Chapter 1S Rotation matrix: Eulerian angles

Masatsugu Sei Suzuki
Department of Physics, State University of New York at Binghgamton

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We consider two types of rotations here; (i) Type-I: the rotation of the orthogonal basis with the position of vector being fixed (passive role of transformation). (ii) Type-II: the rotation of the position vector with the orthogonal basis being fixed (active role of transformation).

1S.1 2D rotation matrix (type-I rotation)

First we consider the type-I rotation for the two-dimensional (2D) system. Suppose that the rotation of the orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ by angle θ around the z axis (counter clock wise) yields to the new orthogonal basis $\{\boldsymbol{e}_1', \boldsymbol{e}_2'\}$ as shown in Fig. We note that the position vector $\boldsymbol{r}$ is fixed under the rotation. This implies that $\boldsymbol{r}$ in the old basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is equal to $\mathbf{r}'$ in the new basis $\{\boldsymbol{e}_1', e_2'\}$; $\boldsymbol{r} = \boldsymbol{r}'$.

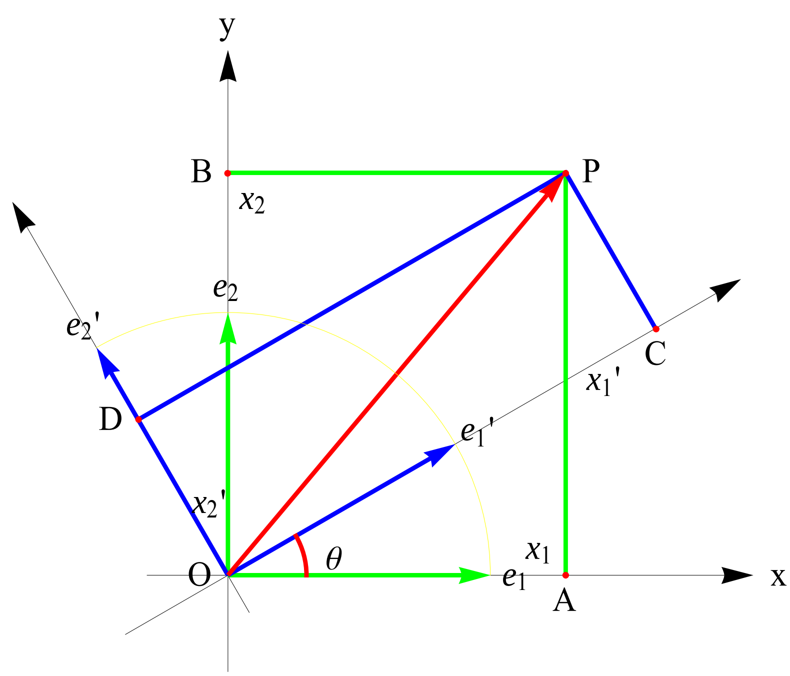


Fig. Rotation of the coordinate axes. $\overrightarrow{OP} = \boldsymbol{r} = \boldsymbol{r}'$. $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$; the old orthogonal basis. $\{\boldsymbol{e}_1', \boldsymbol{e}_2'\}$;. and the new orthogonal basis.

We assume that

$$\mathbf{e}_1' = a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2$$
$$\mathbf{e}_2' = a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2$$

with

$$
\begin{aligned}
a_{11} &= (\mathbf{e}_1.\mathbf{e}_1') = \cos\theta \\
a_{12} &= (\mathbf{e}_2.\mathbf{e}_1') = \sin\theta \\
a_{21} &= (\mathbf{e}_1.\mathbf{e}_2') = -\sin\theta \\
a_{22} &= (\mathbf{e}_1.\mathbf{e}_2') = \cos\theta
\end{aligned}
$$

or

$$
\Re(-\theta) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}
$$

where the matrix elements $\{a_{ij}\}$ are real and $\Re(-\theta)$ is the rotation matrix. We use $(-\theta)$ for convenience. The transpose of the matrix $\Re(-\theta)$ is givrn by

$$
\Re^T(-\theta) = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.
$$

Then we have

$$
\begin{aligned}
\mathbf{e}_1' &= a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \Re^T(-\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\mathbf{e}_2' &= a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2 = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = \Re^T(-\theta)\begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

Suppose that the vector $\boldsymbol{r}$ can be expressed by

$$
\begin{aligned}
\mathbf{r} &= \sum_i x_i \mathbf{e}_i = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \\
\mathbf{r}' &= \sum_i x_i' \mathbf{e}_i' = x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' \\
\mathbf{r} &= \mathbf{r}'
\end{aligned}
$$

in the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ and the basis $\{\boldsymbol{e}_1', \boldsymbol{e}_2'\}$, respectively. Then we have

$$x_1' = \mathbf{e}_1'\cdot\mathbf{r}' = \mathbf{e}_1'\cdot\mathbf{r} = \mathbf{e}_1'\cdot(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = a_{11}x_1 + a_{12}x_2$$
$$x_2' = \mathbf{e}_2'\cdot\mathbf{r}' = \mathbf{e}_2'\cdot\mathbf{r} = \mathbf{e}_2'\cdot(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = a_{21}x_1 + a_{22}x_2$$

or

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \Re(-\theta)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \qquad \text{(A)}$$

