Chapter 16
2D Green's function

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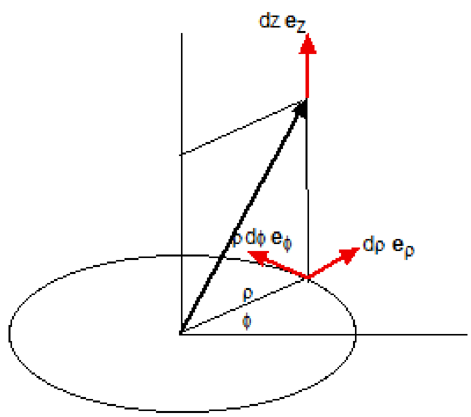
16.1 Summary

Table

	Laplace ∇^2	Helmholtz $\nabla^2 + k^2$	Modified Helmholtz $\nabla^2 - k^2$
2D	$-\dfrac{1}{2\pi}\ln\lvert\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\rvert$	$\dfrac{i}{4}H_0^{(1)}(k\lvert\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\rvert)$	$\dfrac{1}{2\pi}K_0(k\lvert\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\rvert)$

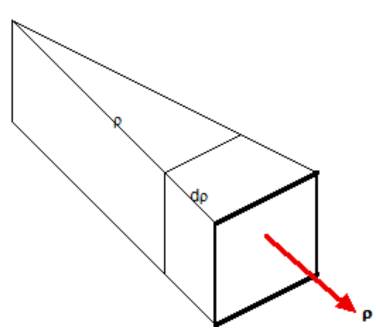
((Note))



Cylindrical co-ordinate:

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

16.2 2D Green's function for the Helmholtz equation



We now consider the Helmholtz equation

$$(\nabla^2 + k^2)G(\boldsymbol{\rho}) = -\delta(\boldsymbol{\rho})$$

Noting that

$$\nabla^2\psi(\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) = \frac{1}{\rho}(\rho\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial\psi}{\partial\rho}) = \frac{1}{\rho}(\rho\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial\psi}{\partial\rho}),$$

we have

$$\rho^2\frac{d^2G}{d\rho^2} + \rho\frac{dG}{d\rho} + (k^2\rho^2 G) = -\rho^2\delta(\rho)$$

For x ($\neq 0$) ($\rho \neq 0$), we put $k\rho = x$

$$x^2\frac{d^2}{dx^2}G + x\frac{d}{dx}G + (x^2 - 0^2)G = 0$$

The solution of this equation is

$$G = H_0^{(1)}(x), H_0^{(2)}(x),$$

Since

$H_0^{(1)}(x) \approx e^{ix}$ Outgoing wave

$H_0^{(2)}(x) \approx e^{-ix}$ incoming wave

We take $G = H_0^{(1)}(x) = aH_0^{(1)}(k\rho)$, where a is constant.

$$\int \nabla^2 G(\rho)d\tau + k^2\int G(\rho)d\tau = -\int \delta(\boldsymbol{\rho})d\tau = -1$$

$$I_1 = \int \nabla^2 G(\rho)d\tau = \int \nabla\cdot\nabla G(\rho)d\tau = \int \nabla G(\rho)\cdot d\mathbf{a}$$
$$= \hat{\rho}\frac{d}{d\rho}G(\rho)\cdot\hat{\rho}(2\pi\rho\times 1) = 2\pi\rho\frac{d}{d\rho}G(\rho)$$

In the limit of $\rho \to 0$, (or $x \to 0$)

$$H_0^{(1)}(x) = i\frac{2}{\pi}\ln(x) + 1 + i\frac{2}{\pi}(\gamma - \ln 2) + ...$$

$$G(\rho) = aH_0^{(1)}(k\rho) = a[i\frac{2}{\pi}\ln(k\rho) + 1 + i\frac{2}{\pi}(\gamma - \ln 2) + ...]$$

