**Chapter 17**
**One dimensional Green's function**
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## 17.1 Summary

Table

| | Laplace | Helmholtz | Modified Helmholtz |
|---|---|---|---|
| | $\nabla^2$ | $\nabla^2 + k^2$ | $\nabla^2 - k^2$ |
| 1D | No solution | $\frac{i}{2k}\exp(ik\lvert x_1 - x_2\rvert)$ | $\frac{1}{2k}\exp(-k\lvert x_1 - x_2\rvert)$ |

## 17.2 Green's function: modified Helmholtz

### ((Arfken 10.5.10)) 1D Green's function

Construct the 1D Green's function for the Helmholtz equation

$$L\psi = \left(\frac{d^2}{dx^2} + k^2\right)\psi ,$$

$$L\psi + g(x) = 0$$

$$\psi = \int G(x,\xi)g(\xi)d\xi$$

$$L\psi = \int LG(x,\xi)g(\xi)d\xi = \int -\delta(x-\xi)g(\xi)d\xi = -g(x).$$

Thus we need $G(x,\xi)$ which satisfies

$$L_x G(x,\xi) = -\delta(x-\xi),$$

$$\left(\frac{d^2}{dx^2} + k^2\right)G(x,\xi) = -\delta(x-\xi),$$

$$\Downarrow$$

$$G(x,\xi) = \frac{i}{2k} e^{ik|x-\xi|}$$

For simplicity, we consider the case ($\xi = 0$),

$$\left( \frac{d^2}{dx^2} + k^2 \right) G(x) = -\delta(x)$$

We define

$$G(q) = \frac{1}{\sqrt{2\pi}} \int G(x) e^{-iqx} dx$$

$$G(x) = \frac{1}{\sqrt{2\pi}} \int G(q) e^{iqx} dq$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iqx} dq$$

Then we have

$$\frac{1}{\sqrt{2\pi}} \int \left( \frac{d^2}{dx^2} + k^2 \right) G(q) e^{iqx} dq = -\frac{1}{2\pi} \int e^{iqx} dq$$

or

$$\frac{1}{\sqrt{2\pi}} \int \left( -q^2 + k^2 \right) G(q) e^{iqx} dq = -\frac{1}{2\pi} \int e^{iqx} dq$$

or

$$(-q^2 + k^2) \sqrt{2\pi} G(q) = -1$$

or

$$G(q) = \frac{1}{\sqrt{2\pi}} \frac{1}{q^2 - k^2}$$

Thus

$$G(x) = \frac{1}{\sqrt{2\pi}} \int G(q) e^{iqx} dq$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \frac{e^{iqx}}{q^2 - (k + i\varepsilon)^2}$$

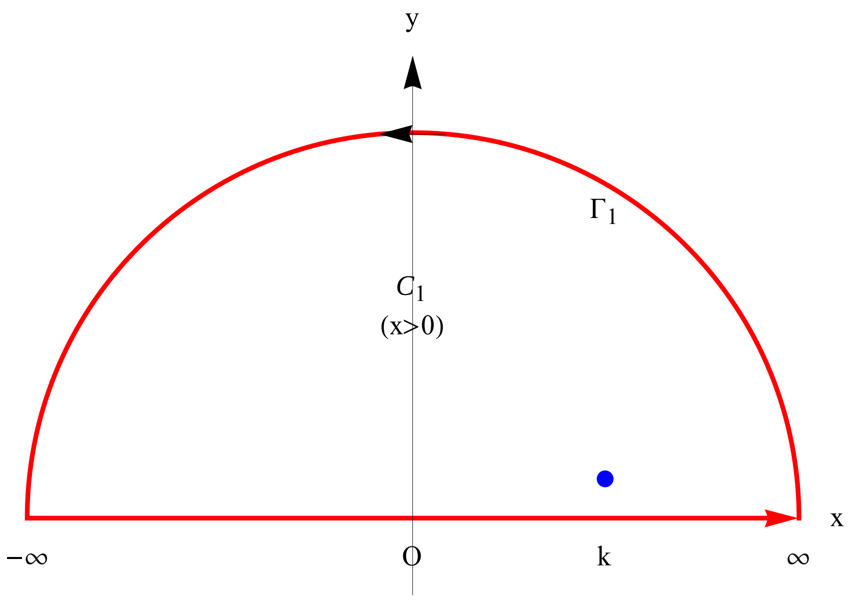
where we change $k$ to $k$+i$\varepsilon$ ($\varepsilon$>0 and $\varepsilon \to 0$). There are two simple poles at $z = k + i\varepsilon$ and $z = -k - i\varepsilon$ in the complex plane.

