i \operatorname{Res}(z=0) = \pi i$$

since

$$\int_{\Gamma} dz \frac{e^{izr}}{z} = 0 \quad (\text{Jordan's lemma, } r > 0).$$

Then we have

$$G_0(\mathbf{r}) = \frac{1}{4\pi^2 i r} \pi i = \frac{1}{4\pi r}$$

or

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

((Note)) Method II

We show that we get the same result using a different contour (method II). Inside this contour, there is a simple pole at $z = 0$.

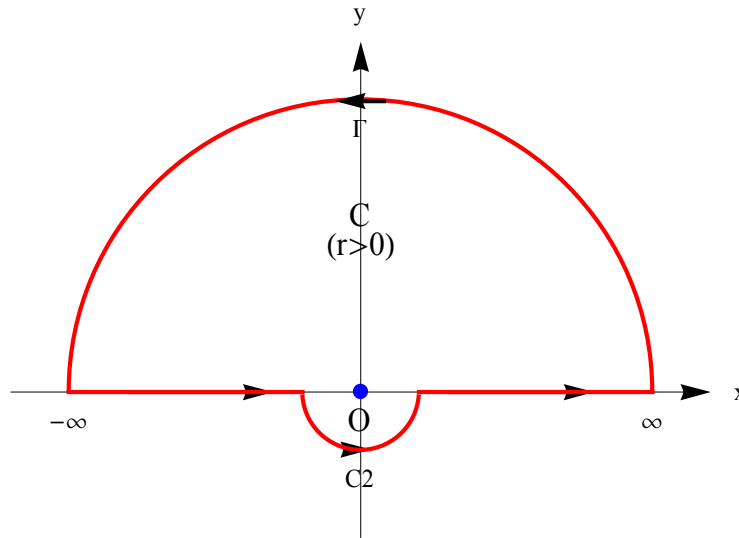


Fig. The contour C_2 has a counter clock-wise rotation.

$$P \int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q} + \int_{\Gamma} dz \frac{e^{izr}}{z} + \int_{C2} dz \frac{e^{izr}}{z} = \oint_C dz \frac{e^{izr}}{z} = 2\pi i \operatorname{Res}(z=0)$$

Since $\int_{\Gamma} dz \frac{e^{izr}}{z} = 0$ from the Jordan's lemma. we have

$$\begin{aligned} P \int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q} &= -\int_{C2} dz \frac{e^{izr}}{z} + 2\pi i \operatorname{Res}(z=0) \\ &= -\pi i \operatorname{Res}(z=0) + 2\pi i \operatorname{Res}(z=0) \\ &= \pi i \operatorname{Res}(z=0) = \pi i \end{aligned}$$

Then we have

$$G_0(\mathbf{r}) = \frac{1}{4\pi^2 i r} \pi i = \frac{1}{4\pi r}.$$

which is the same as that obtained in the method I.

15.3 Derivation of Green's function (vector analysis)

$$\nabla^2 \frac{1}{4\pi r} = -\delta(\mathbf{r}),$$

where

$$\mathbf{r} = (x, y, z), \quad r = \sqrt{x^2 + y^2 + z^2}.$$

We consider a sphere with radius ε ($\varepsilon \rightarrow 0$)

$$\int d\mathbf{r} \nabla \cdot \nabla \frac{1}{r} = \int d\mathbf{r} \Delta \frac{1}{r} = \int d\mathbf{a} \cdot \nabla \frac{1}{r} = \int da (\mathbf{n} \cdot \nabla \frac{1}{r})$$

where

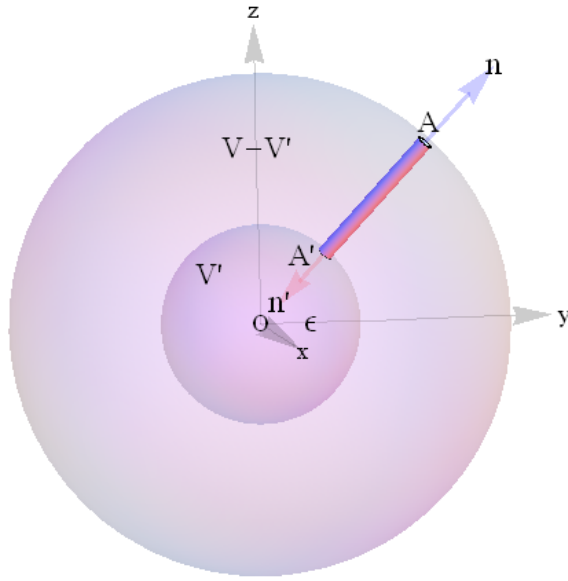
$$r = \sqrt{x^2 + y^2 + z^2}, \quad \mathbf{n} = \frac{\mathbf{r}}{r} = \mathbf{e}_r = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right), \quad d\mathbf{a} = \mathbf{n} da$$

and

$$\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3}, \quad \mathbf{n} \cdot \nabla \frac{1}{r} = \hat{r} \cdot \left(-\frac{\mathbf{r}}{r^3}\right) = -\frac{1}{r^2}$$

$$\nabla \cdot \nabla \left(\frac{1}{r}\right) = 0 \text{ except at the origin.}$$

We now consider the volume integral over the whole volume ($V - V'$) between the surface A and the surface of sphere A' (volume V' , radius $\varepsilon \rightarrow 0$). We note that the outer surface and the inner surface are connected to an appropriate cylinder.



Since $\nabla \cdot \nabla \left(\frac{1}{r} \right) = 0$ over the whole volume $V - V'$ we have

Using the Gauss's law, we get

$$\begin{aligned} \int_{V-V'} d\mathbf{r} \nabla \cdot \nabla \frac{1}{r} &= \int_{V-V'} d\mathbf{r} \nabla^2 \frac{1}{r} \\ &= \int_A da (\mathbf{n} \cdot \nabla \frac{1}{r}) + \int_{A'} da' (\mathbf{n}' \cdot \nabla \frac{1}{r}) = 0 \end{aligned}$$

or

$$\int_A da (\mathbf{n} \cdot \nabla \frac{1}{r}) = - \int_{A'} da' (\mathbf{n}' \cdot \nabla \frac{1}{r}) = \int_{A'} da' (\mathbf{n} \cdot \nabla \frac{1}{r})$$

where $\mathbf{n}' = -\mathbf{n} = -\hat{r}$ and $d\mathbf{r}$ is over the volume integral. Then we have

$$\int_A da (\mathbf{n} \cdot \nabla \frac{1}{r}) = \int da \left(-\frac{1}{r^2} \right) = -4\pi\epsilon^2 \frac{1}{\epsilon^2} = -4\pi = -4\pi \int d\mathbf{r} \delta(\mathbf{r})$$

Using the Gauss's law, we have

$$\int_A d\mathbf{a}(\mathbf{n} \cdot \nabla \frac{1}{r}) = \int_V d\mathbf{r}(\nabla \cdot \nabla \frac{1}{r}) = -4\pi \int_V d\mathbf{r} \delta(\mathbf{r})$$

or

$$\Delta \frac{1}{r} = -4\pi \delta(\mathbf{r}).$$

or

$$\Delta \left(\frac{1}{4\pi r} \right) = -\delta(\mathbf{r}).$$

((Mathematica))

```
Clear["Global`"];
```

```
Needs["VectorAnalysis`"]
```

```
SetCoordinates[Cartesian[x, y, z]]
```

```
Cartesian[x, y, z]
```

```
r1 = {x, y, z}; r = Sqrt[r1.r1]
```

```
Sqrt[x^2 + y^2 + z^2]
```

```
Grad[1/r] // Simplify
```

```
{-x/(x^2 + y^2 + z^2)^(3/2), -y/(x^2 + y^2 + z^2)^(3/2), -z/(x^2 + y^2 + z^2)^(3/2)}
```

```
Laplacian[1/r] // Simplify
```

```
0
```

15.4 3D Green's function (Helmholtz)

We now consider the form of the Green's function (Helmholtz)

$$(\Delta + k^2)G_0(\mathbf{r}) = -\delta(\mathbf{r})$$

The Fourier transform of $G_0(\mathbf{r})$ is defined as

$$G_0(\mathbf{q}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{r})$$

The inverse Fourier transform is also defined as

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{q})$$

where

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}}$$

Then

$$\begin{aligned} (\Delta + k^2)G_0(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} (\Delta + k^2) e^{i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{q}) \\ &= \frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} (-q^2 + k^2) e^{i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{q}) \\ &= -\delta(\mathbf{r}) = -\frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} \end{aligned}$$

or

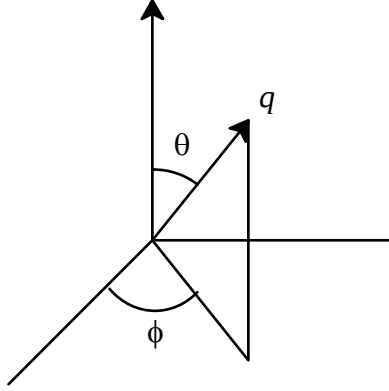
$$\int d\mathbf{q} (-q^2 + k^2) e^{i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{q}) = -\frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}}$$

or

$$G_0(\mathbf{q}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{q^2 - k^2}$$

Thus the Green's function is rewritten as

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{q^2 - k^2}$$



For convenience, we assume that the direction of $\mathbf{r}$ is the z axis. The angle between $\mathbf{r}$ and $\mathbf{q}$ is θ .

$$\mathbf{q} \cdot \mathbf{r} = qr \cos \theta$$

$$d\mathbf{q} = 2\pi q^2 dq \sin \theta d\theta$$

$$G_0^{(\pm)}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int \int 2\pi q^2 dq \sin \theta d\theta e^{iqr \cos \theta} \frac{1}{q^2 - (k^2 \pm i\varepsilon)}$$

Note

$$\int \sin \theta d\theta e^{iqr \cos \theta} = -\frac{i}{qr} [e^{iqr} - e^{-iqr}],$$

we have

$$G_0(\mathbf{r}) = G_0(r) = \frac{1}{(2\pi)^3} \int q^2 dq \left(-\frac{2\pi i}{qr} \right) \frac{e^{iqr} - e^{-iqr}}{q^2 - k^2}$$

since $G_0(\mathbf{r})$ depends only on r .

