Chapter 19 Laplace's equation in spherical coordinate: Green's function

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Addition theorem
Green's function in the spherical coordinate
Dirac delta function in the spherical coordinate
See Chapter 22 for the detail of the Legendre function. Some content in Chapter 22 is the same as that in this Chapter.

19.1 Formal solution of Laplace's equation

We consider the solution of Laplace's equation

$$\nabla^2\Phi(\mathbf{r}) = 0 .$$

where $\Phi(\mathbf{r})$ is a scalar electric potential.

$$\begin{aligned}\nabla^2 &= -\frac{1}{\hbar^2 r^2}\mathbf{L}^2 + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})\\ &= -\frac{1}{\hbar^2 r^2}\mathbf{L}^2 + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})\\ &= -\frac{1}{\hbar^2 r^2}\mathbf{L}^2 + \frac{1}{r}\frac{\partial^2}{\partial r^2}(r)\end{aligned}$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi(\mathbf{r})) - \frac{1}{\hbar^2 r^2}\mathbf{L}^2\Phi(\mathbf{r}) = 0 .$$

Here we assume that

$$\Phi(\mathbf{r}) = U(r)Y_l^m(\theta,\phi) .$$

(separation variable)

$$\frac{1}{r}Y_l^m(\theta,\phi)\frac{\partial^2}{\partial r^2}(rU(r)) - \frac{1}{\hbar^2 r^2}U(r)\mathbf{L}^2 Y_l^m(\theta,\phi) = 0 .$$

We use the relation

$$\mathbf{L}^2 Y_l^m(\theta,\phi) = \hbar^2 l(l+1) Y_l^m(\theta,\phi) .$$

Then we have

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}[rU(r)] - \frac{l(l+1)}{r^2}U(r) = 0 .$$

The solution of $U(r)$ is given by

$$U(r) = Ar^l + Br^{-(l+1)} .$$

The general solution is

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[A_{lm}r^l + B_{lm}r^{-(l+1)}]Y_l^m(\theta,\phi) .$$

19.2. Dirac delta function in the spherical co-ordinate

We define the Dirac delta function as

$$\int \delta(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}' = 1 .$$

Suppose that

$$\delta(\mathbf{r}-\mathbf{r}') = A(r')\delta(r-r')\delta(\phi-\phi')\delta(\mu-\mu') ,$$

with $\mu = \cos\theta$ and $\mu' = \cos\theta$.

From the property of the delta function, we have

$$\delta(\mu-\mu') = \delta[-\sin\theta'(\theta-\theta')] = \frac{1}{\sin\theta'}\delta(\theta-\theta') ,$$

where θ and θ are in the range between 0 and π. Then we have

$$\int \delta(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}' = \int_0^\infty r'^2\,dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} \frac{A(r')}{\sin\theta'}\delta(r-r')\delta(\phi-\phi')\delta(\theta-\theta')$$

$$= \int_0^\infty A(r')r'^2\,dr'\delta(r-r') = A(r)r^2 = 1$$

or

$$A(r) = \frac{1}{r^2}$$

In summary,

$$\boxed{\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{r'^2}\delta(r-r')\delta(\phi-\phi')\delta(\mu-\mu')}.$$

19.3 Green's function

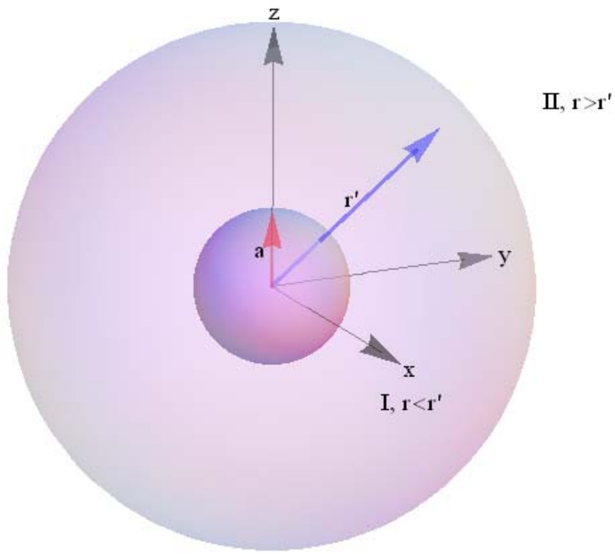
We consider the Green's function given by

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'),$$

with the boundary surfaces which are concentric spheres at $r = a$ and $r = b$ ($b>a$). Note that

$G(\mathbf{r},\mathbf{r}') = 0$ for $r = a$ and for $r = b$.

where $\boldsymbol{r}$ is the variable and $\boldsymbol{r}'$ is fixed.



Within each region (region I ($a<r<r'<b$) and region II ($b>r>r'>a$), we have the simpler equation

$$G(\mathbf{r},\mathbf{r}') = 0\,.$$

The solution of the Green's function is given by the form

$$G(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}(r,r',\theta',\phi')Y_l^m(\theta,\phi)\,.$$

Then the differential equation of the Green's function is given by

$$\sum_{l',m'}\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{l'm'}) - \frac{l'(l'+1)}{r^2}A_{l'm'}]Y_{l'}^{m'}(\theta,\phi) = -\frac{\delta(r-r')}{r^2}\delta(\phi-\phi')\delta(\mu-\mu')\,.$$