Then we have

$$I_1 = (2\pi\rho)ai\frac{2}{\pi}\frac{1}{k\rho}k\,|_{\rho=\varepsilon} = i4a$$

$$I_2 = \int G(\rho)d\tau = \int 2\pi\rho d\rho G(\rho)$$

$$= a\int_0^{\varepsilon} 2\pi\rho d\rho H_0^{(1)}(k\rho)$$

$$= 2\pi a\frac{1}{k^2}\int_0^{\delta} dxxH_0^{(1)}(x)$$

$$= 2\pi a\frac{1}{k^2}\int_0^{\delta} dxx[i\frac{2}{\pi}\ln(x) + 1 + ...]$$

$$= 2\pi a\frac{1}{k^2}\frac{\delta^2}{2\pi}[-i + \pi + 2i\ln\delta] \to 0$$

(when $\delta \to 0$)

Therefore

$i4a = -1$ or $a = \frac{i}{4}$

Green's function is given by

$$G = \frac{i}{4}H_0^{(1)}(k\rho).$$

16.3 2D Green's function for the modified Helmholtz equation

We now consider

$$(\nabla^2 - k^2)G(\boldsymbol{\rho}) = -\delta(\boldsymbol{\rho})$$

Noting that $\nabla^2 G(\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial G}{\partial\rho})$, we have

$$\rho^2\frac{d^2G}{d\rho^2} + \rho\frac{dG}{d\rho} - (k^2\rho^2 G) = -\rho^2\delta(\rho)$$

For x ($\neq 0$) ($\rho \neq 0$), we put $k\rho = x$

$$x^2 \frac{d^2}{dx^2} G + x \frac{d}{dx} G - (x^2 - 0^2) G = 0$$

$$G = K_0(x) \text{ and } I_0(x)$$

The asymptotic forms of these functions are given by

$$I_0(x) = \frac{1}{2} \sqrt{\frac{2}{\pi x}} e^x$$

$$K_0(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$$

Since

$$K_0(x) \approx e^{-x} \qquad \text{Decaying function}$$
$$I_0(x) \approx e^x \qquad \text{diverging function}$$

we take $G = aK_0(x) = aK_0(k\rho)$, where a is constant.

Note that

$$K_0(x) = -\ln x - \gamma + \ln 2 \qquad \text{in the limit of } x \rightarrow 0.$$

$$\int \nabla^2 G(\rho) d\tau - k^2 \int G(\rho) d\tau = -\int \delta(\boldsymbol{\rho}) d\tau = -1$$

$$I_1 = \int \nabla^2 G(\rho) d\tau = \int \nabla \cdot \nabla G(\rho) d\tau = \int \nabla G(\rho) \cdot d\mathbf{a}$$
$$= \hat{\rho} \frac{d}{d\rho} G(\rho) \cdot \hat{\rho}(2\pi\rho \times 1) = 2\pi\rho \frac{d}{d\rho} G(\rho)$$

$$I_2 = \int G(\rho) d\tau = \int 2\pi\rho d\rho G(\rho)$$

In the limit of $\rho \rightarrow 0$,

$$I_1 = 2\pi\rho a(-\frac{1}{k\rho})k = -2\pi a$$

Similarly, we have

$$I_2 = \int G(\rho) d\tau = \int 2\pi\rho d\rho G(\rho)$$

$$= a\int_0^{\varepsilon} 2\pi\rho d\rho K_0(k\rho)$$

$$= 2\pi a \frac{1}{k^2} \int_0^{\delta} dx x K_0(x)$$

$$= 2\pi a \frac{1}{k^2} \int_0^{\delta} dx x[-\ln x - \gamma + \ln 2]$$

$$= 2\pi a \frac{1}{k^2} \frac{\delta^2}{4} [1 - 2\gamma + \ln(4) - 2\ln(\delta)] \to 0$$

Then we have

$$-2\pi a = -1 \qquad \text{or} \qquad a = \frac{1}{2\pi}$$

$$G(\rho) = \frac{1}{2\pi} K_0(k\rho)$$

16.4 2D Green's function of the Laplace equation

We now consider

$$\nabla G(\boldsymbol{\rho}) = -\delta(\boldsymbol{\rho})$$

Noting that $\nabla^2 G(\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial G}{\partial\rho})$, we have