((**Interpretation**))

This is interpreted as an orthogonal transformation as a rotation of the vector, leaving the coordinate system unchanged. We can rotate **r**' clockwise by an angle θ to a new vector *r*'. The component of new vector r' will then be related to the component of old by the same equations (A).

((**Mathematica**))

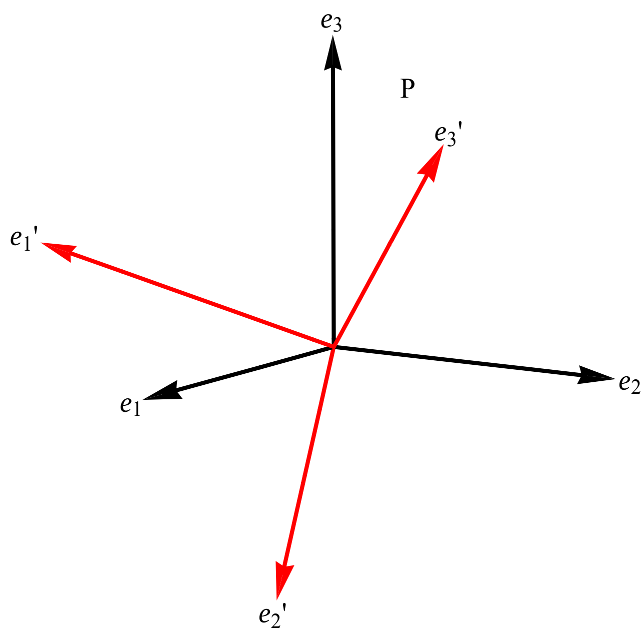
RotationMatrix[ϕ]:

To give the 2D rotation matrix that rotates 2D vectors counter-clockwise by ϕ radians.

In the present case, $\Re(-\theta)$ can be obtained as RotationMatrix[-θ].

```
Clear["Gobal`"];

R = RotationMatrix[-θ] // Simplify; R // MatrixForm

{{Cos[θ], Sin[θ]}, {-Sin[θ], Cos[θ]}}

R[[1]]

{Cos[θ], Sin[θ]}

R[[2]]

{-Sin[θ], Cos[θ]}
```

1S.2 3D rotation matrix (type-I rotation)



We now move to the 3D case.

$$
\begin{aligned}
\mathbf{e}_1' &= a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 + a_{13}\mathbf{e}_3 \\
\mathbf{e}_2' &= a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{23}\mathbf{e}_3 \\
\mathbf{e}_3' &= a_{31}\mathbf{e}_1 + a_{32}\mathbf{e}_2 + a_{33}\mathbf{e}_3
\end{aligned}
$$

The matrix $\Re$ is defined by

$$
\Re = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
$$

The transpose of the matrix $\Re$ is given by

$$
\Re^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}
$$

Then we have

$$\mathbf{e}_1' = a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 + a_{13}\mathbf{e}_3 = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = \Re^T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{e}_2' = a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{23}\mathbf{e}_3 = \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = \Re^T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{e}_1' = a_{31}\mathbf{e}_1 + a_{32}\mathbf{e}_2 + a_{33}\mathbf{e}_3 = \begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix} = \Re^T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Suppose that the vector ***r*** can be expressed by

$$\mathbf{r} = \sum_i x_i \mathbf{e}_i = \mathbf{r}' = \sum_i x_i' \mathbf{e}_i'$$

.

Then we have

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

or

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \Re \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The rotation matrix for the type-I rotation is given by $\Re$ (-θ, **w**).

((**Mathematica**))

RotationMatrix[-θ, ***w***]

To give the 3D rotation matrix for a clockwise rotation (angle θ) around the 3D vector ***w***.

1S3. Eulerian angles (type-I rotation)

We can carry out the transformation from a given Cartesian coordinate system to another one by means of three successive rotations (Eulerian angles) performed in a specific sequence. The sequence is started by rotating the initial system of axes x, y, z, by an angle ϕ counterclockwise around the z axis. The resultant coordinate system is labeled in the ξ, η, ζ axes. In the second stage, the intermediate axes, ξ, η, ζ, are rotated about the ξ axis counterclockwise by an angle θ to produce another intermediate set, the ξ', η', ζ' axes. Finally, the ξ', η', ζ' axes are rotated counterclockwise by an angle y about the z' axis to produce the desired x', y', z' system of axes.