Note that

$$\delta_{l,l'}\delta_{m,m'} = \int d\Omega\langle l',m'|\mathbf{n}\rangle\langle\mathbf{n}|l,m\rangle = \iint \sin\theta d\theta d\phi Y_{l'}^{m'^*}(\theta,\phi)Y_l^m(\theta,\phi)\,,$$

where

$$d\Omega = \sin\theta d\theta d\phi .$$

Then

$$\sum_{l',m'} \int d\Omega Y_l^{m^*}(\theta,\phi) Y_l^m(\theta,\phi) [\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{l'm'}) - \frac{l'(l'+1)}{r^2} A_{l'm'}] = -\int d\Omega Y_l^{m^*}(\theta,\phi) \frac{\delta(r-r')}{r^2} \delta(\phi-\phi')\delta(\mu-\mu')$$

or

$$\sum_{l',m'} [\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{l'm'}) - \frac{l'(l'+1)}{r^2} A_{l'm'}] \delta_{l,l'}\delta_{m,m'} = -\int d\Omega Y_l^{m^*}(\theta,\phi) \frac{\delta(r-r')}{r^2} \delta(\phi-\phi')\delta(\mu-\mu')$$

or

$$\begin{aligned} \frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{lm}) - \frac{l(l+1)}{r^2} A_{lm} &= -\frac{\delta(r-r')}{r^2} \int d\Omega Y_l^{m^*}(\theta,\phi)\delta(\phi-\phi')\delta(\mu-\mu') \\ &= -\frac{\delta(r-r')}{r^2} Y_l^{m^*}(\theta',\phi') \int \sin\theta d\theta d\phi \delta(\phi-\phi')\delta(\mu-\mu') \\ &= -\frac{\delta(r-r')}{r^2} Y_l^{m^*}(\theta',\phi') \end{aligned}$$

Since $Y_l^{m^*}(\theta',\phi')$ is constant, we put

$$g_l(r,r') = \frac{A_{lm}(r,r',\theta',\phi')}{Y_l^{m^*}(\theta',\phi')} .$$

Then we get

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rg_l) - \frac{l(l+1)}{r^2} g_l = -\frac{\delta(r-r')}{r^2} ,$$

with the boundary condition

$$g_l(a,r') = 0 , \qquad g_l(b,r') = 0$$

(i) $g_l(r,r')$ is continuous at $r = r'$.

(ii) $$\frac{\partial g_l(r,r')}{\partial r}|_{r=r'+0} - \frac{\partial g_l(r,r')}{\partial r}|_{r=r'-0} = -\frac{1}{r'^2}$$

Using Mathematica we get the Green's function

$$g_{lI} = \frac{r^{-(l+1)}(r^{2l+1} - a^{2l+1})r'^{-(l+1)}(b^{2l+1} - r'^{2l+1})}{(b^{2l+1} - a^{2l+1})(2l+1)} \qquad \text{for } a<r<r'$$

$$g_{lII} = \frac{r'^{-(l+1)}(r'^{2l+1} - a^{2l+1})r^{-(l+1)}(b^{2l+1} - r^{2l+1})}{(b^{2l+1} - a^{2l+1}+)(2l+1)} \qquad \text{for } b>r>r'$$

or

$$G_I(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\frac{1}{2l+1})\frac{(r^{2l+1} - a^{2l+1})(b^{2l+1} - r'^{2l+1})}{r^{l+1}r'^{l+1}(b^{2l+1} - a^{2l+1})}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(r^l - \frac{a^{2l+1}}{r^{l+1}})(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}})}{(2l+1)[1-\left(\frac{a}{b}\right)^{2l+1}]}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$

$$G_{II}(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(\frac{1}{2l+1})\frac{(r'^{2l+1} - a^{2l+1})(b^{2l+1} - r^{2l+1})}{r^{l+1}r'^{l+1}(b^{2l+1} - a^{2l+1})}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(r'^l - \frac{a^{2l+1}}{r'^{l+1}})(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}})}{(2l+1)[1-\left(\frac{a}{b}\right)^{2l+1}]}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$

Or more simply, we have

$$G(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(r_<^{\,l} - \frac{a^{2l+1}}{r_<^{\,l+1}})(\frac{1}{r_>^{\,l+1}} - \frac{r_>^{\,l}}{b^{2l+1}})}{(2l+1)[1-\left(\frac{a}{b}\right)^{2l+1}]}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi').$$

This means that

$r_< = r$
$r_> = r'$ in the region I ($a<r<r'<b$)

$$r_> = r$$
$$r_< = r'$$ in the region II ($a<r'<r<b$)

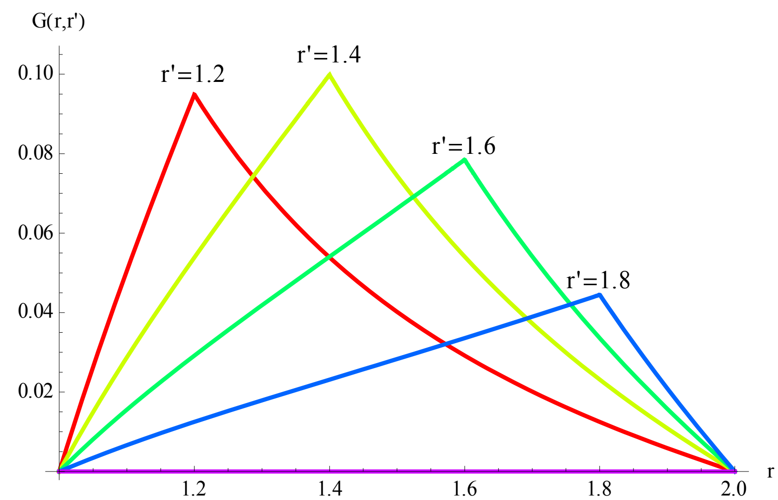


Fig. Plot of the Green's function $g_l(r, r')$ as a function of r. $a = 1$. $b = 2$. $l = 3$. r' is changed as a parameter; $r' = 1.2$, 1.4, 1.6, and 1.8.