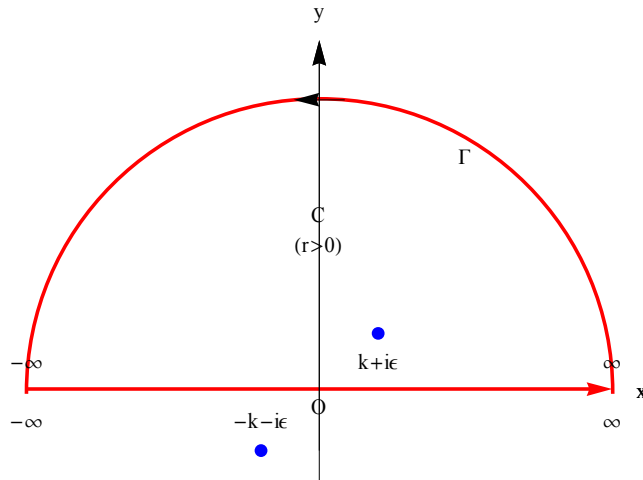
or

$$\begin{aligned} G_0(r) &= \frac{1}{4\pi^2 i r} \int_0^\infty q dq \frac{e^{iqr} - e^{-iqr}}{q^2 - k^2} \\ &= \frac{1}{4\pi^2 i r} \int_{-\infty}^\infty q dq \frac{e^{iqr}}{q^2 - k^2} \\ &= \frac{1}{8\pi^2 i r} \int_{-\infty}^\infty e^{iqr} dq \left(\frac{1}{q - k} + \frac{1}{q + k} \right) \end{aligned}$$

We consider the shift the position of the simple poles by $\pm i\varepsilon$ in the complex plane, where $\varepsilon \rightarrow 0$ and $\varepsilon > 0$. This shift is significant to the calculation, since the contour C is in the upper half plane. In other words, the position of the simple poles from the real axis to the upper half plane or to the lower half plane by $i\varepsilon$.

(i) Retarded Green's function

$$G_0^{(+)}(r) = \frac{1}{8\pi^2 i r} \int_{-\infty}^{\infty} e^{iqr} dq \left(\frac{1}{q - k - i\varepsilon} + \frac{1}{q + k + i\varepsilon} \right)$$



Since $r > 0$, the path of integration can be closed by an infinite semicircle in the upper half-plane (Jordan's lemma).

$$\begin{aligned} G_0^{(+)}(r) &= \frac{1}{8\pi^2 i r} \oint_C e^{izr} dz \left(\frac{1}{z - k - i\varepsilon} + \frac{1}{z + k + i\varepsilon} \right) \\ &= \frac{1}{8\pi^2 i r} 2\pi i \operatorname{Res}(q = k + i\varepsilon) = \frac{1}{4\pi r} e^{ikr} \end{aligned}$$

(retarded Green's function).

This corresponds to the outgoing spherical wave. Formally we get

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|}.$$

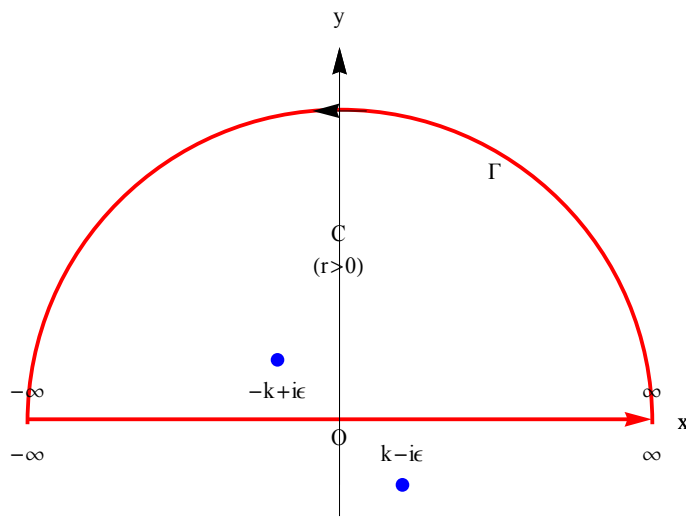
Here we note that

$$\begin{aligned}
\frac{1}{q-k-i\varepsilon} + \frac{1}{q+k+i\varepsilon} &= \frac{1}{q-(k+i\varepsilon)} + \frac{1}{q+(k+i\varepsilon)} \\
&= \frac{2q}{q^2 - (k+i\varepsilon)^2} \\
&= \frac{2q}{q^2 - (k^2 + 2ik\varepsilon)} \\
&= \frac{2q}{q^2 - k^2 - i\delta}
\end{aligned}$$

where $\delta = 2\varepsilon k$ (>0)

(ii) Advanced Green's function

$$G_0^{(-)}(r) = \frac{1}{8\pi^2 i r} \int_{-\infty}^{\infty} e^{iqr} dq \left(\frac{1}{q-k+i\varepsilon} + \frac{1}{q+k-i\varepsilon} \right)$$



$$\begin{aligned}
G_0^{(-)}(r) &= \frac{1}{8\pi^2 i r} \oint_{\Gamma} e^{izr} dz \left(\frac{1}{z-k+i\varepsilon} + \frac{1}{z+k-i\varepsilon} \right) \\
&= \frac{1}{8\pi^2 i r} 2\pi i \operatorname{Res}(q = -k + i\varepsilon) \\
&= \frac{1}{4\pi r} e^{-ikr}
\end{aligned}$$

(Advanced Green's function)

This corresponds to the incoming spherical wave. We note that

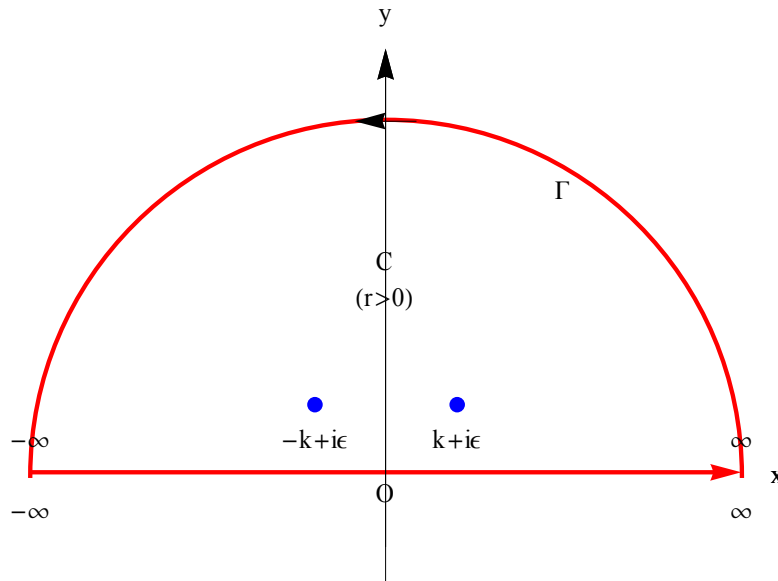
$$\begin{aligned}
 \frac{1}{q-k+i\varepsilon} + \frac{1}{q+k-i\varepsilon} &= \frac{1}{q-(k-i\varepsilon)} + \frac{1}{q+(k-i\varepsilon)} \\
 &= \frac{2q}{q^2 - (k-i\varepsilon)^2} \\
 &= \frac{2q}{q^2 - (k^2 - 2i\varepsilon k)} \\
 &= \frac{2q}{q^2 - k^2 + i\delta}
 \end{aligned}$$

where $\delta = 2\varepsilon k$ (>0).

((Note))

There are two more types of Green's function which depends on the shift of poles

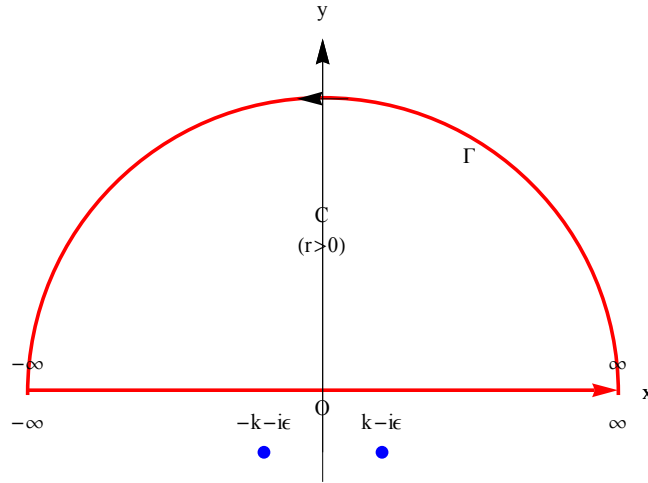
(iii)



$$\begin{aligned}
 G_0(r) &= \frac{1}{8\pi^2 i r} \oint_C e^{izr} dz \left(\frac{1}{z-k-i\varepsilon} + \frac{1}{z+k-i\varepsilon} \right) \\
 &= \frac{1}{8\pi^2 i r} 2\pi i [\text{Res}(q=k+i\varepsilon) + \text{Res}(q=-k+i\varepsilon)] \\
 &= \frac{1}{4\pi r} (e^{-ikr} + e^{ikr})
 \end{aligned}$$

which is the superposition of incoming spherical wave and outgoing spherical wave.

(iv)



There is no pole inside the contour C . then we have

$$G_0(r) = \frac{1}{8\pi^2 i r} \oint_C e^{izr} dz \left(\frac{1}{z - k + i\varepsilon} + \frac{1}{z + k + i\varepsilon} \right) = 0$$

15.5 Calculation of $\nabla^2 G_0^{(\pm)}(r)$

$$\begin{aligned} \nabla^2(fg) &= \nabla \cdot \nabla(fg) = \nabla \cdot [f\nabla g + g\nabla f] = 2\nabla g \cdot \nabla f + f\nabla^2 g + g\nabla^2 f \\ \nabla \cdot (\phi \mathbf{A}) &= \mathbf{A} \cdot \nabla \phi + \phi \nabla \cdot \mathbf{A} \\ \nabla(fg) &= f\nabla g + g\nabla f \end{aligned}$$

We calculate $\nabla^2 G_0^{(\pm)}(r)$.

$$\begin{aligned} \nabla^2 G_0^{(\pm)}(r) &= \nabla^2 \left(\frac{1}{4\pi r} e^{\pm ikr} \right) \\ &= e^{\pm ikr} \nabla^2 \left(\frac{1}{4\pi r} \right) + \left(\frac{1}{4\pi r} \right) \nabla^2 (e^{\pm ikr}) + 2\nabla \left(\frac{1}{4\pi r} \right) \cdot \nabla (e^{\pm ikr}) \end{aligned}$$

Here

$$\begin{aligned} \nabla^2 (e^{\pm ikr}) &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} e^{\pm ikr} \right) \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} [(\pm ikr^2) e^{\pm ikr}] = \left(\pm \frac{2ik}{r} - k^2 \right) e^{\pm ikr} \end{aligned}$$

$$\nabla\left(\frac{1}{4\pi r}\right) = \mathbf{e}_r \frac{\partial}{\partial r}\left(\frac{1}{4\pi r}\right) = -\frac{\mathbf{e}_r}{4\pi r^2},$$

$$\nabla(e^{\pm ikr}) = \mathbf{e}_r \frac{\partial}{\partial r}(e^{\pm ikr}) = \mathbf{e}_r (\pm ikr) e^{\pm ikr}.$$