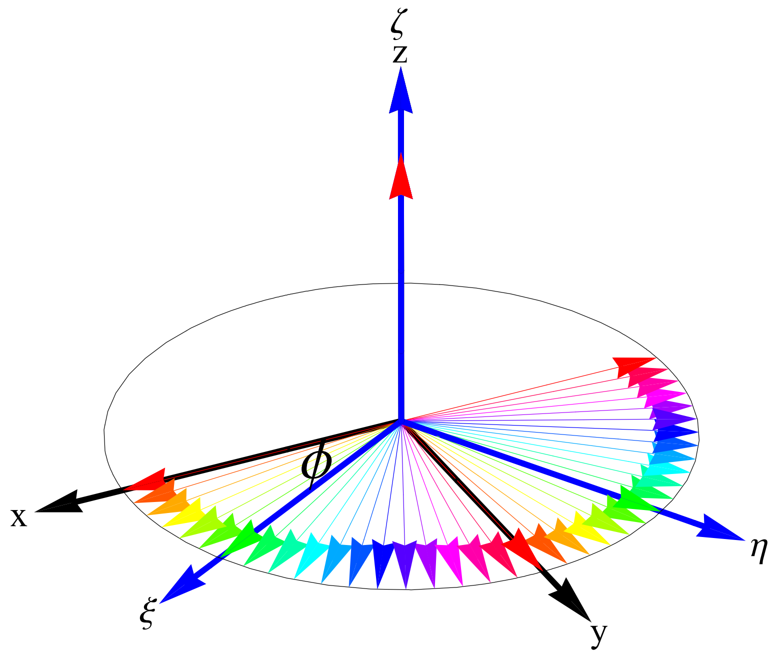
(a) The first rotation

The rotation (through ϕ about z) can be expressed as

$$\Re_z(-\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(x, y, z) \rightarrow (\xi, \eta, \zeta)$$

where $\zeta = z$.



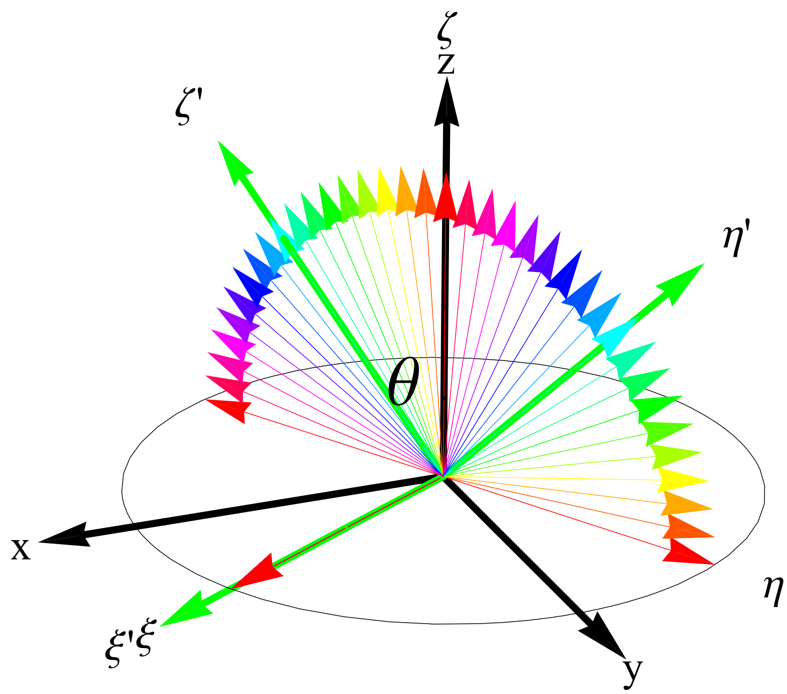
(b) The second rotation

The rotation (through θ about ξ) can be expressed as

$$\Re_x(-\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$(\xi, \eta, \zeta) \rightarrow (\xi', \eta', \zeta')$$

where $\xi' = \xi$.