When $a\to 0$,

$$G(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{r_<^{\,l}\left(\dfrac{1}{r_>^{\,l+1}} - \dfrac{r_>^{\,l}}{b^{2l+1}}\right)}{(2l+1)} Y_l^m(\theta,\phi) Y_l^{m^*}(\theta',\phi')$$

19.4 Special case ($b\to\infty$, $a \neq 0$))

In the limit of $b\to\infty$ (but $a \neq 0$), we have

$$G_I(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(\frac{1}{2l+1}\right) \frac{(r^{2l+1} - a^{2l+1})}{r^{l+1} r'^{l+1}} Y_l^m(\theta,\phi) Y_l^{m^*}(\theta',\phi'),$$

for the region I ($a<r<r'$), and

$$G_{II}(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(\frac{1}{2l+1}\right) \frac{(r'^{2l+1} - a^{2l+1})}{r^{l+1} r'^{l+1}} Y_l^m(\theta,\phi) Y_l^{m^*}(\theta',\phi'),$$

for the region II ($r>r'>a$). Or more simply, we have

$$G(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(r_<^{2l+1} - a^{2l+1})}{(2l+1)r_>^{l+1}r_<^{l+1}}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi') .$$

Further we assume that $a\to 0$.

$$G(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^{l}}{r_>^{l+1}}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi') .$$

19.5 Mathematica

Find the Green's function for

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rg_l) - \frac{l(l+1)}{r^2}g_l = -\frac{\delta(r-r')}{r^2},$$

with the boundary condition

$$g_l(a,r') = 0, \qquad g_l(b,r') = 0 .$$

```
Clear["Global`*"];

eq1 = D[r G[r], {r, 2}] - L (L + 1)/r G[r] == -1/r DiracDelta[r - r1];

eq2 = DSolve[eq1, G[r], r] // FullSimplify[#, {L > 0, 0 < a < r1 < b}] &
```

$$\left\{\left\{G[r] \to \frac{(r\, r1)^{-L}\left(r1^{L}\left(r^{1+2L}C[1]+C[2]\right)+\frac{\left(-r^{1+2L}+r1^{1+2L}\right)\text{HeavisideTheta}[r-r1]}{r1+2L\,r1}\right)}{r}\right\}\right\}$$

```
GI[r_] = G[r] /. eq2[[1]] // Simplify[#, 0 < a < r < r1 < b] &
```

$$r^{L}C[1]+r^{-1-L}C[2]$$

```
GII[r_] = G[r] /. eq2[[1]] // Simplify[#, b > r > r1 > a] &
```

$$\frac{(r\, r1)^{-L}\left(\frac{-r^{1+2L}+r1^{1+2L}}{r1+2L\,r1}+r1^{L}\left(r^{1+2L}C[1]+C[2]\right)\right)}{r}$$

```
eq3 = Solve[{GII[b] == 0, GI[a] == 0}, {C[1], C[2]}] // Simplify
```

$$\left\{\left\{C[1]\to\frac{r1^{-1-L}\left(-b^{1+2L}+r1^{1+2L}\right)}{\left(a^{1+2L}-b^{1+2L}\right)(1+2L)},\ C[2]\to\frac{a^{1+2L}\,r1^{-1-L}\left(b^{1+2L}-r1^{1+2L}\right)}{\left(a^{1+2L}-b^{1+2L}\right)(1+2L)}\right\}\right\}$$

```
GI1[r_] = GI[r] /. eq3[[1]] // Simplify
```

$$\frac{r^{-1-L}\left(-a^{1+2L}+r^{1+2L}\right)r1^{-1-L}\left(-b^{1+2L}+r1^{1+2L}\right)}{\left(a^{1+2L}-b^{1+2L}\right)(1+2L)}$$

```
GII1[r_] = GII[r] /. eq3[[1]] // Simplify
```

$$\frac{\left(b^{1+2L}-r^{1+2L}\right)(r\, r1)^{-1-L}\left(a^{1+2L}-r1^{1+2L}\right)}{\left(a^{1+2L}-b^{1+2L}\right)(1+2L)}$$

19.6 Addition theorem for spherical harmonics

The Green's function $G(\boldsymbol{r}, \boldsymbol{r}')$ for the differential equation

$$\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')$$

is given by

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

This leads to the relation

$$\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$

or

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\frac{r_<^{\,l}}{r_>^{\,l+1}}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$

Here we note that

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{r_<^{\,l}}{r_>^{\,l+1}}P_l(\cos\gamma)$$

where γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$.