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial G}{\partial\rho}) = -\delta(\rho)$$

For $\rho \neq 0$, we have

$$\frac{\partial}{\partial\rho}(\rho\frac{\partial G}{\partial\rho}) = 0$$

$$\rho\frac{\partial G}{\partial\rho} = a$$

$$G = a\int \frac{d\rho}{\rho} = a\ln\rho$$

We note that

$$\int \nabla^2 G(\rho) d\tau = -\int \delta(\boldsymbol{\rho}) d\tau = -1$$

$$I = \int \nabla^2 G(\rho) d\tau = \int \nabla \cdot \nabla G(\rho) d\tau = \int \nabla G(\rho) \cdot d\mathbf{a}$$
$$= \hat{\rho} \frac{d}{d\rho} G(\rho) \cdot \hat{\rho}(2\pi\rho \times 1) = 2\pi\rho \frac{d}{d\rho} G(\rho)$$

In the limit of $\rho \rightarrow 0$,

$$I = 2\pi\rho \frac{a}{\rho} = 2\pi a = -1, \qquad \text{or} \qquad a = -\frac{1}{2\pi}$$

$$G(\rho) = -\frac{1}{2\pi} \ln \rho$$

16.5 Derivation of the Green function for the 2D Helmholtz equation: Fourier transformation

We start with

$$(\nabla^2 + k^2) G(\boldsymbol{\rho}) = -\delta(\boldsymbol{\rho})$$

$$G(\boldsymbol{\rho}) = \frac{1}{2\pi} \int e^{i\mathbf{K}\cdot\boldsymbol{\rho}} G(\mathbf{K}) d\mathbf{K}$$

$$\frac{1}{2\pi} \int (\nabla^2 + k^2) e^{i\mathbf{K}\cdot\boldsymbol{\rho}} G(\mathbf{K}) d\mathbf{K} = -\frac{1}{(2\pi)^2} \int e^{i\mathbf{K}\cdot\boldsymbol{\rho}} d\mathbf{K}$$

or

$$\frac{1}{2\pi} \int (-\mathbf{K}^2 + k^2) e^{i\mathbf{K}\cdot\boldsymbol{\rho}} G(\mathbf{K}) d\mathbf{K} = -\frac{1}{(2\pi)^2} \int e^{i\mathbf{K}\cdot\boldsymbol{\rho}} d\mathbf{K}$$

or

$$G(\mathbf{K}) = \frac{1}{2\pi} \frac{1}{K^2 - k^2}$$

Now we return to the calculation of the Green's function

$$G(\boldsymbol{\rho}) = \frac{1}{2\pi}\int e^{i\mathbf{K}\cdot\boldsymbol{\rho}} G(\mathbf{K}) d\mathbf{K}$$

$$= \frac{1}{(2\pi)^2}\int e^{i\mathbf{K}\cdot\boldsymbol{\rho}} \frac{1}{K^2 - k^2} d\mathbf{K}$$

$$= \frac{1}{(2\pi)^2}\int_0^\infty \frac{KdK}{K^2 - k^2}\int_0^{2\pi} e^{iK\rho\cos\theta} d\theta$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{KJ_0(K\rho)dK}{K^2 - k^2}$$

$$= \frac{1}{2\pi} K_0(-i\rho k) = \frac{1}{2\pi}\frac{\pi}{2} iH_0^{(1)}(\rho k) = \frac{i}{4} H_0^{(1)}(\rho k)$$