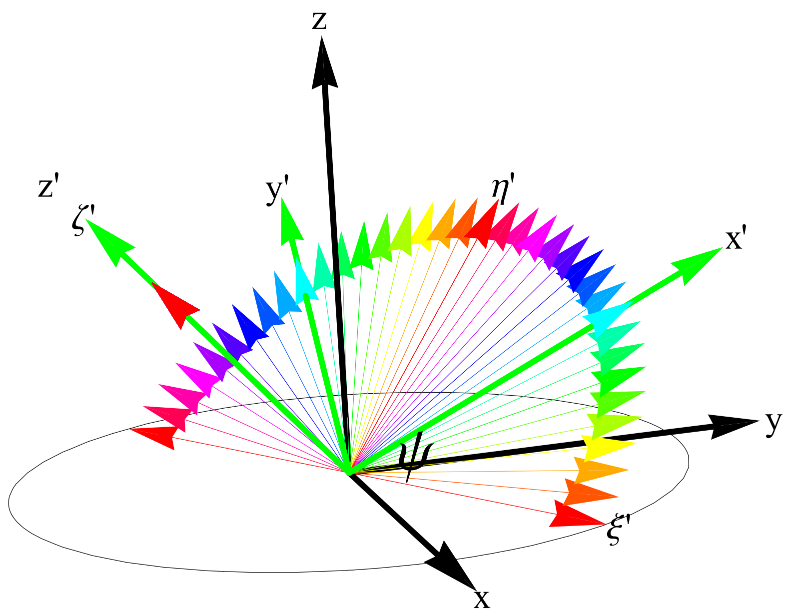
(c) The third rotation

The rotation through ψ about ζ' can be expressed as

$$\Re_z(-\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\xi',\ \eta',\ \zeta') \to (x',\ y',\ z')$$

where $\zeta' = z'$.



3D rotation matrix is given by

$$\Re = \Re_z(-\psi)\Re_x(-\theta)\Re_z(-\phi) = \begin{pmatrix} \Re_{11} & \Re_{12} & \Re_{13} \\ \Re_{21} & \Re_{22} & \Re_{23} \\ \Re_{31} & \Re_{32} & \Re_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\theta\sin\phi\cos\psi - \cos\phi\sin\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{e}_{x'} \\ \mathbf{e}_{y'} \\ \mathbf{e}_{z'} \end{pmatrix} = \Re \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \Re_{11} & \Re_{12} & \Re_{13} \\ \Re_{21} & \Re_{22} & \Re_{23} \\ \Re_{31} & \Re_{32} & \Re_{33} \end{pmatrix} \begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix}$$

Suppose that the position vector $\boldsymbol{r}$ in the orthogonal basis $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$ is the same as the position vector $\boldsymbol{r}'$ in the orthogonal basis $\{\boldsymbol{e}_x', \boldsymbol{e}_y', \boldsymbol{e}_z'\}$. Then r' can be described by

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \Re \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Re \mathbf{r}\,.$$

((**Mathematica**))

```
Clear["Gobal`"];

Rφ = RotationMatrix[-φ, {0, 0, 1}] // Simplify; Rφ // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\phi] & \text{Sin}[\phi] & 0 \\ -\text{Sin}[\phi] & \text{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
Rθ = RotationMatrix[-θ, {1, 0, 0}] // Simplify; Rθ // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\theta] & \text{Sin}[\theta] \\ 0 & -\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

```
Rψ = RotationMatrix[-ψ, {0, 0, 1}] // Simplify; Rψ // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\psi] & \text{Sin}[\psi] & 0 \\ -\text{Sin}[\psi] & \text{Cos}[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
S = Rψ.Rθ.Rφ // Simplify; S // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\phi]\,\text{Cos}[\psi] - \text{Cos}[\theta]\,\text{Sin}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\psi]\,\text{Sin}[\phi] + \text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Sin}[\theta]\,\text{Sin}[\psi] \\ -\text{Cos}[\theta]\,\text{Cos}[\psi]\,\text{Sin}[\phi] - \text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\theta]\,\text{Cos}[\phi]\,\text{Cos}[\psi] - \text{Sin}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\psi]\,\text{Sin}[\theta] \\ \text{Sin}[\theta]\,\text{Sin}[\phi] & -\text{Cos}[\phi]\,\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}$$

```
S[[1]]

{Cos[φ] Cos[ψ] - Cos[θ] Sin[φ] Sin[ψ], Cos[ψ] Sin[φ] + Cos[θ] Cos[φ] Sin[ψ], Sin[θ] Sin[ψ]}

S[[2]]

{-Cos[θ] Cos[ψ] Sin[φ] - Cos[φ] Sin[ψ], Cos[θ] Cos[φ] Cos[ψ] - Sin[φ] Sin[ψ], Cos[ψ] Sin[θ]}

S[[3]]

{Sin[θ] Sin[φ], -Cos[φ] Sin[θ], Cos[θ]}
```

1S4. Angular velocity

We derive the angular velocity in the new coordinate (x', y', z'). The rotation matrix is given by

$$\Re(t) = \Re_z(-\psi(t))\Re_x(-\theta(t))\Re_z(-\phi(t))$$

where the Euler angles are dependent on time t.