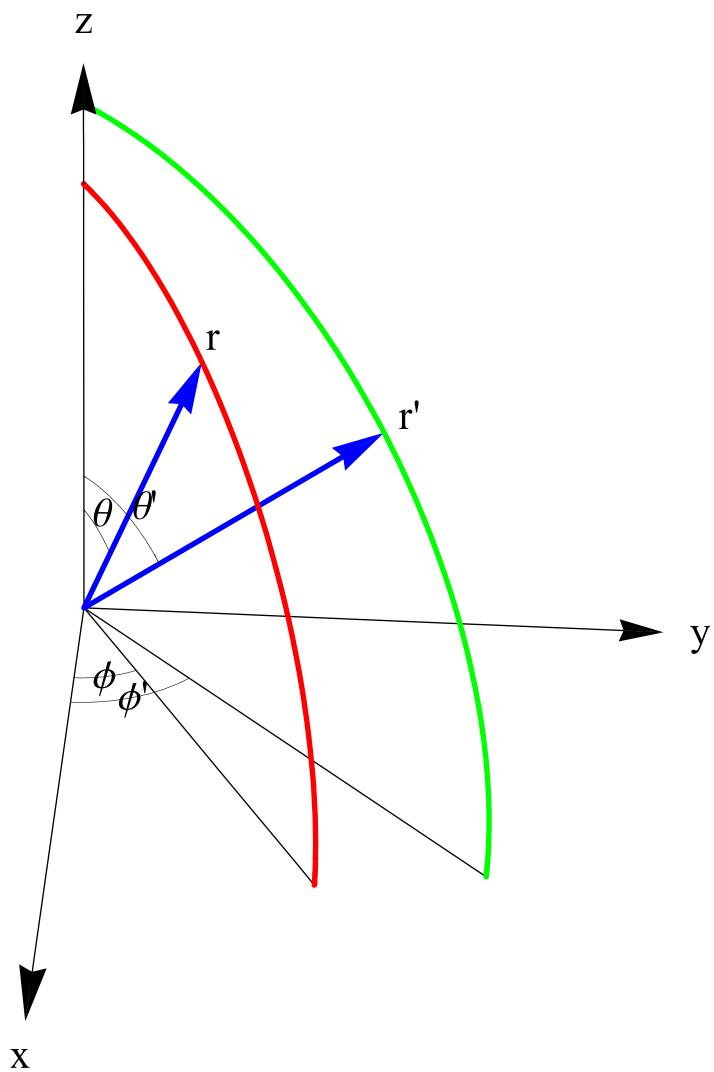


Fig. The angle γ is the angle between the vectors ***r*** and **r**'.

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

$$\mathbf{r}' = r'(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$$

$$\begin{aligned}\mathbf{r}\cdot\mathbf{r}' &= rr'\cos\gamma = rr'(\sin\theta\cos\phi\sin\theta'\cos\phi' + \sin\theta\sin\phi\sin\theta'\sin\phi' + \cos\theta\cos\theta' + \\ &= rr'[\sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta']\end{aligned}$$

or

$$\cos\gamma = \sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'$$

From

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^{\,l}}{r_>^{\,l+1}} P_l(\cos\gamma) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(\frac{4\pi}{2l+1}\right) \frac{r_<^{\,l}}{r_>^{\,l+1}} Y_l^m(\theta,\phi) Y_l^{m^*}(\theta',\phi')$$

we have

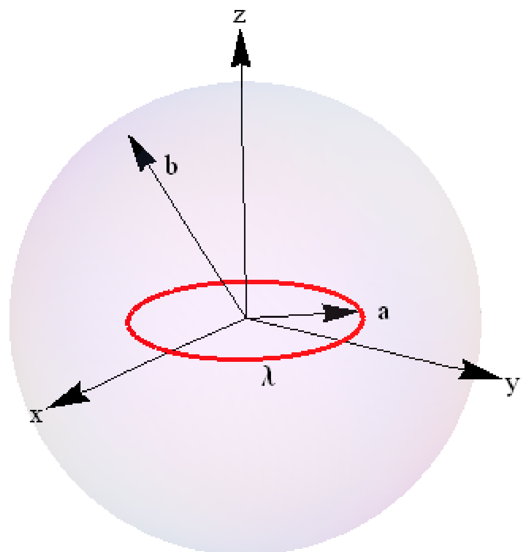
$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta,\phi) Y_l^{m^*}(\theta',\phi')$$

(addition theorem)

19.7 Electric potential - I

Jackson: Classical Electrodynamics

We consider a hollow grounded sphere of radius b with a concentric ring of charge of radius a and the uniform charge density λ (= $Q/(2\pi a)$). Q is the total charge. The ring of charge is located in the x-y plane. We discuss the distribution of the electric potential Φ.



The electric potential Φ is described by a Poisson equation,

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon_0}$$

Using the Green's function, the electric potential Φ can be obtained as

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\int_V G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')d^3\mathbf{r}'$$

where

$$\rho(\mathbf{r}') = A\delta(r'-a)\delta(\mu') .$$

Note that the constant A is determined as

$$2\pi a\lambda = \int_V \rho(\mathbf{r}')d^3\mathbf{r}' = \int_0^\infty A\delta(r'-a)r'^2\,dr'\int_{-1}^{1}\delta(\mu')d\mu'\int_0^{2\pi}d\phi'$$
$$= 2\pi A a^2$$

or

$$A = \frac{\lambda}{a} = \frac{Q}{2\pi a^2},$$

since the total charge Q is $(2\pi a\lambda)$. Here we use the Green's function ($a\rightarrow 0$);

$$G(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r_<^{\,l}\left(\frac{1}{r_>^{\,l+1}} - \frac{r_>^{\,l}}{b^{2l+1}}\right)}{(2l+1)}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$

Then the electric potential $\Phi(\mathbf{r})$ is obtained as

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\int_V \frac{\lambda}{a}\delta(r'-a)\delta(\mu')\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r_<^{\,l}\left(\frac{1}{r_>^{\,l+1}} - \frac{r_>^{\,l}}{b^{2l+1}}\right)}{(2l+1)}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')r'^2\,dr'\,d\mu'\,d\phi'$$