(i) For $x$>0, we have

$$\frac{1}{2\pi} \int_{C_1} \frac{e^{iqx}}{z^2 - (k + i\varepsilon)^2} dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \frac{e^{iqx}}{q^2 - (k + i\varepsilon)^2} + \frac{1}{2\pi} \int_{\Gamma_1} \frac{e^{iqx}}{z^2 - (k + i\varepsilon)^2} dz$$
$$= \frac{1}{2\pi} 2\pi i \operatorname{Res}(z = k + i\varepsilon) = \frac{i}{2k} e^{ikx}$$

or

$$G(x) = \frac{i}{2k} e^{ikx}$$



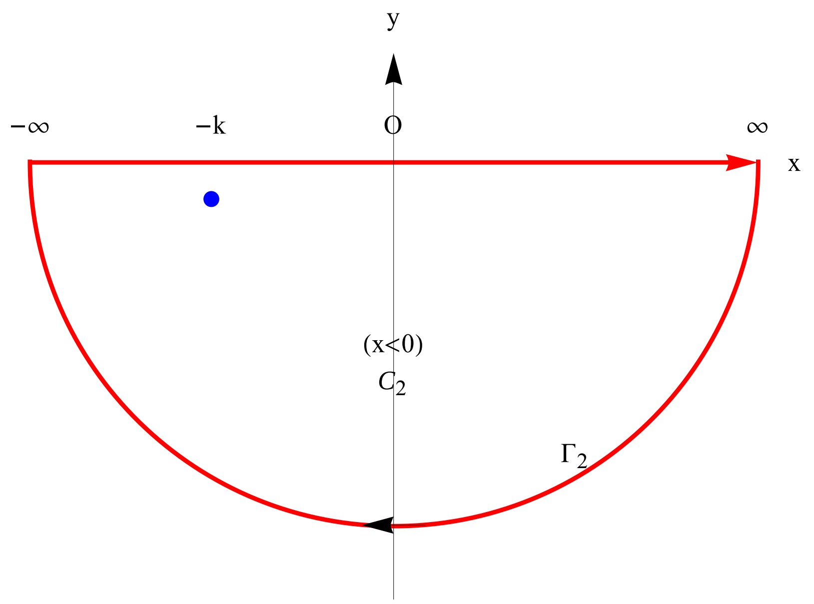

using the residue theorem. Note that the contour $C_1$ is the contour in the upper-half plane (counter-clock wise). According to the Jordan's theorem, the integral along the path $\Gamma_1$ is equal to zero. There is a simple pole of $q = k$+i $\varepsilon$ ($\varepsilon$>0) inside the contour $C_1$.

(ii) For $x < 0$, we have

$$\frac{1}{2\pi}\int_{C_2}\frac{e^{iqx}}{z^2-(k+i\varepsilon)^2}dz=\frac{1}{2\pi}\int_{-\infty}^{\infty}dq\frac{e^{iqx}}{q^2-(k+i\varepsilon)^2}+\frac{1}{2\pi}\int_{\Gamma_2}\frac{e^{iqx}}{z^2-(k+i\varepsilon)^2}dz$$
$$=\frac{1}{2\pi}2\pi i\,\mathrm{Res}(z=-k-i\varepsilon)=\frac{i}{2k}e^{-ikx}$$

or

$$G(x)=\frac{i}{2k}e^{-ikx}$$




using the residue theorem. Note that the contopur $C_2$ is in the lower-half plane (clock-wise). There is only a simple pole at $z$ = -$k$-i$\varepsilon$ inside the contour $C_2$. The integral along the path $\Gamma_2$ is equal to zero according to the Jordan's lemma.