$$\mathbf{\Omega}_{\phi} = \Re(t)\begin{pmatrix} 0 \\ 0 \\ \dot{\phi}(t) \end{pmatrix}_{x,y,z} = \begin{pmatrix} \sin\theta(t)\sin\psi(t)\dot{\phi}(t) \\ \sin\theta(t)\cos\psi(t)\dot{\phi}(t) \\ \cos\theta(t)\dot{\phi}(t) \end{pmatrix}_{x',y',z'}$$

where $\dot{\phi}(t)$ is directed along the z axis.

$$\mathbf{\Omega}_{\theta} = \Re_z(-\psi(t))\Re_x(-\theta(t)\begin{pmatrix} \dot{\theta}(t) \\ 0 \\ 0 \end{pmatrix}_{\xi,\eta,\varsigma} = \begin{pmatrix} \cos\psi(t)\dot{\theta}(t) \\ -\sin\psi(t)\dot{\theta}(t) \\ 0 \end{pmatrix}_{x',y',z'}$$

where $\dot{\theta}(t)$ is directed along the ξ axis.

$$\mathbf{\Omega}_{\psi} = \Re_z(-\psi(t))\begin{pmatrix} 0 \\ 0 \\ \dot{\psi}(t) \end{pmatrix}_{\xi',\eta',\varsigma'} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}(t) \end{pmatrix}_{x',y,z'}$$

where $\dot{\psi}(t)$ is directed along the ζ' axis. Then we have the angular velocity in the new coordiante (x', y', z') as

$$\begin{aligned} \mathbf{\Omega}_{x'y'z'} &= \mathbf{\Omega}_{\phi} + \mathbf{\Omega}_{\theta} + \mathbf{\Omega}_{\psi} \\ &= \begin{pmatrix} \Omega_{x'} \\ \Omega_{y'} \\ \Omega_{z'} \end{pmatrix} = \begin{pmatrix} \cos\psi(t)\dot{\theta}(t) + \sin\theta(t)\sin\psi(t)\dot{\phi}(t) \\ -\sin\psi(t)\dot{\theta}(t) + \sin\theta(t)\cos\psi(t)\dot{\phi}(t) \\ \cos\theta(t)\dot{\phi}(t) + \dot{\psi}(t) \end{pmatrix}_{x',y',z'} \end{aligned}$$

The angular velocity in the original (x, y, z) coordinate is obtained as

$$\begin{aligned} \mathbf{\Omega}_{x,y,z} &= \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix} \\ &= \Re^{-1}(t)\mathbf{\Omega}_{x'y'z'} = \Re^{T}(t)\mathbf{\Omega}_{x'y'z'} \\ &= \begin{pmatrix} \cos\phi(t)\dot{\theta}(t) + \sin\theta(t)\sin\phi(t)\dot{\psi}(t) \\ \sin\phi(t)\dot{\theta}(t) - \sin\theta(t)\cos\phi(t)\dot{\psi}(t) \\ \dot{\phi}(t) + \cos\theta(t)\dot{\psi}(t) \end{pmatrix}_{x,y,z} \end{aligned}$$

((**Mathematica**))

D1: the rotation with the angle ϕ around the z axis
C1: the rotation with the angle θ around the ξ axis
B1: the rotation with the angle ψ around the ζ' axis
A1: the resultant rotation is described by a matrix
A1 = B1 C1 D1

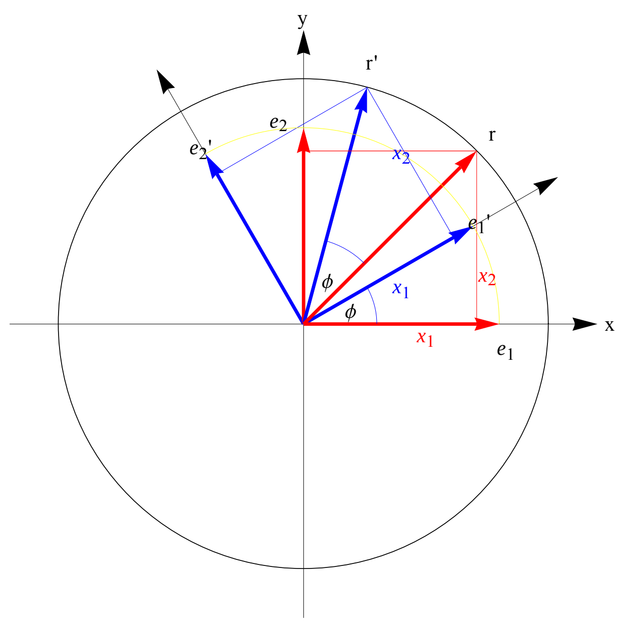
```
Clear["Global`*"];

D1 = RotationMatrix[-ϕ[t], {0, 0, 1}];
C1 = RotationMatrix[-θ[t], {1, 0, 0}];
B1 = RotationMatrix[-ψ[t], {0, 0, 1}];
A1 = B1.C1.D1 // Simplify;

Ωϕ1 = A1.{0, 0, ϕ'[t]}

{Sin[θ[t]] Sin[ψ[t]] ϕ′[t],
 Cos[ψ[t]] Sin[θ[t]] ϕ′[t], Cos[θ[t]] ϕ′[t]}

Ωθ1 = B1.C1.{θ'[t], 0, 0}

{Cos[ψ[t]] θ′[t], -Sin[ψ[t]] θ′[t], 0}

Ωψ1 = B1.{0, 0, ψ'[t]}

{0, 0, ψ′[t]}

Ω1 = Ωϕ1 + Ωθ1 + Ωψ1

{Cos[ψ[t]] θ′[t] + Sin[θ[t]] Sin[ψ[t]] ϕ′[t],
 -Sin[ψ[t]] θ′[t] + Cos[ψ[t]] Sin[θ[t]] ϕ′[t],
 Cos[θ[t]] ϕ′[t] + ψ′[t]}
```