Here $Y_l^m(\theta,\phi)$ can be also expressed by

$$Y_l^m(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta),$$

where $P_l^m(\cos\theta)$ is the associated Legendre function. We note that

$$\int\int \delta(\mu')Y_l^{m^*}(\theta',\phi')d\mu'\,d\phi' = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\int\delta(\mu')P_l^m(\mu')d\mu'\int_0^{2\pi}e^{-im\phi'}d\phi'$$
$$= 2\pi(-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(0)\delta_{m,0}$$
$$= 2\pi\sqrt{\frac{(2l+1)}{4\pi}}P_l(0)\delta_{m,0}$$

$$\Phi(\mathbf{r}) = \frac{2\pi}{\varepsilon_0}\int_0^\infty dr'\frac{\lambda}{a}r'^2\,\delta(r'-a)\sum_{l=0}^{\infty}\frac{r_<^{\,l}\left(\frac{1}{r_>^{\,l+1}}-\frac{r_>^{\,l}}{b^{2l+1}}\right)}{2l+1}\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)\sqrt{\frac{2l+1}{4\pi}}P_l(0)$$

$$= \frac{1}{2\varepsilon_0}\frac{\lambda}{a}\int_0^\infty drr'^2\,\delta(r'-a)\sum_{l=0}^{\infty}P_l(0)r_<^{\,l}\left(\frac{1}{r_>^{\,l+1}}-\frac{r_>^{\,l}}{b^{2l+1}}\right)P_l(\cos\theta)$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}P_l(0)r_<^{\,l}\left(\frac{1}{r_>^{\,l+1}}-\frac{r_>^{\,l}}{b^{2l+1}}\right)P_l(\cos\theta)$$

where

$r_> = r$, $r_< = a$ for $r>a$

$r_> = a$, $r_< = r$ for $r<a$

For $r<a$, $L_{\max} = 5$ (the highest term in the summation),

$$\Phi(\mathbf{r}) =$$

$$\frac{1}{4\pi\epsilon 0}Q\left(\frac{1}{a}-\frac{1}{b}-\frac{1}{4}\left(\frac{1}{a^3}-\frac{a^2}{b^5}\right)r^2\left(-1+3\,\mathrm{Cos}[\theta]^2\right)+\frac{3}{64}\left(\frac{1}{a^5}-\frac{a^4}{b^9}\right)r^4\left(3-30\,\mathrm{Cos}[\theta]^2+35\,\mathrm{Cos}[\theta]^4\right)\right)$$

For $a<r<b$, $L_{\max} = 5$ (the highest term in the summation),

$$\Phi(\mathbf{r}) =$$

$$\frac{1}{4\pi\epsilon 0}Q\left(-\frac{1}{b}+\frac{1}{r}-\frac{1}{4}a^2\left(\frac{1}{r^3}-\frac{r^2}{b^5}\right)\left(-1+3\,\mathrm{Cos}[\theta]^2\right)+\frac{3}{64}a^4\left(\frac{1}{r^5}-\frac{r^4}{b^9}\right)\left(3-30\,\mathrm{Cos}[\theta]^2+35\,\mathrm{Cos}[\theta]^4\right)\right)$$

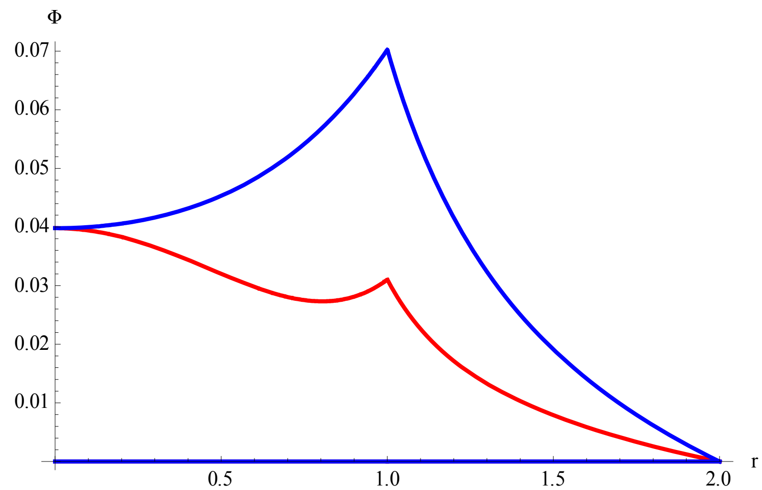


Fig. Plot of the electric potential Φ as a function of r. red ($\theta = 0$) and blue ($\theta = \pi/2$). $a = 1$. $b = 2$. $Q = 1$. $\varepsilon_0 = 1$. $L_{\max} = 5$.