Then

$$\nabla^2 G_0^{(\pm)}(r) = e^{\pm ikr} \nabla^2\left(\frac{1}{4\pi r}\right) + \left(\frac{1}{4\pi r}\right) (\pm \frac{2ik}{r} - k^2) e^{\pm ikr} - \frac{1}{2\pi r^2} (\pm ikr) e^{\pm ikr},$$

or

$$\nabla^2 G_0^{(\pm)}(r) = -\delta(\mathbf{r}) - k^2 \left(\frac{1}{4\pi r}\right) e^{\pm ikr} = -\delta(\mathbf{r}) - k^2 G^{(\pm)}(r),$$

or

$$(\nabla^2 + k^2) G_0^{(\pm)}(r) = -\delta(\mathbf{r}).$$

15.6 Derivation of Green's function (Helmholtz)

We assume that $G(r)$ is only dependent on r .

$$(\nabla^2 + k^2)G(r) = -\delta(\mathbf{r}),$$

where

$$\nabla^2 G(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial G(r)}{\partial r} \right) = \frac{1}{r} \frac{\partial}{\partial r^2} (rG(r)).$$

Then

$$\frac{1}{r} \frac{\partial}{\partial r^2} (rG(r)) + k^2 G(r) = -\delta(\mathbf{r}).$$

For $r \neq 0$, we have

$$\frac{\partial}{\partial r^2} (rG(r)) + k^2 rG(r) = 0.$$

Then we have

$$G_{\pm}(r) = A_{\pm} \frac{e^{\pm ikr}}{r}$$

where $A_{\pm}$ is constant. We show that

$$A_{\pm} = \frac{1}{4\pi}.$$

We use the formula

$$\nabla^2(fg) = 2\nabla g \cdot \nabla f + f\nabla^2 g + g\nabla^2 f,$$

with

$$f = \frac{1}{r}, \quad \text{and} \quad g = A_{\pm} e^{\pm ikr}.$$

Then

$$\begin{aligned} \nabla^2(G_{\pm}) &= 2A_{\pm} \nabla e^{\pm ikr} \cdot \nabla \frac{1}{r} + \frac{A_{\pm}}{r} \nabla^2 e^{\pm ikr} + A_{\pm} e^{\pm ikr} \nabla^2 \frac{1}{r} \\ &= A_{\pm} e^{\pm ikr} [2\mathbf{e}_r(\pm ikr) \cdot (-\mathbf{e}_r \frac{1}{r^2}) + \frac{1}{r} (\pm \frac{2ik}{r} - k^2) - 4\pi\delta(\mathbf{r})] \\ &= A_{\pm} e^{\pm ikr} [-\frac{k^2}{r} - 4\pi\delta(\mathbf{r})] \end{aligned}$$

or

$$(\nabla^2 + k^2)G_{\pm} = -4\pi\delta(\mathbf{r})A_{\pm}e^{\pm ikr} = -4\pi\delta(\mathbf{r})A_{\pm} = -\delta(\mathbf{r}).$$

Then we have

$$A_{\pm} = \frac{1}{4\pi}.$$

Note that

$$\nabla^2(e^{\pm ikr}) = (\pm \frac{2ik}{r} - k^2)e^{\pm ikr}$$

$$\nabla\left(\frac{1}{r}\right) = \mathbf{e}_r \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\mathbf{e}_r \frac{1}{r^2},$$

$$\nabla(e^{\pm ikr}) = \mathbf{e}_r(\pm ikr)e^{\pm ikr}$$

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$$

15.7 Derivation of the Green's function from the Green's theorem

$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi) d\tau = \int_S (\psi \nabla \phi - \phi \nabla \psi) \cdot d\mathbf{a}$$

((**Proof**)) In the Gauss's theorem, we put

$$\mathbf{A} = \psi \nabla \phi$$

Then we have

$$I_1 = \int_V \nabla \cdot \mathbf{A} d\tau = \int_V \nabla \cdot (\psi \nabla \phi) d\tau = \int_S (\psi \nabla \phi) \cdot d\mathbf{a}.$$

Noting that

$$\nabla \cdot (\psi \nabla \phi) = \psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi,$$

we have

$$I_1 = \int_V (\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi) d\tau = \int_S (\psi \nabla \phi) \cdot d\mathbf{a}.$$

By replacing $\psi \leftrightarrow \phi$, we also have

$$I_2 = \int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d\tau = \int_S (\phi \nabla \psi) \cdot d\mathbf{a},$$

Thus we find the Green's theorem

$$I_1 - I_2 = \int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi) d\tau = \int_S (\psi \nabla \phi - \phi \nabla \psi) \cdot d\mathbf{a},$$

or

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d\tau = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} da,$$

We now consider the formula

$$\int_V [\phi(\mathbf{r}') \nabla'^2 \psi(\mathbf{r}') - \psi(\mathbf{r}') \nabla'^2 \phi(\mathbf{r}')] d^3 \mathbf{r}' = \int_S [\phi(\mathbf{r}') \nabla' \psi(\mathbf{r}') - \psi(\mathbf{r}') \nabla' \phi(\mathbf{r}')] \cdot \mathbf{n} da',$$

where $\mathbf{r}$ is the observation point and $\mathbf{r}'$ is the integration variable. Here we choose

$$\nabla'^2 \phi(\mathbf{r}') = -\frac{\rho(\mathbf{r}')}{\epsilon_0},$$

$$\psi = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} = G(\mathbf{r}, \mathbf{r}'),$$

$$\nabla'^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'),$$

Then we have

$$\int_V [-\delta(\mathbf{r} - \mathbf{r}') \phi(\mathbf{r}') + \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \frac{\rho(\mathbf{r}')}{\epsilon_0}] d^3 \mathbf{r}' = \int_S [\phi(\mathbf{r}') \nabla' \left(\frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \right) - \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \nabla' \phi(\mathbf{r}')] \cdot \mathbf{n} da'$$

If the point $\mathbf{r}$ lies in the volume V , we obtain

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\rho(\mathbf{r}')}{\epsilon_0} d^3 \mathbf{r}' + \frac{1}{4\pi} \int_S \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla' \phi(\mathbf{r}') - \phi(\mathbf{r}') \nabla' \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] \cdot \mathbf{n} da'$$

15.8 3D Green's function (modified Helmholtz)

We now consider the form of the Green's function (modified Helmholtz)

$$(\Delta - k^2)G_0(\mathbf{r}) = -\delta(\mathbf{r}).$$

The Fourier transform of $G_0(\mathbf{r})$ is defined as

$$G_0(\mathbf{q}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} G_0(\mathbf{r}).$$

The inverse Fourier transform is also defined as

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{q}),$$

where

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}}.$$

Then

$$\begin{aligned} \frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} (\Delta - k^2) e^{i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{q}) &= \frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} (-q^2 - k^2) e^{i\mathbf{q}\cdot\mathbf{r}} G_0(\mathbf{q}) \\ &= -\delta(\mathbf{r}) = -\frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}}, \end{aligned}$$

or

$$G_0(\mathbf{q}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{q^2 + k^2}.$$

Thus the Green's function is rewritten as

$$\begin{aligned} G_0(\mathbf{r}) &= \frac{1}{(2\pi)^3} \int d\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{r}} \frac{1}{q^2 + k^2} \\ G_0(\mathbf{r}) &= \frac{1}{(2\pi)^3} \iint 2\pi q^2 dq \sin\theta d\theta e^{iqr \cos\theta} \frac{1}{q^2 + k^2}. \end{aligned}$$

Note

$$\int \sin\theta d\theta e^{iqr \cos\theta} = -\frac{i}{qr} [e^{iqr} - e^{-iqr}],$$

we have

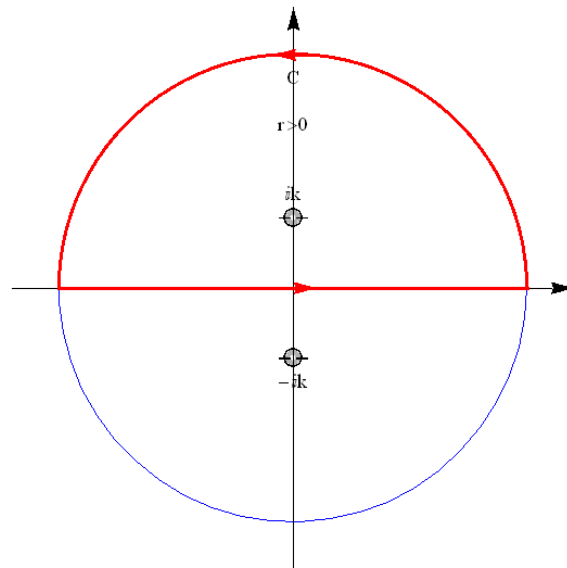
$$G_0(\mathbf{r}) = G_0(r) = \frac{1}{(2\pi)^3} \int q^2 dq \left(-\frac{2\pi i}{qr}\right) \frac{e^{iqr} - e^{-iqr}}{q^2 + k^2},$$

since $G_0(\mathbf{r})$ depends only on r . This equation can be rewritten as

$$G_0(r) = \frac{1}{4\pi^2 i r} \int_0^\infty q dq \frac{e^{iqr} - e^{-iqr}}{q^2 + k^2} = \frac{1}{4\pi^2 i r} \int_{-\infty}^\infty q dq \frac{e^{iqr}}{q^2 + k^2}$$

This function has a single pole at

$$q = \pm ik$$



Since $r > 0$, the path of integration can be closed by an infinite semicircle in the upper half-plane (Jordan's lemma).

$$G_0(r) = \frac{1}{4\pi^2 i r} \oint_C q dq \frac{e^{iqr}}{q^2 + k^2} = \frac{1}{4\pi^2 i r} 2\pi i \operatorname{Res}(q = ik) = \frac{1}{4\pi r} e^{-kr}$$

15.9 Derivation of Green's function (modified Helmholtz)

$$(\nabla^2 - k^2)G(r) = -\delta(\mathbf{r})$$

$$\nabla^2 G(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial G(r)}{\partial r} \right) = \frac{1}{r} \frac{\partial}{\partial r^2} (rG(r))$$

$$\frac{1}{r} \frac{\partial}{\partial r^2} (rG(r)) - k^2 G(r) = -\delta(\mathbf{r})$$