In summary we have the Green's function as

$$G(x)=\frac{i}{2k}e^{ik|x|}$$

or

$$G(x,\xi)=G(x-\xi)=\frac{i}{2k}e^{ik|x-\xi|}$$

((**Mathematica**))

```
Clear["Global`*"]

eq1 = G''[x] + k^2 G[x] == -DiracDelta[x - ξ];

eq2 = DSolve[eq1, G[x], x] // Simplify
```

$$\left\{\left\{G[x] \to C[1]\,\mathrm{Cos}[k\,x] + C[2]\,\mathrm{Sin}[k\,x] - \frac{\mathrm{HeavisideTheta}[x-\xi]\,\mathrm{Sin}[k\,(x-\xi)]}{k}\right\}\right\}$$

```
G1[x_] = G[x] /. eq2[[1]] // Simplify[#, x < ξ] & // TrigToExp
```

$$\frac{1}{2} e^{-i k x} C[1] + \frac{1}{2} e^{i k x} C[1] + \frac{1}{2} i\, e^{-i k x} C[2] - \frac{1}{2} i\, e^{i k x} C[2]$$

```
Collect[G1[x], {e^(i k x), e^(-i k x)}]
```

$$e^{i k x}\left(\frac{C[1]}{2} - \frac{1}{2} i\, C[2]\right) + e^{-i k x}\left(\frac{C[1]}{2} + \frac{1}{2} i\, C[2]\right)$$

```
G2[x_] = G[x] /. eq2[[1]] // Simplify[#, x > ξ] & // TrigToExp
```

$$-\frac{i\, e^{-i k (x-\xi)}}{2 k} + \frac{i\, e^{i k (x-\xi)}}{2 k} + \frac{1}{2} e^{-i k x} C[1] + \frac{1}{2} e^{i k x} C[1] + \frac{1}{2} i\, e^{-i k x} C[2] - \frac{1}{2} i\, e^{i k x} C[2]$$

The condition of the Green ' s function
G1[x] has the form of exp[-i k x]
G2[x] has the form of exp[i k x]

$$\mathrm{eq3} = \left(\frac{C[1]}{2} - \frac{1}{2} i\, C[2]\right) == 0;\ \mathrm{eq4} = -\frac{i\, e^{i k \xi}}{2 k} + \frac{1}{2} C[1] + \frac{1}{2} i\, C[2] == 0;$$

```
eq5 = Solve[{eq3, eq4}, {C[1], C[2]}]
```

$$\left\{\left\{C[1] \to \frac{i\, e^{i k \xi}}{2 k},\ C[2] \to \frac{e^{i k \xi}}{2 k}\right\}\right\}$$

```
G1[x] /. eq5[[1]] // Simplify
```

$$\frac{i\, e^{-i k (x-\xi)}}{2 k}$$

```
G2[x] /. eq5[[1]] // Simplify
```

$$\frac{i\, e^{i k (x-\xi)}}{2 k}$$

## 17.3 Green's function: modified Helmholtz equation

### ((Arflken 10.5.11)) 1D Green's function

Construct the 1D Green's function for the modified Helmholtz equation

$$L\psi = \left(\frac{d^2}{dx^2} - k^2\right)\psi$$

$$L\psi = f(x)$$

$$\psi = -\int G(x,\xi) f(\xi) d\xi$$

$$L\psi = -\int LG(x,\xi) f(\xi) d\xi = \int \delta(x-\xi) f(\xi) d\xi = f(x)$$

Thus we need $G(x,\xi)$ which satisfies

$$L_x G(x,\xi) = -\delta(x-\xi)$$

$$\left(\frac{d^2}{dx^2} - k^2\right) G(x,\xi) = -\delta(x-\xi)$$

$$\Downarrow$$

$$G(x,\xi) = \frac{1}{2k} e^{-k|x-\xi|}$$

We consider the case at $\xi = 0$

$$\left(\frac{d^2}{dx^2} - k^2\right) G(x) = -\delta(x)$$

$$G(q) = \frac{1}{\sqrt{2\pi}} \int G(x) e^{-iqx} dx$$

$$G(x) = \frac{1}{\sqrt{2\pi}} \int G(q) e^{iqx} dq$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iqx} dq$$

$$\frac{1}{\sqrt{2\pi}} \int \left(\frac{d^2}{dx^2} - k^2\right) G(q) e^{iqx} dq = -\frac{1}{2\pi} \int e^{iqx} dq$$

or

$$\frac{1}{\sqrt{2\pi}}\int\left(-q^2-k^2\right)G(q)e^{iqx}dq=-\frac{1}{2\pi}\int e^{iqx}dq\,.$$