The angular velocity with respect to the body axes

```
Ω = Inverse[A1].Ω1 // Simplify

{Cos[ϕ[t]] θ′[t] + Sin[θ[t]] Sin[ϕ[t]] ψ′[t],
 Sin[ϕ[t]] θ′[t] - Cos[ϕ[t]] Sin[θ[t]] ψ′[t],
 ϕ′[t] + Cos[θ[t]] ψ′[t]}
```

1S5. 2D rotation matrix (type-II rotation)

Suppose that the vector $\boldsymbol{r}$ is rotated through θ (counter-clock wise) around the z axis. The position vector $\boldsymbol{r}$ is changed into $\boldsymbol{r}'$ in the same orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.



In this Fig, we have

$$\mathbf{e}_1 \cdot \mathbf{e}_1' = \cos\phi \qquad \mathbf{e}_2 \cdot \mathbf{e}_1' = \sin\phi$$
$$\mathbf{e}_1 \cdot \mathbf{e}_2' = -\sin\phi, \qquad \mathbf{e}_2 \cdot \mathbf{e}_2' = \cos\phi$$

We define $\boldsymbol{r}$ and $\boldsymbol{r}'$ as

$$\mathbf{r}' = x_1'\mathbf{e}_1 + x_2'\mathbf{e}_2 = x_1\mathbf{e}_1' + x_2\mathbf{e}_2',$$

and

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2.$$

Using the relation

$$\mathbf{e}_1 \cdot \mathbf{r}' = \mathbf{e}_1 \cdot (x_1'\mathbf{e}_1 + x_2'\mathbf{e}_2) = \mathbf{e}_1 \cdot (x_1\mathbf{e}_1' + x_2\mathbf{e}_2')$$
$$\mathbf{e}_2 \cdot \mathbf{r}' = \mathbf{e}_2 \cdot (x_1'\mathbf{e}_1 + x_2'\mathbf{e}_2) = \mathbf{e}_2 \cdot (x_1\mathbf{e}_1' + x_2\mathbf{e}_2')$$

we have

$$x_1' = \mathbf{e}_1 \cdot (x_1\mathbf{e}_1' + x_2\mathbf{e}_2') = x_1\cos\phi - x_2\sin\phi$$
$$x_2' = \mathbf{e}_2 \cdot (x_1\mathbf{e}_1' + x_2\mathbf{e}_2') = x_1\sin\phi + x_2\cos\phi$$

or

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \Re(\phi)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

((**Mathematica**))

$\Re(\phi)$ is obtained in the Mathematica as RotationMatrix[ϕ] to gives the 2D rotation matrix that rotates 2D vectors counterclockwise by ϕ radians.

((**Note**))

Rotation around the *z* axis in the complex plane

$$x' + iy' = e^{i\phi}(x + iy) = (\cos\phi + i\sin\phi)(x + iy) = x\cos\phi - y\sin\phi + i(x\sin\phi + y\cos\phi)$$

$$\begin{pmatrix} x' \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

1S.6 3D Rotation matrix (type-II rotation)

We discuss the three-dimensional (3D) case,

$$\mathbf{r} = \sum_{j=1}^{3} x_j\mathbf{e}_j, \qquad \mathbf{r}' = \sum_{j=1}^{3} x_j'\mathbf{e}_j = \sum_{j=1}^{3} x_j\mathbf{e}_j'$$

$$\mathbf{r}' = \Re_z(\phi)\mathbf{r} = \Re_z(\phi)(\sum_{j=1}^{3} x_j\mathbf{e}_j) = \sum_{j=1}^{3} x_j\Re_z(\phi)\mathbf{e}_j = \sum_{j=1}^{3} x_j\mathbf{e}_j'$$

where

$$\Re_z(\phi)\mathbf{e}_j = \mathbf{e}_j'$$

Thus we have

$$(\sum_{j=1}^{3} x_j \mathbf{e}_j') \cdot \mathbf{e}_i = (\sum_{j=1}^{3} x_j' \mathbf{e}_j) \cdot \mathbf{e}_i$$

or

$$\sum_{j=1}^{3} x_j' \delta_{j,i} = x_i' = \sum_{j=1}^{3} (\mathbf{e}_i \cdot \mathbf{e}_j') x_j = \sum_{j=1}^{3} \Re_{ij} x_j$$

where

$$\Re_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j'$$

(i) The rotation around the z axis.