For $r \neq 0$,

$$\frac{\partial}{\partial r^2} (rG(r)) - k^2 rG(r) = 0$$

Then we have

$$G_{\pm}(r) = A_{\pm} \frac{e^{\pm kr}}{r}$$

Here we show that

$$A_{\pm} = \frac{1}{4\pi}.$$

((Proof))

$$\begin{aligned} \nabla^2(G_{\pm}) &= 2A_{\pm} \nabla e^{\pm kr} \cdot \nabla \frac{1}{r} + \frac{A_{\pm}}{r} \nabla^2 e^{\pm kr} + A_{\pm} e^{\pm kr} \nabla^2 \frac{1}{r} \\ &= A_{\pm} e^{\pm kr} [2\mathbf{e}_r(\pm kr) \cdot (-\mathbf{e}_r \frac{1}{r^2}) + \frac{1}{r} (\pm \frac{2k}{r} + k^2) - 4\pi\delta(\mathbf{r})] \\ &= A_{\pm} e^{\pm kr} [\frac{k^2}{r} - 4\pi\delta(\mathbf{r})] \end{aligned}$$

or

$$(\nabla^2 - k^2)G_{\pm} = -4\pi\delta(\mathbf{r})A_{\pm}e^{\pm kr} = -4\pi\delta(\mathbf{r})A_{\pm} = -\delta(\mathbf{r}).$$

Then we have

$$A_{\pm} = \frac{1}{4\pi}.$$

Note that

$$\nabla^2(e^{\pm kr}) = (\pm \frac{2k}{r} + k^2)e^{\pm kr}$$

$$\nabla(\frac{1}{r}) = \mathbf{e}_r \frac{\partial}{\partial r}(\frac{1}{r}) = -\mathbf{e}_r \frac{1}{r^2},$$

$$\nabla(e^{\pm kr}) = \mathbf{e}_r(\pm kr)e^{\pm kr}$$

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r})$$

15.10 Classical electrodynamics

Using

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0},$$

and

$$\mathbf{E} = -\nabla \phi,$$

we obtain a Poisson equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}.$$

The solution of ϕ can be obtained using the Green's function

$$\phi = \int G(\mathbf{r}, \mathbf{r}') \frac{\rho(\mathbf{r}')}{\epsilon_0} d^3 \mathbf{r}',$$

$$\nabla^2 \phi = \int \nabla^2 G(\mathbf{r}, \mathbf{r}') \frac{\rho(\mathbf{r}')}{\epsilon_0} d^3 \mathbf{r}' = -\frac{\rho}{\epsilon_0},$$

where

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'),$$

with

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|}.$$

Then we have

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'.$$

15.11 Ampere's law and Biot-Savart law

We start with the Maxwell's equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$

$$\nabla \cdot \mathbf{B} = 0.$$

$\mathbf{B}$ is expressed in terms of the vector potential $\mathbf{A}$ as

$$\mathbf{B} = \nabla \times \mathbf{A},$$

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J},$$

Here we choose a Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0.$$

Then we get

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.$$

The solution of $\mathbf{A}$ is obtained using the Green's function as

$$\mathbf{A}(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}') \mu_0 \mathbf{J}(\mathbf{r}') d^3 \mathbf{r}',$$

$$\nabla^2 \mathbf{A} = \int \nabla^2 G(\mathbf{r}, \mathbf{r}') \mu_0 \mathbf{J}(\mathbf{r}') d^3 \mathbf{r}' = -\mu_0 \mathbf{J},$$

where

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'),$$

with

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}'|}.$$

Then we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'.$$

$\mathbf{B}$ can be calculated as

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) d^3 \mathbf{r}'.$$

Since

$$\nabla \times \left(\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \right) = -\mathbf{J}(\mathbf{r}') \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\mathbf{J}(\mathbf{r}') \times \left(-\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right),$$

we have the Bio-Savart law,

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}'.$$

We note that

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left[\frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right] d^3\mathbf{r}'.$$

Here

$$\begin{aligned} \nabla \times \left[\frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right] &= \mathbf{J}(\mathbf{r}') \left[\nabla \cdot \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right] - [\mathbf{J}(\mathbf{r}') \cdot \nabla] \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \\ &= 4\pi \mathbf{J}(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') - [\mathbf{J}(\mathbf{r}') \cdot \nabla] \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \end{aligned}$$

Then we have

$$\begin{aligned} \nabla \times \mathbf{B} &= \frac{\mu_0}{4\pi} \int [4\pi \mathbf{J}(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') - (\mathbf{J}(\mathbf{r}') \cdot \nabla) \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}] d^3\mathbf{r}' \\ &= \mu_0 \mathbf{J}(\mathbf{r}) + \frac{\mu_0}{4\pi} \int (\mathbf{J}(\mathbf{r}') \cdot \nabla) \frac{(\mathbf{r} - \mathbf{r}')^3}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{r}' \\ &= \mu_0 \mathbf{J}(\mathbf{r}) - \frac{\mu_0}{4\pi} \int (\mathbf{J}(\mathbf{r}') \cdot \nabla') \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' \end{aligned}$$

Note that

$$(\mathbf{J}(\mathbf{r}') \cdot \nabla') \frac{(\mathbf{r} - \mathbf{r}')_x}{|\mathbf{r} - \mathbf{r}'|^3} = \nabla' \cdot \left[\frac{(\mathbf{r} - \mathbf{r}')_x}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{J}(\mathbf{r}') \right] - \frac{(\mathbf{r} - \mathbf{r}')_x}{|\mathbf{r} - \mathbf{r}'|^3} (\nabla' \cdot \mathbf{J}(\mathbf{r}')).$$

For the steady current, $\nabla' \cdot \mathbf{J}(\mathbf{r}') = 0$. Then

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu_0 \mathbf{J}(\mathbf{r}) - \frac{\mu_0}{4\pi} \int_V \left\{ \hat{x} \nabla' \cdot \left[\frac{(\mathbf{r} - \mathbf{r}')_x}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{J}(\mathbf{r}') \right] + \hat{y} \nabla' \cdot \left[\frac{(\mathbf{r} - \mathbf{r}')_y}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{J}(\mathbf{r}') \right] + \hat{z} \nabla' \cdot \left[\frac{(\mathbf{r} - \mathbf{r}')_z}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{J}(\mathbf{r}') \right] \right\} d^3\mathbf{r}' \\ &= \mu_0 \mathbf{J}(\mathbf{r}) - \frac{\mu_0}{4\pi} \int_A \left\{ \hat{x} \left[\frac{(\mathbf{r} - \mathbf{r}')_x}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{J}(\mathbf{r}') \right] + \hat{y} \left[\frac{(\mathbf{r} - \mathbf{r}')_y}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{J}(\mathbf{r}') \right] + \hat{z} \left[\frac{(\mathbf{r} - \mathbf{r}')_z}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{J}(\mathbf{r}') \right] \right\} d\mathbf{a}' \\ &= \mu_0 \mathbf{J}(\mathbf{r}) \end{aligned}$$

where we use the Gauss's law and we assume that $\mathbf{J}(\mathbf{r}') = 0$ on the surface A (large enough to include all the currents).

((Note)) formula

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times \nabla f ,$$

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f ,$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) ,$$

$$\nabla \frac{1}{r} = -\frac{\mathbf{r}}{r^3} ,$$

$$\nabla \cdot \nabla \frac{1}{r} = \nabla^2 \frac{1}{r} = -\nabla \cdot \frac{\mathbf{r}}{r^3} = -4\pi\delta(\mathbf{r} - \mathbf{r}') .$$

15.12 Electromagnetism and d'Alembertian operator

15.12.1 Maxwell's equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$

where

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} .$$

15.12.2 Vector potential $\mathbf{A}$ and scalar potential ϕ

$$\mathbf{B} = \nabla \times \mathbf{A} ,$$

since $\nabla \cdot \mathbf{B} = 0$.

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{A} ,$$

or

$$\nabla \times (\mathbf{E} + \frac{\partial}{\partial t} \mathbf{A}) = 0,$$

or

$$\mathbf{E} + \frac{\partial}{\partial t} \mathbf{A} = -\nabla \phi.$$

Then we have

$$\mathbf{E} = -\frac{\partial}{\partial t} \mathbf{A} - \nabla \phi, \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

We now calculate

$$\begin{aligned} \nabla \times \mathbf{B} &= \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\ &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left(-\frac{\partial}{\partial t} \mathbf{A} - \nabla \phi \right) \end{aligned}$$

or

$$(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}) \mathbf{A} = -\mu_0 \mathbf{J} + \nabla(\nabla \cdot \mathbf{A} + \mu_0 \varepsilon_0 \frac{\partial \phi}{\partial t})$$

Similarly we have

$$\nabla \cdot \left(-\frac{\partial}{\partial t} \mathbf{A} - \nabla \phi \right) = \frac{\rho}{\varepsilon_0}$$

or

$$-\nabla^2 \phi + \varepsilon_0 \mu_0 \frac{\partial^2 \phi}{\partial t^2} - \varepsilon_0 \mu_0 \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = \frac{\rho}{\varepsilon_0}$$

or

$$(\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}) \phi = -\frac{\rho}{\varepsilon_0} - \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A} + \varepsilon_0 \mu_0 \frac{\partial \phi}{\partial t})$$

15.12.3 Gauge transformation

We have a gauge transformation

$$\mathbf{A}' = \mathbf{A} + \nabla \lambda,$$

$$\phi' = \phi - \frac{\partial \lambda}{\partial t},$$

where

$$\mathbf{E} = -\frac{\partial}{\partial t} \mathbf{A} - \nabla \phi \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Let us calculate

$$-\frac{\partial}{\partial t} \mathbf{A}' - \nabla \phi' = -\frac{\partial}{\partial t} (\mathbf{A} + \nabla \chi) - \nabla (\phi - \frac{\partial \chi}{\partial t}) = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi = \mathbf{E},$$

$$\nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} + \nabla \chi) = \nabla \times \mathbf{A}.$$

Therefore $(\mathbf{A}', \phi')$ and $(\mathbf{A}, \phi)$ gives the same expression for $\mathbf{E}$ and $\mathbf{B}$.