Then we have

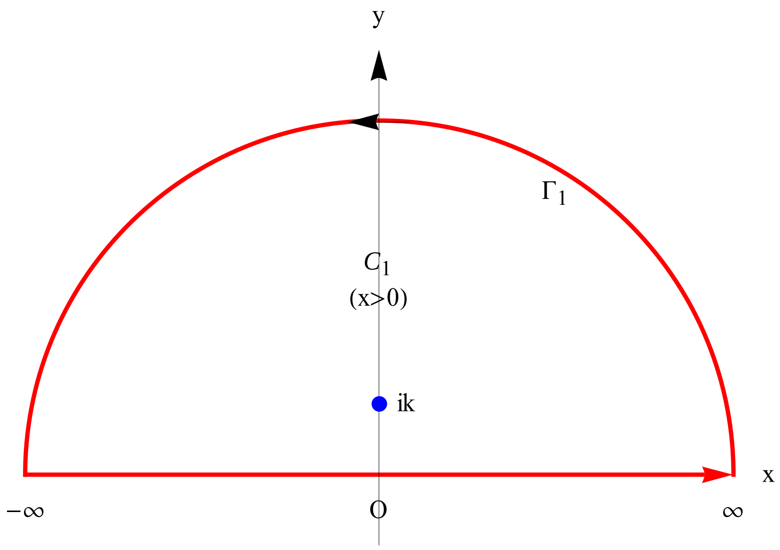
$$(-q^2-k^2)G(q)=-\frac{1}{\sqrt{2\pi}} \qquad \text{or} \qquad G(q)=\frac{1}{\sqrt{2\pi}}\frac{1}{q^2+k^2}$$

$$G(x)=\frac{1}{2\pi}\int\frac{e^{iqx}}{q^2+k^2}dq$$

(i) For $x>0$

$$G(x)=\frac{1}{2\pi}\int_{C_1}\frac{e^{iqx}}{q^2+k^2}dq=\frac{1}{2k}e^{-kx}$$

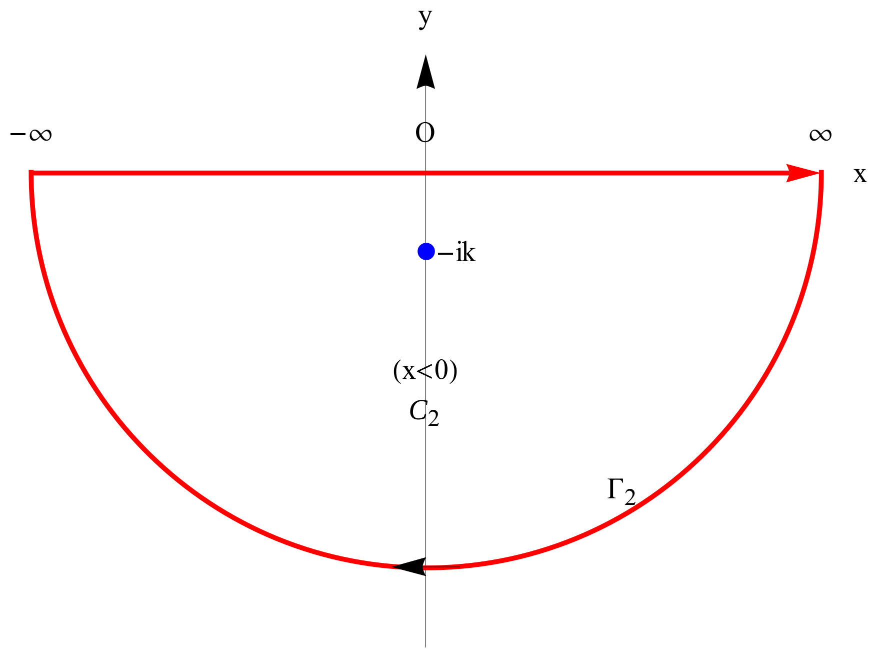
where $C_1$ is in the lower-half plane (counter-clock wise). There is a simple pole of $z = \mathrm{i}\,k$ inside the contour $C_1$. We use the Residue theorem.




(ii) For $x<0$

$$G(x)=\frac{1}{2\pi}\int_{C_2}\frac{e^{iqx}}{q^2+k^2}dq=\frac{1}{2k}e^{kx}$$

where $C_2$ is in the lower-half plane (clock-wise). There is a simple pole of $z$ = -i $k$ inside the contour $C_2$. We use the residue theorem.