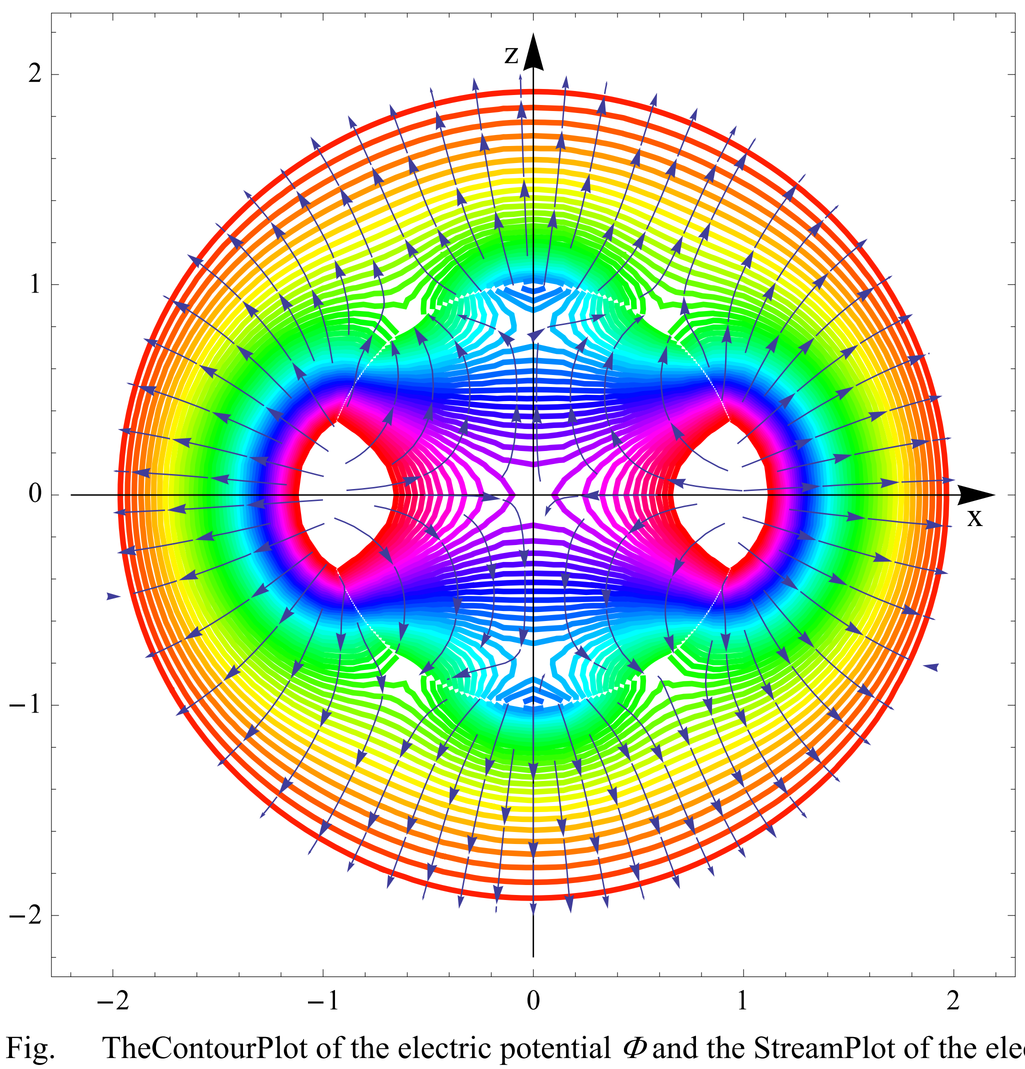
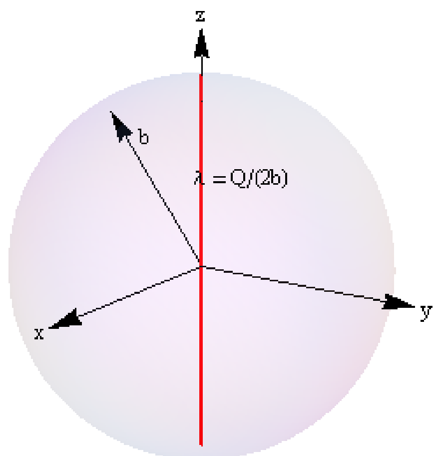


Fig. TheContourPlot of the electric potential Φ and the StreamPlot of the electric field. a = 1., $b = 2$, $\varepsilon_0 = 1$. $Q = 1$. We use $L_{\max} = 5$.

19.8 Electric potential - II

Jackson: Classical Electrodynamics



We consider a hollow grounded sphere of radius b with a uniform line charge of total charge Q, located on the z axis between the north and south poles of the sphere. We discuss the distribution of the electric potential Φ. The volume charge density is given by

$$\rho(\mathbf{r}') = \frac{Q}{4\pi b r'^2}[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)].$$

Note that

$$\begin{aligned}\int_0^b r'^2\,dr'\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \rho(\mathbf{r}') &= \int_0^b r'^2\,dr'\int_0^{2\pi} d\phi' \int_0^{\pi} \sin\theta' d\theta' \frac{Q}{4\pi b r'^2}[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1)] \\ &= \frac{Q}{4\pi b}\int_0^b dr'\int_0^{2\pi} d\phi' \int_{-1}^{1}{}' d\mu' [\delta(\mu' - 1) + \delta(\mu' + 1)] \\ &= \frac{2Q}{4\pi b} b(2\pi) = Q\end{aligned}$$

The electric potential $\Phi(\mathbf{r})$ can be described by

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\int_V \frac{Q}{4\pi b r'^2}[\delta(\mu'-1)+\delta(\mu'+1)]$$

$$\times\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r_<^{\,l}(\frac{1}{r_>^{\,l+1}}-\frac{r_>^{\,l}}{b^{2l+1}})}{(2l+1)}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')r'^2\,dr'd\mu'd\phi'$$

Here we note

$$\begin{aligned}\int Y_l^{m^*}(\theta',\phi')[\delta(\mu'-1)+\delta(\mu'+1)]d\mu'd\phi' &= (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\int[\delta(\mu'-1)+\delta(\mu'+1)]P_l^m(\mu')d\mu'\int_0^{2\pi}e^{-im\phi'}d\phi' \\ &= 2\pi(-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}[P_l^m(1)+P_l^m(-1)]\delta_{m,0} \\ &= 2\pi\sqrt{\frac{2l+1}{4\pi}}[P_l(1)+P_l(-1)]\delta_{m,0}\end{aligned}$$

$$Y_l^0(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

and

$$P_l(1) = 1, \qquad P_l(-1) = (-1)^l .$$

Then we have

$$\Phi(\mathbf{r}) = \frac{Q}{8\pi\varepsilon_0 b}\sum_{l=0}^{\infty}P_l(\cos\theta)[P_l(1)+P_l(-1)]\int_0^b dr'\,r_<^{\,l}(\frac{1}{r_>^{\,l+1}}-\frac{r_>^{\,l}}{b^{2l+1}})$$