We adopt the Lorentz gauge

$$\nabla \cdot \mathbf{A} + \varepsilon_0 \mu_0 \frac{\partial \phi}{\partial t} = 0 \quad (\text{Lorentz gauge})$$

Then we have

$$(\nabla^2 - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}) \mathbf{A} = -\mu_0 \mathbf{J}$$

or

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \mathbf{A} = -\mu_0 \mathbf{J}$$

and

$$(\nabla^2 - \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}) \phi = -\frac{\rho}{\varepsilon_0}$$

or

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})\phi = -\frac{\rho}{\epsilon_0}$$

The d'Alembertian operator or simply d'Alembertian is the differential operator

$$\square = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}.$$

Using this notation, we get the following expression

$$\square \phi = -\frac{\rho}{\epsilon_0}, \quad \text{and} \quad \square \mathbf{A} = -\mu_0 \mathbf{J}$$

The Green's function associated with the d'Alembertian satisfies the differential equation

$$\square G(\mathbf{r}, t) = -\delta(\mathbf{r})\delta(t).$$

15.12.4. Fourier transform

Fourier transform

$$\mathbf{A}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{A}(\mathbf{r}, t) e^{i\omega t} dt$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{A}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

$$\mathbf{J}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{J}(\mathbf{r}, t) e^{i\omega t} dt$$

$$\mathbf{J}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{J}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

$$\phi(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\mathbf{r}, t) e^{i\omega t} dt$$

$$\phi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\mathbf{r}, \omega) e^{-i\omega t} d\omega,$$

$$\rho(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(\mathbf{r}, t) e^{i\omega t} dt$$

$$\rho(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(\mathbf{r}, \omega) e^{-i\omega t} d\omega,$$

15.13 Retarded vector potential $\mathbf{A}(\mathbf{r}, t)$

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \mathbf{A} = -\mu_0 \mathbf{J}$$

where

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{A}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

$$\mathbf{J}(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{J}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

Then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \mathbf{A}(\mathbf{r}, \omega) e^{-i\omega t} d\omega = -\mu_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{J}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

or

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\nabla^2 + \frac{\omega^2}{c^2}) \mathbf{A}(\mathbf{r}, \omega) e^{-i\omega t} d\omega = -\mu_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{J}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

or

$$(\nabla^2 + \frac{\omega^2}{c^2}) \mathbf{A}(\mathbf{r}, \omega) = -\mu_0 \mathbf{J}(\mathbf{r}, \omega)$$

We can solve this using the Green's function (Helmholtz)

$$\mathbf{A}(\mathbf{r}, \omega) = \int G(\mathbf{r}, \mathbf{r}') \mu_0 \mathbf{J}(\mathbf{r}', \omega) d^3 \mathbf{r}'$$

with

$$(\nabla^2 + \frac{\omega^2}{c^2}) G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$$

$$G(\mathbf{r}, \mathbf{r}') = \frac{\exp[i\frac{\omega}{c}|\mathbf{r} - \mathbf{r}'|]}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

Thus we have

$$\mathbf{A}(\mathbf{r}, \omega) = \mu_0 \int d^3\mathbf{r}' \frac{\exp[i\frac{\omega}{c}|\mathbf{r} - \mathbf{r}'|]}{4\pi|\mathbf{r} - \mathbf{r}'|} \mathbf{J}(\mathbf{r}', \omega)$$

or

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \mu_0 \int d^3\mathbf{r}' \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \frac{1}{\sqrt{2\pi}} \int d\omega \exp[-i\omega(t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|)] \mathbf{J}(\mathbf{r}', \omega) \\ &= \frac{\mu_0}{4\pi} \int d^3\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}', t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} \end{aligned}$$

The retarded time is defined as

$$t_r = t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|.$$

15.14 Retarded potential $\phi(\mathbf{r}, t)$

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})\phi = -\frac{\rho}{\epsilon_0}$$

where

$$\phi(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

$$\rho(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

Then

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \phi(\mathbf{r}, \omega) e^{-i\omega t} d\omega = -\frac{1}{\epsilon_0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

or

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\nabla^2 + \frac{\omega^2}{c^2}) \phi(\mathbf{r}, \omega) e^{-i\omega t} d\omega = -\frac{1}{\varepsilon_0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

or

$$(\nabla^2 + \frac{\omega^2}{c^2}) \phi(\mathbf{r}, \omega) = -\frac{1}{\varepsilon_0} \rho(\mathbf{r}, \omega)$$

We can solve this using the Green's function (Helmholtz)

$$\phi(\mathbf{r}, \omega) = \int G(\mathbf{r}, \mathbf{r}') \frac{1}{\varepsilon_0} \rho(\mathbf{r}', \omega) d^3 \mathbf{r}'$$

with

$$(\nabla^2 + \frac{\omega^2}{c^2}) G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$$

$$G(\mathbf{r}, \mathbf{r}') = \frac{\exp[i \frac{\omega}{c} |\mathbf{r} - \mathbf{r}'|]}{4\pi |\mathbf{r} - \mathbf{r}'|}$$

Thus we have

$$\phi(\mathbf{r}, \omega) = \frac{1}{4\pi\varepsilon_0} \int d^3 \mathbf{r}' \frac{\exp[i \frac{\omega}{c} |\mathbf{r} - \mathbf{r}'|]}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}', \omega)$$

or

$$\begin{aligned} \phi(\mathbf{r}, t) &= \frac{1}{4\pi\varepsilon_0} \int d^3 \mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{1}{\sqrt{2\pi}} \int d\omega \exp[-i\omega(t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'|)] \rho(\mathbf{r}', \omega) \\ &= \frac{1}{4\pi\varepsilon_0} \int d^3 \mathbf{r}' \frac{\rho(\mathbf{r}', t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} \end{aligned}$$

15.15 Jefimenko's equation

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} - \nabla\phi \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

For simplicity , we define

$$t_r = t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|, \quad \text{and} \quad \mathbf{R} = \mathbf{r} - \mathbf{r}'$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho(\mathbf{r}', t_r)\mathbf{R}}{R^3} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{cR^2}\mathbf{R} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2 R} \right] d^3\mathbf{r}'$$

and

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{R^3} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{cR^2} \right] \times \mathbf{R} d^3\mathbf{r}' .$$

15.16 Green's function of the d'Alembertian

We consider the equation

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})\phi = -\frac{\rho}{\epsilon_0}$$

The Green's function of the d'Alembertian is defined as

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2})G(\mathbf{r}, t) = -\delta(\mathbf{r})\delta(t)$$

Here we introduce the Fourier transforms,

$$G(\mathbf{r}, t) = \frac{1}{(\sqrt{2\pi})^4} \int d\mathbf{k} \int_{-\infty}^{\infty} G(\mathbf{k}, \omega) e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} d\omega$$

$$\delta(\mathbf{r})\delta(t) = \frac{1}{(2\pi)^4} \int e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega$$

Then

$$\frac{1}{(2\pi)^2} \iiint (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) G(\mathbf{k}, \omega) e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} d\mathbf{k} d\omega = -\frac{1}{(2\pi)^4} \iiint e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} d\mathbf{k} d\omega$$

or

$$(-\mathbf{k}^2 + \frac{\omega^2}{c^2}) G(\mathbf{k}, \omega) = -\frac{1}{(2\pi)^2},$$

The solution to this equation is

$$G(\mathbf{k}, \omega) = \frac{1}{(2\pi)^2} \frac{1}{\frac{\omega^2}{c^2} - \mathbf{k}^2}.$$

The inverse Fourier transform:

$$G(\mathbf{r}, t) = \frac{1}{(2\pi)^2} \frac{1}{(2\pi)^2} \iiint d\mathbf{k} d\omega \frac{e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}}{\frac{\omega^2}{c^2} - \mathbf{k}^2}$$

For convenience, we assume that the direction of $\mathbf{r}$ is the z axis. The angle between $\mathbf{r}$ and $\mathbf{k}$ is θ .

$$\mathbf{k} \cdot \mathbf{r} = kr \cos \theta,$$

$$d\mathbf{k} = 2\pi k^2 dk \sin \theta d\theta,$$

$$G(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \iiint d\mathbf{k} d\omega \frac{e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}}{\frac{\omega^2}{c^2} - \mathbf{k}^2}$$

$$G(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega \int_0^{\infty} 2\pi k^2 dk \int_0^{\pi} \sin \theta d\theta e^{ikr \cos \theta} \frac{1}{\frac{\omega^2}{c^2} - k^2},$$

Since

$$\int \sin \theta d\theta e^{ikr \cos \theta} = -\frac{i}{kr} (e^{ikr} - e^{-ikr}),$$

$$\begin{aligned}
G(\mathbf{r}, t) &= \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega \int_0^{\infty} 2\pi k^2 dk \frac{1}{\frac{\omega^2}{c^2} - k^2} \left(-\frac{i}{kr}\right) (e^{ikr} - e^{-ikr}) \\
&= \frac{1}{(2\pi)^3} \left(-\frac{i}{r}\right) \int_0^{\infty} k dk (e^{ikr} - e^{-ikr}) \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\frac{\omega^2}{c^2} - k^2} d\omega, \\
&= \frac{1}{(2\pi)^3} \left(-\frac{c^2 i}{r}\right) \int_0^{\infty} k dk (e^{ikr} - e^{-ikr}) \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2 - c^2 k^2}
\end{aligned}$$

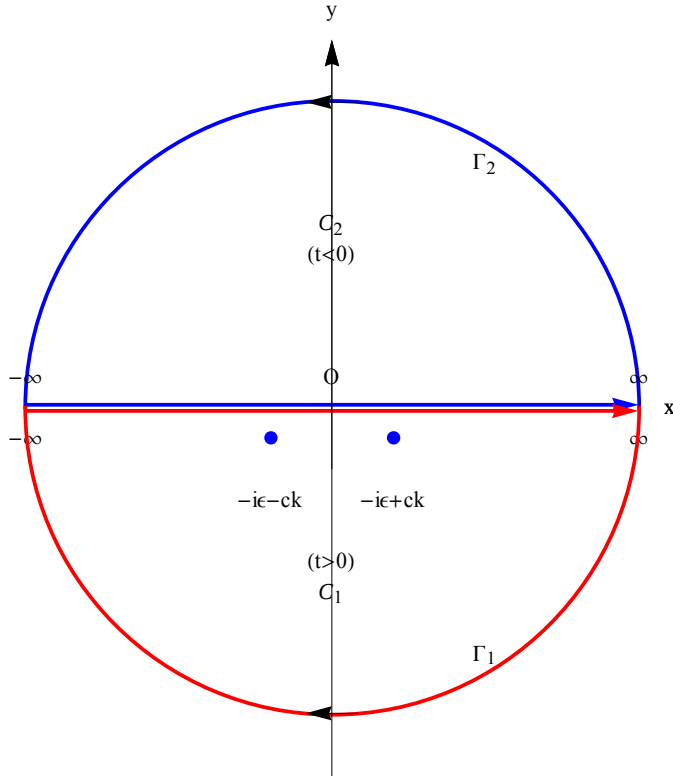
where

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2 - c^2 k^2} \\
&= \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \frac{1}{2ck} \left(\frac{1}{\omega - ck} - \frac{1}{\omega + ck} \right) \\
&= \oint dz e^{-izt} \frac{1}{2ck} \left(\frac{1}{z - ck} - \frac{1}{z + ck} \right)
\end{aligned}$$

(i) Retarded Green function

We calculate the intergral given by

$$I_1 = \oint dz e^{-izt} \frac{1}{2ck} \left(\frac{1}{z - ck + i\varepsilon} - \frac{1}{z + ck + i\varepsilon} \right)$$



For positive value of t , we need to choose the contour C_1 in the lower half plane. The complex exponential $\exp(-izt)$ only decays at infinity if the imaginary of z is negative. According to the Jordan's lemma, the integral along the path Γ_1 is zero. There are two simple poles inside the contour C_1 . Since the path is taken with the clock-wise (negative) direction, we find that for $t > 0$,

$$\begin{aligned}
 I_1 &= \oint_{C_1} dz e^{-izt} \frac{1}{2ck} \left(\frac{1}{z - ck + i\varepsilon} - \frac{1}{z + ck + i\varepsilon} \right) \\
 &= \frac{-2\pi i}{2ck} [\text{Re } s(z = ck - i\varepsilon) - \text{Re } s(z = -ck - i\varepsilon)] \\
 &= \frac{\pi i}{ck} (e^{ickt} - e^{-ickt})
 \end{aligned}$$