In summary we have the Green's function

$$G(x) = \frac{1}{2k} e^{-k|x|}$$

or

$$G(x,\xi) = \frac{1}{2k} e^{-k|x-\xi|}$$

((**Mathematica**))

```
Clear["Global`*"];

eq1 = DSolve[{G''[x] - k^2 G[x] == -DiracDelta[x - ξ]}, G[x], x] // Simplify;

G[x_] = G[x] /. eq1[[1]] // ExpandAll
```

$$e^{k x} C[1] + e^{-k x} C[2] - \frac{e^{k x - k \xi} \text{HeavisideTheta}[x-\xi]}{2 k} + \frac{e^{-k x + k \xi} \text{HeavisideTheta}[x-\xi]}{2 k}$$

```
G2[x_] = Simplify[G[x], x > ξ] // ExpandAll
```

$$-\frac{e^{k x - k \xi}}{2 k} + \frac{e^{-k x + k \xi}}{2 k} + e^{k x} C[1] + e^{-k x} C[2]$$

```
G1[x_] = Simplify[G[x], x < ξ] // FullSimplify
```

$$e^{k x} C[1] + e^{-k x} C[2]$$

From the boundary condition
G[x] must vanish for x →∞ and x →-∞,

It is required that C[2] = 0, C[1] = $\frac{1}{2k} e^{-k\xi}$

```
G22[x_] = G2[x] /. {C[2] → 0, C[1] → 1/(2 k) e^(-k ξ)} // Simplify
```

$$\frac{e^{k(-x+\xi)}}{2 k}$$

```
G11[x_] = G1[x] /. {C[2] → 0, C[1] → 1/(2 k) e^(-k ξ)} // Simplify
```

$$\frac{e^{k(x-\xi)}}{2 k}$$

## 17.4 Heat conduction: Laplace transform

We start with the equation for heat conduction,

$$\frac{\partial}{\partial t} T(x,t) - D \frac{\partial^2}{\partial x^2} T(x,t) = 0$$

with the initial condition $T(x, 0) = f(x)$. We apply the Laplace transform to this differential equation.

$$sT(x,s) - T(x,t=0) - D \frac{\partial^2}{\partial x^2} T(x,s) = 0$$

where $T(x, s)$ is the Laplace transform of $T(x, t)$. Then we have

$$D\frac{\partial^2}{\partial x^2}T(x,s) - sT(x,s) = -f(x)\,,$$

or

$$(\frac{\partial^2}{\partial x^2} - k^2)T(x,s) = -\frac{f(x)}{D}$$

where

$$k = \sqrt{\frac{s}{D}}$$

We use the Green's function defined by

$$(\frac{\partial^2}{\partial x^2} - k^2)G(x,\xi,s) = -\delta(x-\xi)\,.$$

The solution of $G(x,\xi,s)$ is already given as

$$G(x,\xi,s) = \frac{1}{2k}e^{-k|x-\xi|} = \frac{\sqrt{D}}{2\sqrt{s}}\exp(-\sqrt{\frac{s}{D}}\,|\,x-\xi\,|)$$

Then the solution of $T(x, s)$ is obtained as

$$\begin{aligned} T(x,s) &= \int_{-\infty}^{\infty} G(x,\xi,s)(-\frac{1}{D})f(\xi)d\xi \\ &= \int_{-\infty}^{\infty} \frac{\sqrt{D}}{2\sqrt{s}}(-\frac{1}{D})\exp(-\sqrt{\frac{s}{D}}\,|\,x-\xi\,|)f(\xi)d\xi \\ &= -\frac{1}{2\sqrt{Ds}}\int_{-\infty}^{\infty}\exp(-\sqrt{\frac{s}{D}}\,|\,x-\xi\,|)f(\xi)d\xi \end{aligned}$$

Here we use the formula for the inverse Laplace transform;

$$L^{-1}[\frac{1}{\sqrt{s}}\exp(-\sqrt{s}\frac{|x-\xi|}{\sqrt{D}}] = \frac{1}{\sqrt{\pi t}}\exp(-\frac{a^2}{4t})$$

where

$$a = \frac{|x-\xi|}{\sqrt{D}}$$

Then

$$T(x,t) = -\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}\exp[-\frac{(x-\xi)^2}{4tD}]f(\xi)d\xi$$

((**Mathematica**))

```
InverseLaplaceTransform[1/Sqrt[s] Exp[-a Sqrt[s]], s, t] //
  Simplify[#, a > 0] &
```

$$\frac{e^{-\frac{a^2}{4t}}}{\sqrt{\pi}\,\sqrt{t}}$$