For the integrand,

$r_< = r$, $r >= r'$ for $0<r<r'<b$

$r_< = r'$, $r >= r$ for $0<r'<r< <b$

Then we get

$$I = \int_0^b dr' r_<'^l (\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}) = \int_0^r (\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}) dr' + \int_r^b dr' r^l (\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}})$$

$$= (\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}) \int_0^r r'^l \, dr' + r^l \int_r^b dr' (\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}})$$

$$= \frac{2l+1}{l(l+1)} [1 - \left(\frac{r}{b}\right)^l]$$

For $l = 0$,

$$\lim_{l \to 0} I = \ln\left(\frac{b}{r}\right)$$

Thus we get

$$\Phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 b} \{\ln(\frac{b}{r}) + \sum_{l=1}^{\infty} P_l(\cos\theta) \frac{[1+(-1)^l]}{2} \frac{2l+1}{l(l+1)} [1 - \left(\frac{r}{b}\right)^l]\}$$

where

$$P_0(\cos\theta) = 1.$$

19.9 Electric potential due to charge distribution

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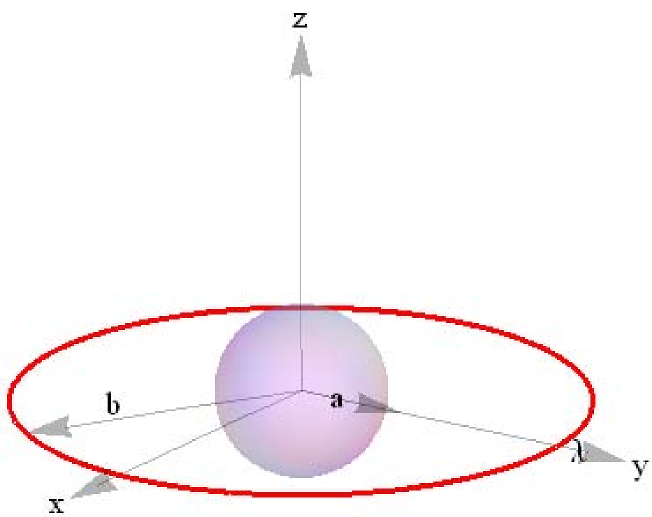

We consider a grounded sphere of radius a and a ring of charge of radius b with the uniform charge density λ. The ring of charge is located in the x-y plane. We discuss the distribution of the electric potential Φ.

The electric potential is described by a Poisson equation,

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}$$

Using the Green's function, the electric potential Φ can be obtained as

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\int_V G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')d^3\mathbf{r}'$$

where

$$\rho(\mathbf{r}') = A\delta(r'-b)\delta(\mu').$$

with

$$A = \frac{\lambda}{b}.$$

Note that the constant A is determined as

$$Q = 2\pi b\lambda = \int_V \rho(\mathbf{r}')d^3\mathbf{r}' = \int_0^\infty A\delta(r'-b)r'^2\,dr'\int_{-1}^{1}\delta(\mu')d\mu'\int_0^{2\pi}d\phi'$$
$$= 2\pi Ab^2$$

or

$$A = \frac{\lambda}{b},$$

since the total charge is $(2\pi b\lambda)$.

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon_0}\int_V \frac{\lambda}{b}\delta(r'-b)\delta(\mu')\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(r_<{}^{2l+1} - a^{2l+1})}{(2l+1)r_>{}^{l+1}r_<{}^{l+1}}Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')r'^2\,dr'd\mu'd\phi'$$

Here

$Y_l^m(\theta,\phi)$ can be also expressed by

$$Y_l^m(\theta,\phi) = (-1)^m\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta)$$

where $P_l^m(\cos\theta)$ is the associated Legendre function.

Note that

$$\int\int \delta(\mu')Y_l^{m^*}(\theta',\phi')d\mu' d\phi' = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\int \delta(\mu')P_l^m(\mu')d\mu' \int_0^{2\pi} e^{-im\phi'}d\phi'$$

$$= 2\pi\sqrt{\frac{2l+1}{4\pi}}P_l(0)\delta_{m,0}$$

$$\Phi(\mathbf{r}) = \frac{2\pi}{\varepsilon_0}\int_0^\infty dr' \frac{\lambda}{b} r'^2\, \delta(r'-b)\sum_{l=0}^{\infty}\frac{(r_<^{\;2l+1}-a^{2l+1})}{(2l+1)r_>^{\;l+1}r_<^{\;l+1}}\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)\sqrt{\frac{2l+1}{4\pi}}P_l(0)$$

$$= \frac{1}{2\varepsilon_0}\frac{\lambda}{b}\int_0^\infty drr'^2\, \delta(r'-b)\sum_{l=0}^{\infty}\frac{(r_<^{\;2l+1}-a^{2l+1})}{r_>^{\;l+1}r_<^{\;l+1}}P_l(\cos\theta)P_l(0)$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{(r_<^{\;2l+1}-a^{2l+1})}{r^{l+1}b^{l+1}}P_l(\cos\theta)P_l(0)$$