Then we have

$$\begin{aligned}
G_{ret}(\mathbf{r}, t) &= -\frac{1}{(2\pi)^2} \frac{c}{2r} \int_0^\infty dk (e^{ikr} - e^{-ikr})(e^{-ickt} - e^{ickt}) \\
&= -\frac{1}{(2\pi)^2} \frac{c}{2r} \int_0^\infty dk [e^{ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)} + e^{-ik(r-ct)}] \\
&= -\frac{1}{(2\pi)^2} \frac{c}{2r} \int_0^\infty dk [e^{ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)} + e^{-ik(r-ct)}] \\
&= -\frac{1}{(2\pi)^2} \frac{c}{2r} \int_{-\infty}^\infty dk [e^{ik(r-ct)} - e^{ik(r+ct)}] \\
&= -\frac{c}{4\pi r} [\delta(r-ct) - \delta(r+ct)]
\end{aligned}$$

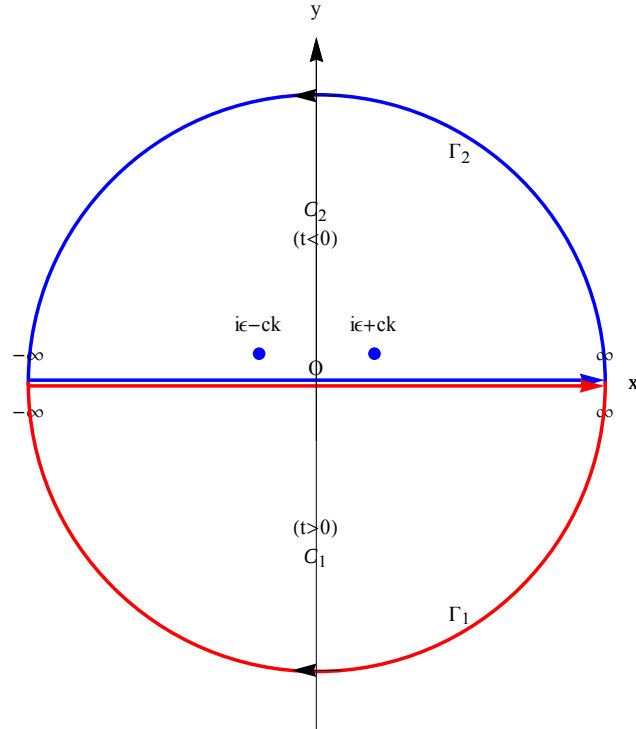
Since $r > 0$ and $t > 0$, we have

$$G_{ret}(\mathbf{r}, t) = -\frac{c}{4\pi r} \pi [\delta(r-ct)].$$

For negative value of t , we need to choose the contour C_2 in the upper half plane. There is no poles inside the contour C_2 . Then we find that for $t < 0$,

$$G_{ret}(\mathbf{r}, t) = 0$$

(ii) Advanced Green function



For positive value of t , we need to choose the contour C_1 in the lower half plane. There is no pole inside the contour C_1 . So we find that for $t > 0$,

$$I_2 = \oint_{C_1} dz e^{-izt} \frac{1}{2ck} \left(\frac{1}{z - ck - i\varepsilon} - \frac{1}{z + ck - i\varepsilon} \right) = 0$$

or

$$G_{adv}(\mathbf{r}, t) = 0$$

For negative value of t , we need to choose the contour C_2 in the upper half plane. The complex exponential $\exp(-izt)$ only decays at infinity if the imaginary of z is positive. According to the Jordan's lemma, the integral along the path Γ_2 is zero. There are two simple poles inside the contour C_2 . Since the path is taken with the clock-wise (positive) direction, we find that for $t < 0$,

$$\begin{aligned}
I_2 &= \oint_{C_2} dz e^{-izt} \frac{1}{2ck} \left(\frac{1}{z - ck - i\varepsilon} - \frac{1}{z + ck - i\varepsilon} \right) \\
&= \frac{2\pi i}{2ck} [\text{Re } s(z = ck + i\varepsilon) - \text{Re } s(z = -ck + i\varepsilon)] \\
&= \frac{\pi i}{ck} (e^{-ickt} - e^{ickt})
\end{aligned}$$

$$\begin{aligned}
G_{adv}(\mathbf{r}, t) &= \frac{1}{(2\pi)^2} \frac{c}{2r} \int_0^\infty dk (e^{ikr} - e^{-ikr}) (e^{-ickt} - e^{ickt}) \\
&= \frac{1}{(2\pi)^2} \frac{c}{2r} \int_0^\infty dk [e^{ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)} + e^{-ik(r-ct)}] \\
&= \frac{1}{(2\pi)^2} \frac{c}{2r} \int_0^\infty dk [e^{ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)} + e^{-ik(r-ct)}] \\
&= \frac{1}{(2\pi)^2} \frac{c}{2r} \int_{-\infty}^\infty dk [e^{ik(r-ct)} - e^{ik(r+ct)}] \\
&= \frac{c}{4\pi r} [\delta(r-ct) - \delta(r+ct)]
\end{aligned}$$

Since $r > 0$ and $t < 0$, we have

$$G_{adv}(\mathbf{r}, t) = -\frac{c}{4\pi r} \pi [\delta(r+ct)].$$

((Note))

The retarded Green's function is represented by a spherical shell emitted at $t = 0$ and with increasing radius $r = ct$.

15.17 Green's function for the Klein-Gordon equation

We start with the Einstein's relation

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4$$

In quantum mechanics, we use the operators

$$E \rightarrow i\hbar \frac{\partial}{\partial t} \quad \mathbf{p} \rightarrow \frac{\hbar}{i} \nabla$$

The Green's function is defined by

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{m^2 c^2}{\hbar^2})G(\mathbf{r}, t) = -\delta(t)\delta(\mathbf{r})$$

The Green's function is expressed by the inverse Fourier transform as

$$G(\mathbf{r}, t) = \frac{1}{(2\pi)^{4/2}} \int e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} G(\mathbf{k}, \omega)$$

From the Klein-Gordon equation, we have

$$\begin{aligned} (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{m^2 c^2}{\hbar^2})G(\mathbf{r}, t) &= \frac{1}{(2\pi)^{4/2}} \int d^3\mathbf{k} d\omega [e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} (-k^2 + \frac{\omega^2}{c^2} - \frac{m^2 c^2}{\hbar^2}) G(\mathbf{k}, \omega)] \\ &= -\frac{1}{(2\pi)^4} \int d^3\mathbf{k} d\omega e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \end{aligned}$$

or

$$(-k^2 + \frac{\omega^2}{c^2} - \frac{m^2 c^2}{\hbar^2})G(\mathbf{k}, \omega) = -\frac{1}{(2\pi)^2}$$

or

$$G(\mathbf{k}, \omega) = \frac{1}{(2\pi)^2} \frac{1}{k^2 + \frac{m^2 c^2}{\hbar^2} - \frac{\omega^2}{c^2}}$$

Then

$$\begin{aligned}
G(\mathbf{r}, t) &= \frac{1}{(2\pi)^4} \int d^3\mathbf{k} \int d\omega e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \frac{1}{k^2 + \frac{m^2 c^2}{\hbar^2} - \frac{\omega^2}{c^2}} \\
&= \frac{1}{(2\pi)^4} \int e^{i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k} \int d\omega \frac{1}{k^2 + \frac{m^2 c^2}{\hbar^2} - \frac{\omega^2}{c^2}} \\
&= -\frac{c^2}{(2\pi)^4} \int e^{i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{k} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2 - \omega_0^2}
\end{aligned}$$

where

$$\omega_0 = (c^2 k^2 + \frac{m^2 c^4}{\hbar^2})^{1/2} = \frac{1}{\hbar} (\hbar^2 c^2 k^2 + m^2 c^4) = \frac{1}{\hbar} E_0$$

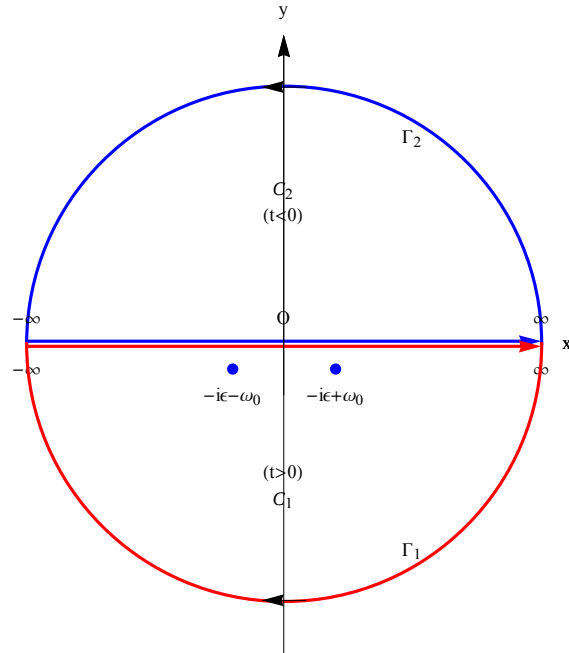
We now calculate

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2 - \omega_0^2} \\
&= \frac{1}{2\omega_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left(\frac{1}{\omega - \omega_0} - \frac{1}{\omega + \omega_0} \right)
\end{aligned}$$

(i) Retarded case

$$I_1 = \frac{1}{2\omega_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left(\frac{1}{\omega - \omega_0 + i\varepsilon} - \frac{1}{\omega + \omega_0 + i\varepsilon} \right)$$

Two simple poles are located in the lower half plane.