$$\Re_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1' = \cos\phi, \quad \Re_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2' = -\sin\phi, \quad \Re_{13} = \mathbf{e}_1 \cdot \mathbf{e}_3' = 0$$
$$\Re_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1' = \sin\phi, \quad \Re_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2' = \cos\phi, \quad \Re_{23} = \mathbf{e}_2 \cdot \mathbf{e}_3' = 0$$
$$\Re_{31} = \mathbf{e}_3 \cdot \mathbf{e}_1' = 0, \quad \Re_{32} = \mathbf{e}_3 \cdot \mathbf{e}_2' = 0, \quad \Re_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3' = 1$$

$$\mathbf{r}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \Re_{11} & \Re_{12} & \Re_{13} \\ \Re_{21} & \Re_{22} & \Re_{23} \\ \Re_{31} & \Re_{32} & \Re_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where

$$\Re_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) Rotation around the x axis

$$\Re_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

(iii) Rotation around the y axis

$$\Re_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

APPENDIX

In the Mathematica

The matrix R is represented by

R={{R_{11}, R_{12}, R_{13}},{R_{21}, R_{22}, R_{23}},{R_{31}, R_{32}, R_{33}}}.

Thus $\boldsymbol{e}_1$' is simply expressed as R[[1]] (={R_{11}, R_{12}, R_{13}}). $\boldsymbol{e}_2$' is simply expressed as R[[2]] (={R_{21}, R_{22}, R_{23}}). $\boldsymbol{e}_3$' is simply expressed as R[[3]] (={R_{31}, R_{32}, R_{33}}).

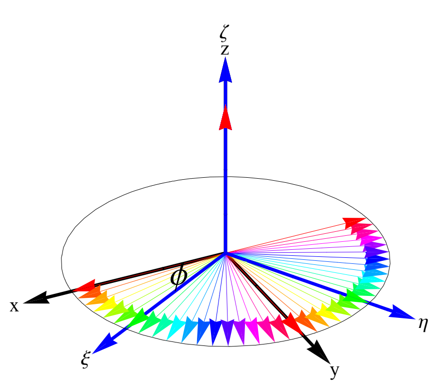
Rotation Matrix for the 3 D system
Euler angles

RotationMatrix[θ,w]
gives the 3D rotation matrix for a counterclockwise rotation around the 3D vector w

```
Clear["Gobal`"];

Rϕ = RotationMatrix[-ϕ, {0, 0, 1}] // Simplify; Rθ = RotationMatrix[-θ, {1, 0, 0}] // Simplify;
Rψ = RotationMatrix[-ψ, {0, 0, 1}] // Simplify;
S = Rψ.Rθ.Rϕ // Simplify;

S1 = S /. {θ → 0, ϕ → 0, ψ → 0};

g1 = Graphics3D[{Black, Thick, Arrow[{{0, 0, 0}, 1.3 S1[[1]]}], Arrow[{{0, 0, 0}, 1.3 S1[[2]]}],
    Arrow[{{0, 0, 0}, 1.3 S1[[3]]}], Text[Style["x", 12, Black], 1.35 S1[[1]]],
    Text[Style["y", 12, Black], 1.35 S1[[2]]], Text[Style["z", 12, Black], 1.35 S1[[3]]]},
   Boxed → False];

S2 = S /. {θ → 0, ψ → 0};

g2 =
  Graphics3D[
   {Table[{Hue[ϕ/(π/2)], Thin, Arrow[{{0, 0, 0}, S2[[1]]}], Arrow[{{0, 0, 0}, S2[[2]]}],
      Arrow[{{0, 0, 0}, S2[[3]]}]}, {ϕ, 0, π/2, π/(2×18)}]}, Boxed → False];

S21 = S /. {θ → 0, ψ → 0, ϕ → π/6};

g21 = Graphics3D[{Blue, Thick, Arrow[{{0, 0, 0}, 1.3 S21[[1]]}], Arrow[{{0, 0, 0}, 1.3 S21[[2]]}],
    Arrow[{{0, 0, 0}, 1.3 S21[[3]]}], Text[Style["ξ", 12, Black], 1.35 S21[[1]]],
    Text[Style["η", 12, Black], 1.35 S21[[2]]], Text[Style["ξ", 12, Black], 1.45 S21[[3]]],
    Text[Style["ϕ", Black, 20], {0.35, 0.1, 0}]}];

f2 = ParametricPlot3D[{Cos[α], Sin[α], 0}, {α, 0, 2 π}, PlotStyle → {Black, Thin}, Boxed → False];
```