where

$$P_l(0) = 1$$

$r_> = r$
$r_< = b$ for $r>b$

$r_> = b$
$r_< = r$ for $r<b$

For $a<r<b$,

$$\frac{1}{4\pi\epsilon 0}Q\left(\frac{-a+r}{b\,r} - \frac{(-a^5+r^5)\,(-1+3\,\mathrm{Cos}[\theta]^2)}{4\,b^3\,r^3} + \frac{3\,(-a^9+r^9)\,(3-30\,\mathrm{Cos}[\theta]^2+35\,\mathrm{Cos}[\theta]^4)}{64\,b^5\,r^5} - \frac{5\,(-a^{13}+r^{13})\,(-5+105\,\mathrm{Cos}[\theta]^2-315\,\mathrm{Cos}[\theta]^4+231\,\mathrm{Cos}[\theta]^6)}{256\,b^7\,r^7} + \frac{35\,(-a^{17}+r^{17})\,(35-1260\,\mathrm{Cos}[\theta]^2+6930\,\mathrm{Cos}[\theta]^4-12\,012\,\mathrm{Cos}[\theta]^6+6435\,\mathrm{Cos}[\theta]^8)}{16\,384\,b^9\,r^9} - \frac{1}{65\,536\,b^{11}\,r^{11}}63\,(-a^{21}+r^{21})\,(-63+3465\,\mathrm{Cos}[\theta]^2 - 30\,030\,\mathrm{Cos}[\theta]^4+90\,090\,\mathrm{Cos}[\theta]^6-109\,395\,\mathrm{Cos}[\theta]^8+46\,189\,\mathrm{Cos}[\theta]^{10})\right)$$

For $r>b$,

$$\frac{1}{4\pi\epsilon0}Q\left(\frac{-a+b}{b\,r}-\frac{(-a^5+b^5)\,(-1+3\,\mathrm{Cos}[\theta]^2)}{4\,b^3\,r^3}+\frac{3\,(-a^9+b^9)\,(3-30\,\mathrm{Cos}[\theta]^2+35\,\mathrm{Cos}[\theta]^4)}{64\,b^5\,r^5}-\frac{5\,(-a^{13}+b^{13})\,(-5+105\,\mathrm{Cos}[\theta]^2-315\,\mathrm{Cos}[\theta]^4+231\,\mathrm{Cos}[\theta]^6)}{256\,b^7\,r^7}+\frac{35\,(-a^{17}+b^{17})\,(35-1260\,\mathrm{Cos}[\theta]^2+6930\,\mathrm{Cos}[\theta]^4-12\,012\,\mathrm{Cos}[\theta]^6+6435\,\mathrm{Cos}[\theta]^8)}{16\,384\,b^9\,r^9}-\frac{1}{65\,536\,b^{11}\,r^{11}}63\,(-a^{21}+b^{21})\,(-63+3465\,\mathrm{Cos}[\theta]^2-30\,030\,\mathrm{Cos}[\theta]^4+90\,090\,\mathrm{Cos}[\theta]^6-109\,395\,\mathrm{Cos}[\theta]^8+46\,189\,\mathrm{Cos}[\theta]^{10})\right)$$

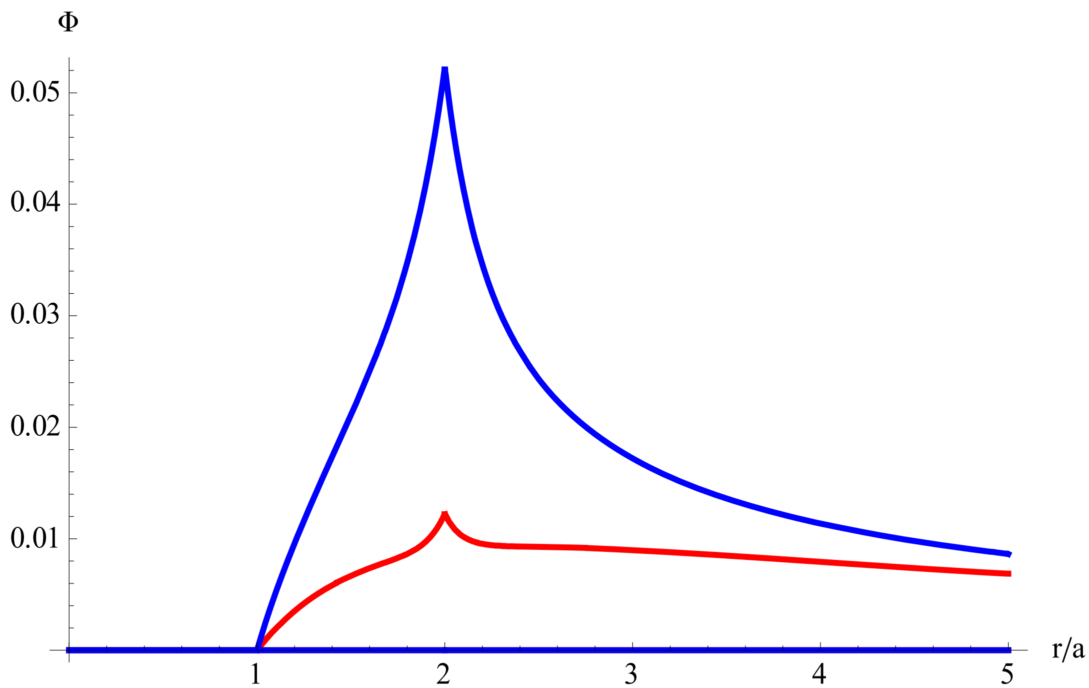

Fig. Plot of the electric potential Φ as a function of *r* (the first 20 terms, $l \le 20$). red ($\theta = 0$) and blue ($\theta = \pi/2$). $a = 1$. $b = 2$. $Q = 1$. $\varepsilon_0 = 1$.

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APPENDIX

From the Green's law, we have

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\rho(\mathbf{r}')}{\varepsilon_0} d^3\mathbf{r}' + \frac{1}{4\pi}\int_S [\frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\phi(\mathbf{r}') - \phi(\mathbf{r}')\nabla'(\frac{1}{|\mathbf{r}-\mathbf{r}'|})]\cdot\mathbf{n}da'$$