For $t > 0$

$$\begin{aligned}
 I_1 &= \frac{1}{2\omega_0} \oint_{C_1} dz e^{-izt} \left(\frac{1}{z - \omega_0 + i\varepsilon} - \frac{1}{z + \omega_0 + i\varepsilon} \right) \\
 &= \frac{1}{2\omega_0} (-2\pi i) (e^{-it\omega_0} - e^{it\omega_0}) \\
 &= -\frac{\pi i}{\omega_0} (e^{-it\omega_0} - e^{it\omega_0})
 \end{aligned}$$

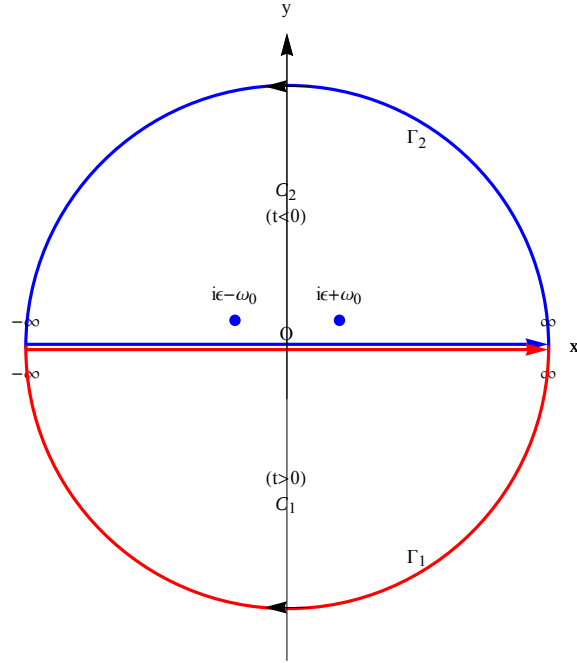
For $t < 0$

$$I_1 = \frac{1}{2\omega_0} \oint_{C_2} dz e^{-izt} \left(\frac{1}{z - \omega_0 + i\varepsilon} - \frac{1}{z + \omega_0 + i\varepsilon} \right) = 0$$

(ii) Advanced case

$$I_2 = \frac{1}{2\omega_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left(\frac{1}{\omega - \omega_0 - i\varepsilon} - \frac{1}{\omega + \omega_0 - i\varepsilon} \right)$$

There are two simple poles in the upper half plane.



For $t > 0$

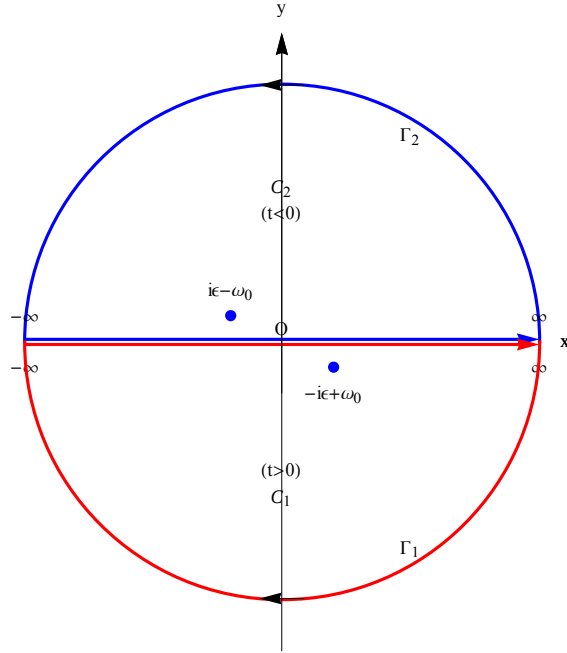
$$I_2 = \frac{1}{2\omega_0} \oint_{C_1} dz e^{-izt} \left(\frac{1}{z - \omega_0 - i\epsilon} - \frac{1}{z + \omega_0 - i\epsilon} \right) = 0$$

For $t < 0$,

$$\begin{aligned} I_2 &= \frac{1}{2\omega_0} \oint_{C_2} dz e^{-izt} \left(\frac{1}{z - \omega_0 - i\epsilon} - \frac{1}{z + \omega_0 - i\epsilon} \right) \\ &= \frac{1}{2\omega_0} (2\pi i) (e^{-it\omega_0} - e^{it\omega_0}) \\ &= \frac{\pi i}{\omega_0} (e^{-it\omega_0} - e^{it\omega_0}) \end{aligned}$$

(iii)

$$I_3 = \frac{1}{2\omega_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left(\frac{1}{\omega - \omega_0 + i\epsilon} - \frac{1}{\omega + \omega_0 - i\epsilon} \right)$$



For $t > 0$,

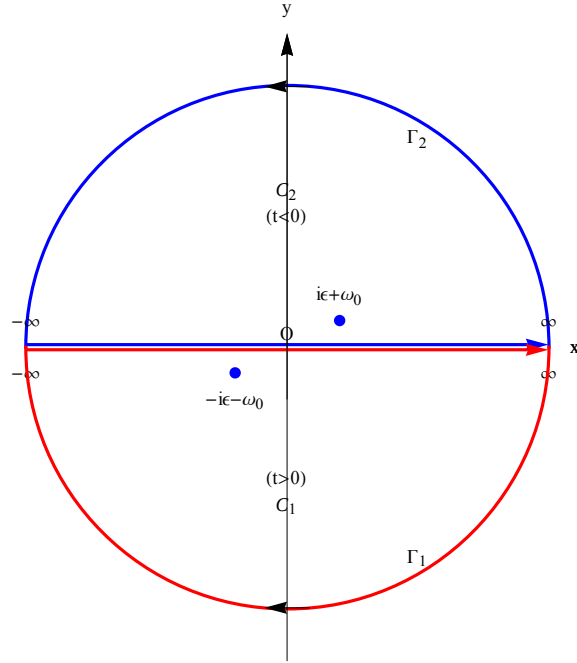
$$\begin{aligned}
 I_3 &= \frac{1}{2\omega_0} \oint_{C_1} dz e^{-izt} \left(\frac{1}{z - \omega_0 - i\varepsilon} - \frac{1}{z + \omega_0 + i\varepsilon} \right) \\
 &= \frac{1}{2\omega_0} (-2\pi i) (-e^{-it\omega_0}) = \frac{\pi i}{\omega_0} e^{-it\omega_0}
 \end{aligned}$$

For $t < 0$,

$$\begin{aligned}
 I_3 &= \frac{1}{2\omega_0} \oint_{C_2} dz e^{-izt} \left(\frac{1}{z - \omega_0 - i\varepsilon} - \frac{1}{z + \omega_0 + i\varepsilon} \right) \\
 &= \frac{1}{2\omega_0} (2\pi i) (e^{it\omega_0}) = \frac{\pi i}{\omega_0} e^{it\omega_0}
 \end{aligned}$$

(iv) Feynman propagator

$$I_4 = \frac{1}{2\omega_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left(\frac{1}{\omega - \omega_0 - i\varepsilon} - \frac{1}{\omega + \omega_0 + i\varepsilon} \right)$$

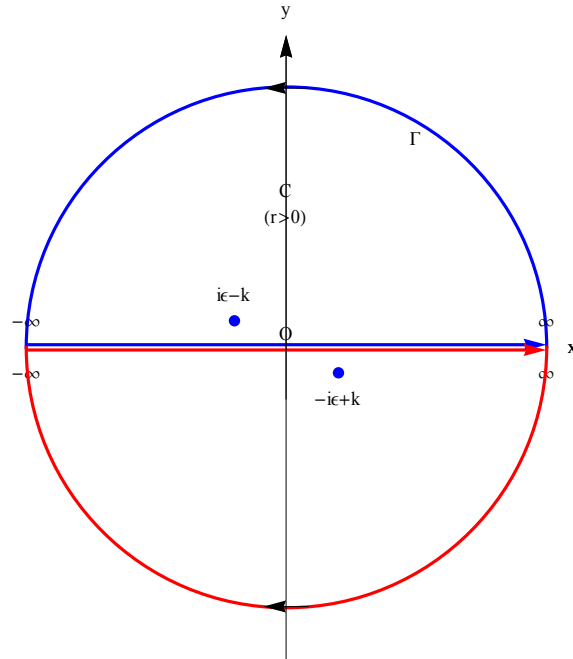


For $t > 0$,

$$\begin{aligned}
 I_4 &= \frac{1}{2\omega_0} \oint_{C_1} dz e^{-izt} \left(\frac{1}{z - \omega_0 - i\varepsilon} - \frac{1}{z + \omega_0 + i\varepsilon} \right) \\
 &= \frac{1}{2\omega_0} (-2\pi i) (-e^{it\omega_0}) = \frac{\pi i}{\omega_0} e^{it\omega_0}
 \end{aligned}$$

For $t < 0$,

$$\begin{aligned}
 I_4 &= \frac{1}{2\omega_0} \oint_{C_2} dz e^{-izt} \left(\frac{1}{z - \omega_0 - i\varepsilon} - \frac{1}{z + \omega_0 + i\varepsilon} \right) \\
 &= \frac{1}{2\omega_0} (2\pi i) (e^{-it\omega_0}) = \frac{\pi i}{\omega_0} e^{-it\omega_0}
 \end{aligned}$$



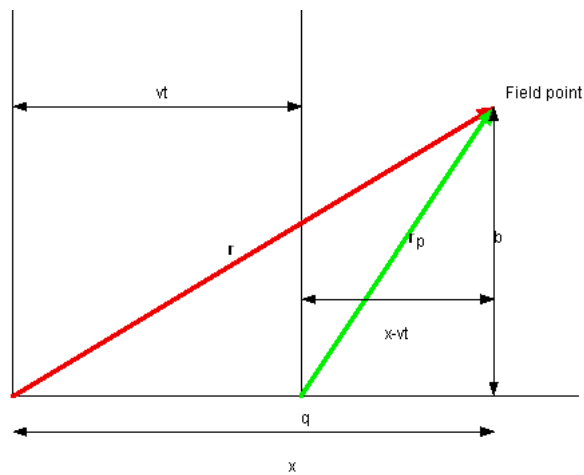
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APPENDIX.

A.1 Lienard-Wiechert potential

We now consider the Lienard-Wiechert potential



In the S' frame:

$$\phi' = \frac{q}{4\pi\epsilon_0} \frac{1}{r'}$$

$$\mathbf{A}' = 0$$

$$\begin{aligned} A_1 &= \frac{A_1' + \frac{v}{c^2} \phi'}{\sqrt{1 - \beta^2}} & A_1 &= \frac{\frac{v}{c^2} \phi'}{\sqrt{1 - \beta^2}} \\ A_2 &= A_2' & \text{or} & & A_2 &= 0 \\ A_3 &= A_3' & & & A_3 &= 0 \\ \phi &= \frac{v A_1' + \phi'}{\sqrt{1 - \beta^2}} & \phi &= \frac{\phi'}{\sqrt{1 - \beta^2}} = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{1 - \beta^2}} \frac{1}{r'} \end{aligned}$$

Then we get

$$\phi = \frac{\phi'}{\sqrt{1 - \beta^2}} = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{1 - \beta^2}} \frac{1}{\sqrt{x_1'^2 + x_2'^2 + x_3'^2}}$$

where

$$\begin{aligned} x_1' &= \gamma (x_1 - vt) \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}$$