Rotation-1
Rotation around z axis by an angle ϕ ;
$x - y - z \rightarrow \xi - \eta - \zeta$
where $\zeta = z$.

```
Show[g1, g2, g21, f2]
```



```
S3 = S /. {ϕ → π/6, ψ → 0};

g3 =
  Graphics3D[
   {Table[{Hue[θ/(π/2)], Thin, Arrow[{{0, 0, 0}, S3[[1]]}], Arrow[{{0, 0, 0}, S3[[2]]}],
      Arrow[{{0, 0, 0}, S3[[3]]}]}, {θ, 0, π/2, π/(2 × 18)}]}, Boxed → False];

S31 = S /. {θ → π/4, ψ → 0, ϕ → π/6};

g21 = Graphics3D[{Green, Thick, Arrow[{{0, 0, 0}, 1.3 S31[[1]]}], Arrow[{{0, 0, 0}, 1.3 S31[[2]]}],
    Arrow[{{0, 0, 0}, 1.3 S31[[3]]}], Text[Style["ξ", 12, Black], 1.35 S21[[1]]],
    Text[Style["η", 12, Black], 1.20 S21[[2]]], Text[Style["ζ", 12, Black], 1.45 S21[[3]]],
    Text[Style["ξ'", 12, Black], 1.45 S31[[1]]], Text[Style["η'", 12, Black], 1.45 S31[[2]]],
    Text[Style["ζ'", 12, Black], 1.45 S31[[3]]], Text[Style["θ", 20, Black], {0.05, -0.2, 0.25}]}];
```

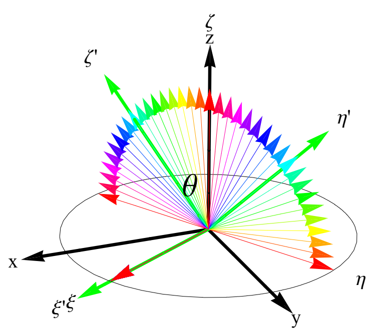
Rotation-2

Rotation around ξ (= ξ') axis by an angle θ ;

ξ- η - ζ → ξ' - η' - ζ'

where $\xi' = \xi$.

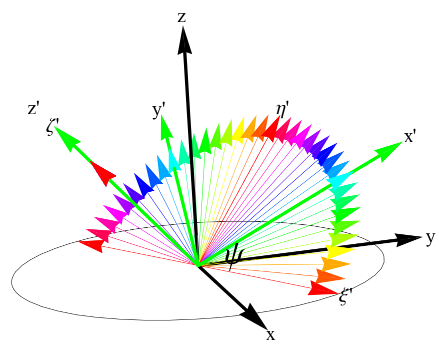
```
Show[g1, g21, g3, f2]
```



```
S4 = S /. {ϕ → π/6, θ → π/4};
g4 =
 Graphics3D[
  {Table[{Hue[ψ/(π/2)], Thin, Arrow[{{0, 0, 0}, S4[[1]]}], Arrow[{{0, 0, 0}, S4[[2]]}],
     Arrow[{{0, 0, 0}, S4[[3]]}]}, {ψ, 0, π/2, π/(2×18)}]}, Boxed → False];

S41 = S /. {θ → π/4, ψ → π/4, ϕ → π/6};

g41 = Graphics3D[{Green, Thick, Arrow[{{0, 0, 0}, 1.3 S41[[1]]}], Arrow[{{0, 0, 0}, 1.3 S41[[2]]}],
    Arrow[{{0, 0, 0}, 1.3 S41[[3]]}], Text[Style["x'", 12, Black], 1.35 S41[[1]]],
    Text[Style["y'", 12, Black], 1.35 S41[[2]]], Text[Style["z'", 12, Black], 1.45 S41[[3]]],
    Text[Style["ξ'", 12, Black], 1.05 S31[[1]]], Text[Style["η'", 12, Black], 1.05 S31[[2]]],
    Text[Style["ζ'", 12, Black], 1.30 S31[[3]]], Text[Style["ψ", 20, Black], {0.05, 0.2, 0.05}]}];
```

Rotation-3
Rotation around z axis by an angle ψ ;
$\xi' - \eta' - \zeta' \rightarrow$ x' - y' - z'
where $\zeta' =$ z'.

```
Show[g1, g4, g41, f2]
```



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