Note that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{iK\rho\cos\theta} d\theta = J_0(K\rho)$$

$$\int_0^\infty \frac{KJ_0(K\rho)dK}{K^2 - k^2} = K_0(-ik\rho)$$

16.6 Derivation of the Green function for the 2D modified Helmholtz equation: Fourier transformation

We start from

$$(\nabla^2 - k^2)G(\boldsymbol{\rho}) = -\delta(\boldsymbol{\rho})$$

$$G(\boldsymbol{\rho}) = \frac{1}{2\pi}\int e^{i\mathbf{K}\cdot\boldsymbol{\rho}} G(\mathbf{K}) d\mathbf{K}$$

$$\frac{1}{2\pi}\int (\nabla^2 - k^2) e^{i\mathbf{K}\cdot\boldsymbol{\rho}} G(\mathbf{K}) d\mathbf{K} = -\frac{1}{(2\pi)^2}\int e^{i\mathbf{K}\cdot\boldsymbol{\rho}} d\mathbf{K}$$

or

$$\frac{1}{2\pi}\int (-\mathbf{K}^2 - k^2) e^{i\mathbf{K}\cdot\boldsymbol{\rho}} G(\mathbf{K}) d\mathbf{K} = -\frac{1}{(2\pi)^2}\int e^{i\mathbf{K}\cdot\boldsymbol{\rho}} d\mathbf{K}$$

or

$$G(\mathbf{K}) = \frac{1}{2\pi}\frac{1}{K^2 + k^2}$$

Now we return to the calculation of the Green's function

$$\begin{aligned}
G(\boldsymbol{\rho}) &= \frac{1}{2\pi}\int e^{i\mathbf{K}\cdot\boldsymbol{\rho}} G(\mathbf{K}) d\mathbf{K} \\
&= \frac{1}{(2\pi)^2}\int e^{i\mathbf{K}\cdot\boldsymbol{\rho}} \frac{1}{K^2+k^2} d\mathbf{K} \\
&= \frac{1}{(2\pi)^2}\int_0^\infty \frac{KdK}{K^2+k^2}\int_0^{2\pi} e^{iK\rho\cos\theta} d\theta \\
&= \frac{1}{2\pi}\int_0^\infty \frac{KJ_0(K\rho)dK}{K^2+k^2} \\
&= \frac{1}{2\pi} K_0(\rho k)
\end{aligned}$$

Note that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{iK\rho\cos\theta} d\theta = J_0(K\rho)$$

$$\int_0^\infty \frac{KJ_0(K\rho)dK}{K^2+k^2} = K_0(k\rho)$$

16.7 Mathematica

```
Clear["Global`*"]

Integrate[Exp[ⅈ x ρ Cos[θ]], {θ, 0, 2 π}] //
 Simplify[#, x ρ ∈ Reals] &

2 π BesselJ[0, x ρ]

Integrate[x BesselJ[0, x ρ]/(x^2 + k^2), {x, 0, ∞}] //
 Simplify[#, {k > 0, ρ > 0}] &

BesselK[0, k ρ]

Integrate[x BesselJ[0, x ρ]/(x^2 - k^2), {x, 0, ∞}] //
 Simplify[#, {ρ > 0, k^2 ∉ Reals || Re[k^2] ≤ 0}] &
```

$$\text{BesselK}\left[0, \frac{\rho}{\sqrt{-\frac{1}{k^2}}}\right]$$

Appendix

((Mathematica))

The four Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ are the best known and most frequently used special functions. That is why we will devote a slightly longer section to them and present a couple of applications. Following *Mathematica*'s naming convention, they are written as follows.

BesselJ[n,z] for $J_n(z)$
BesselI[n,z] for $I_n(z)$
BesselK[n,z] for $K_n(z)$
BesselY[n,z] for $N_n(z)$ (or $Y_n(z)$)

Hankel function

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$$
$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$$

where $\nu > 0$, $\nu =$ integral and nonintegral values.

In the limit of $x \approx 0$,

$$H_0^{(1)}(x) = i\frac{2}{\pi}\ln x + 1 + i\frac{2}{\pi}(\gamma - \ln 2) + \ldots\ldots..$$

$$H_\nu^{(1)}(x) = -i\frac{(\nu-1)!}{\pi}(\frac{2}{x})^\nu + \ldots\ldots..$$

$$H_0^{(2)}(x) = -i\frac{2}{\pi}\ln x + 1 - i\frac{2}{\pi}(\gamma - \ln 2) + \ldots\ldots..$$

$$H_\nu^{(2)}(x) = i\frac{(\nu-1)!}{\pi}(\frac{2}{x})^\nu + \ldots\ldots..$$