The scalar potential ϕ is given by

$$\begin{aligned} \phi &= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{1 - \beta^2}} \frac{1}{\sqrt{\gamma^2 (x_1 - vt)^2 + x_2^2 + x_3^2}} \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(x_1 - vt)^2 + (1 - \beta^2)(x_2^2 + x_3^2)}} \end{aligned}$$

or

$$\phi = \frac{q}{4\pi\epsilon_0} \frac{1}{R^*}$$

with

$$R^* = \sqrt{(x_1 - vt)^2 + (1 - \beta^2)(x_2^2 + x_3^2)}$$

Similarly we have for the vector potential

$$\mathbf{A} = (A_1, 0, 0)$$

with

$$A_1 = \frac{\frac{v}{c^2} \phi'}{\sqrt{1 - \beta^2}} = \frac{v}{c^2} \frac{q}{4\pi\epsilon_0} \frac{1}{R^*} = \frac{qv\mu_0}{4\pi} \frac{1}{R^*}$$

The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are given by

$$\mathbf{E} = -\frac{\partial}{\partial t} \mathbf{A} - \nabla \phi \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} (1 - \beta^2) \frac{\mathbf{R}}{R^{*3}}$$

$$\mathbf{B} = \frac{v}{c^2} \times \mathbf{E}$$

where

$$\mathbf{R} = (x - vt, y, z)$$

((**Mathematica**))

```
Clear["Global`*"]
```

```
<< "VectorAnalysis`"
```

```
SetCoordinates[Cartesian[x, y, z]]
```

```
Cartesian[x, y, z]
```

$$R = \sqrt{(x - vt)^2 + (1 - \beta^2)(y^2 + z^2)}$$

$$\sqrt{(-tv + x)^2 + (y^2 + z^2)(1 - \beta^2)}$$

$$\phi = \frac{q}{4\pi\epsilon_0} \frac{1}{R}$$

$$\frac{q}{4\pi\sqrt{(-tv + x)^2 + (y^2 + z^2)(1 - \beta^2)}} \epsilon_0$$

$$A1 = \frac{qv\mu_0}{4\pi} \frac{1}{R}$$

$$\frac{qv\mu_0}{4\pi\sqrt{(-tv + x)^2 + (y^2 + z^2)(1 - \beta^2)}}$$

$$A = \{A1, 0, 0\}$$

$$\left\{ \frac{qv\mu_0}{4\pi\sqrt{(-tv + x)^2 + (y^2 + z^2)(1 - \beta^2)}}, 0, 0 \right\}$$

B1 = Curl[A] // FullSimplify

$$\left\{0, \frac{q v z (-1 + \beta^2) \mu_0}{4 \pi \left((-t v + x)^2 - (y^2 + z^2) (-1 + \beta^2) \right)^{3/2}}, \right. \\ \left. - \frac{q v y (-1 + \beta^2) \mu_0}{4 \pi \left((-t v + x)^2 - (y^2 + z^2) (-1 + \beta^2) \right)^{3/2}} \right\}$$

Electric field in the frame S

E1 = -Grad[phi] - D[A, t] /. {mu0 -> 1/(epsilon0 c^2)} // FullSimplify

$$\left\{ \frac{q (-c^2 + v^2) (t v - x)}{4 c^2 \pi \left((-t v + x)^2 - (y^2 + z^2) (-1 + \beta^2) \right)^{3/2} \epsilon_0}, \right. \\ - \frac{q y (-1 + \beta^2)}{4 \pi \left((-t v + x)^2 - (y^2 + z^2) (-1 + \beta^2) \right)^{3/2} \epsilon_0}, \\ \left. - \frac{q z (-1 + \beta^2)}{4 \pi \left((-t v + x)^2 - (y^2 + z^2) (-1 + \beta^2) \right)^{3/2} \epsilon_0} \right\}$$

V1 = {v, 0, 0}

{v, 0, 0}

eq1 = 1/c^2 Cross[V1, E1] // Simplify

$$\left\{0, \frac{q v z (-1 + \beta^2)}{4 c^2 \pi \left((-t v + x)^2 - (y^2 + z^2) (-1 + \beta^2) \right)^{3/2} \epsilon_0}, \right. \\ \left. - \frac{q v y (-1 + \beta^2)}{4 c^2 \pi \left((-t v + x)^2 - (y^2 + z^2) (-1 + \beta^2) \right)^{3/2} \epsilon_0} \right\}$$

eq1 - B1 /. {mu0 -> 1/(c^2 epsilon0)} // Simplify

{0, 0, 0}

A.2 Distribution of the electric field

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} (1 - \beta^2) \frac{\mathbf{R}_p}{R^{*3}}$$

where

$$R^* = \sqrt{(x - vt)^2 + (1 - \beta^2)(y^2 + z^2)}$$

$$\mathbf{R}^* = (x - vt, \sqrt{1 - \beta^2}y, \sqrt{1 - \beta^2}z)$$

$$\mathbf{R}_p = (x - vt, y, z)$$

$\mathbf{R}_p$ is the relative coordinate of the field point and the charge point. The electric field is along the position vector $\mathbf{R}_p$. $\mathbf{R}_p$ is a vector from the instantaneous location of the charge in S to the point where E is measured in S.

((Mathematica))

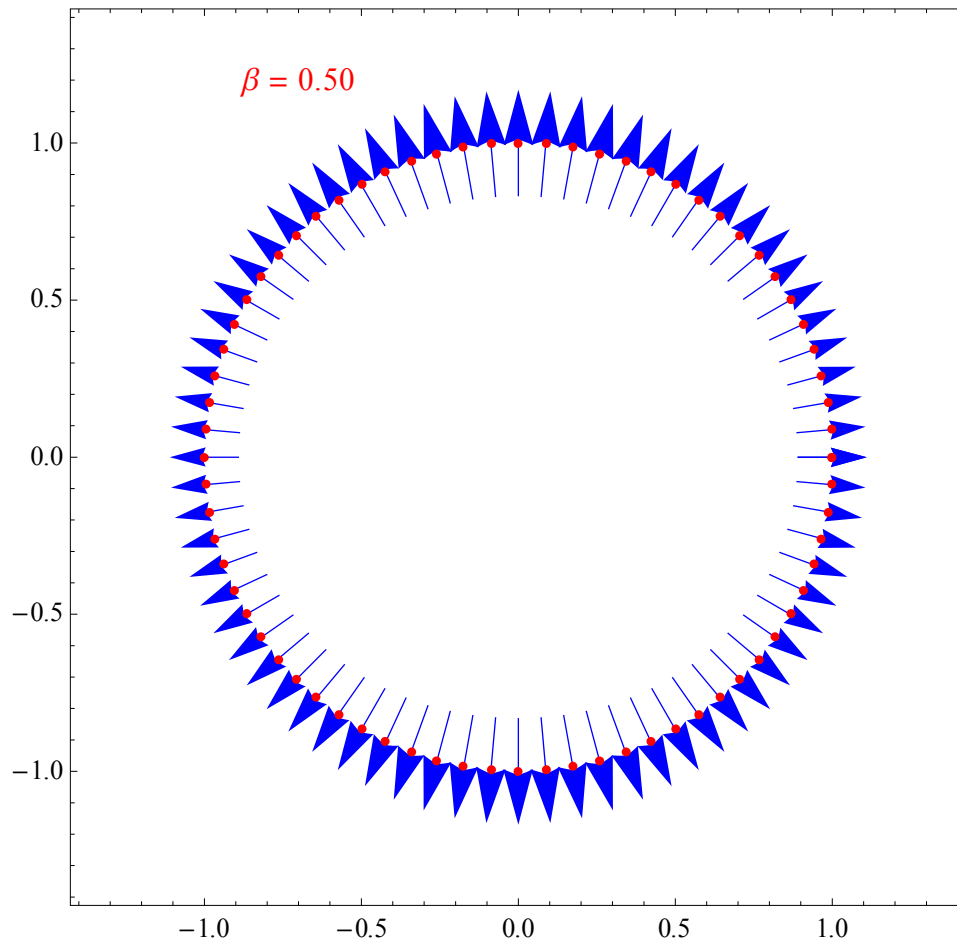


Fig. The distribution of the electric field E for $\beta = v/c = 0.5$. The direction of the velocity is the x axis.

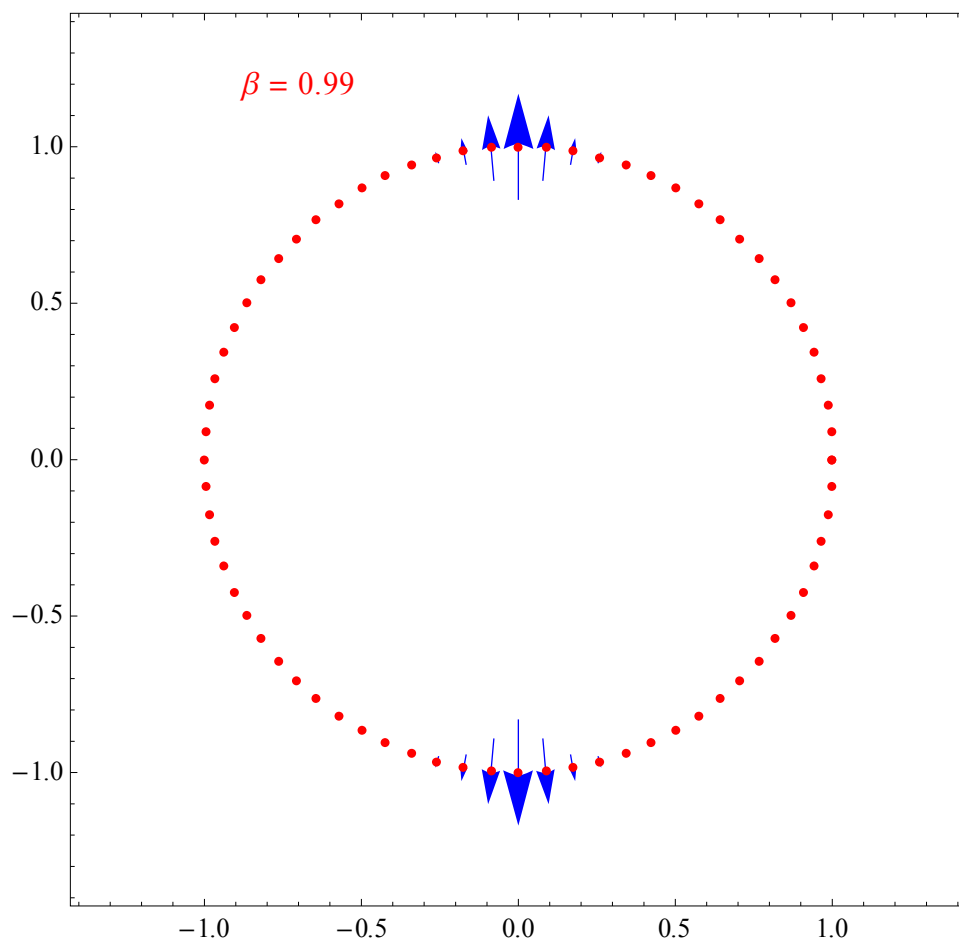


Fig. The distribution of the electric field E for $\beta = v/c = 0.99$. The direction of the velocity is the x axis.