Chapter 1
Vector Analysis
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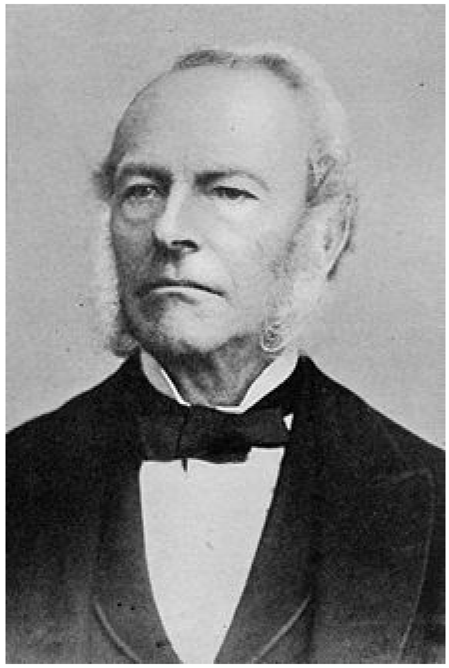
(Date: August 31, 2010)

((Note)) You may find original Mathematica programs in my web site
http://bingweb.binghamton.edu/~suzuki/

Johann Carl Friedrich Gauss (30 April 1777 – 23 February 1855) was a German mathematician and scientist who contributed significantly to many fields, including number theory, statistics, analysis, differential geometry, geodesy, geophysics, electrostatics, astronomy and optics.

http://en.wikipedia.org/wiki/Carl_Friedrich_Gauss

Sir George Gabriel Stokes, 1st Baronet FRS (13 August 1819–1 February 1903), was a mathematician and physicist, who at Cambridge made important contributions to fluid dynamics (including the Navier–Stokes equations), optics, and mathematical physics (including Stokes' theorem). He was secretary, then president, of the Royal Society.



http://en.wikipedia.org/wiki/Sir_George_Stokes,_1st_Baronet

1.1 Fundamentals

1.1.1 Definition of vectors

Vectors are usually indicated by boldface letters, such as A, and we will follow this most common convention. Alternative notation is a small arrow over the letters such as $\vec{A}$. The magnitude of a vector is also often expressed by $A = |\mathbf{A}|$. The displacement vector serves as a prototype for all other vectors. Any quantity that has magnitude and direction and that behaves mathematically like he displacement vector is a **vector**.

((**Example**))

velocity, acceleration, force,
linear momentum, angular momentum, torque
electric field, magnetic field, current density, magnetization, polarization
electric dipole moment, magnetic moment

By contrast, any quantity that has a magnitude but no direction is called a **scalar**.

((**Example**))

length, time, mass, area, volume, density, temperature, energy

A unit vector is a vector of unit length; a unit vector in the direction of ***A*** is written with a caret as $\hat{\mathbf{A}}$, which we read as "A hat."

$$\mathbf{A} = A\hat{\mathbf{A}} = \hat{\mathbf{A}}A$$

(A) A vector *r*

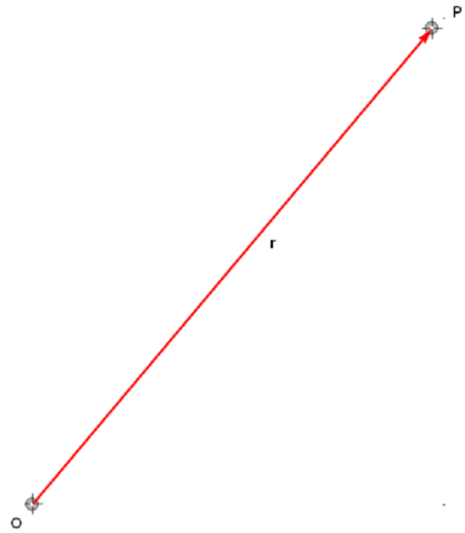


Fig. The vector ***r*** represents the position of a point P relative to another point O as origin.

(B) Negative vector: - *r*

The negative of a given vector ***r*** is a vector of the same magnitude, but opposite direction.

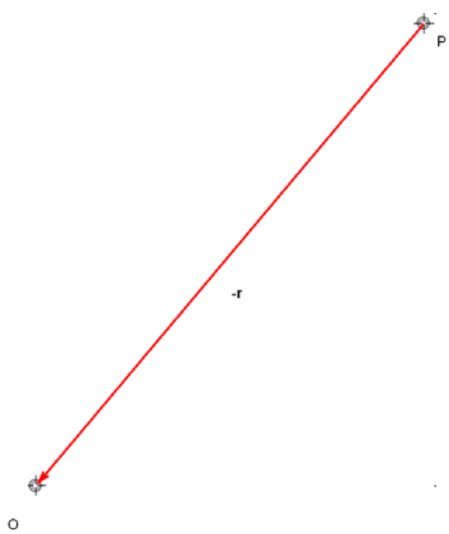


Fig. The vector –***r*** is equal in magnitude but opposite in direction to ***r***.

(C) The multiplication of the vector by a scalar

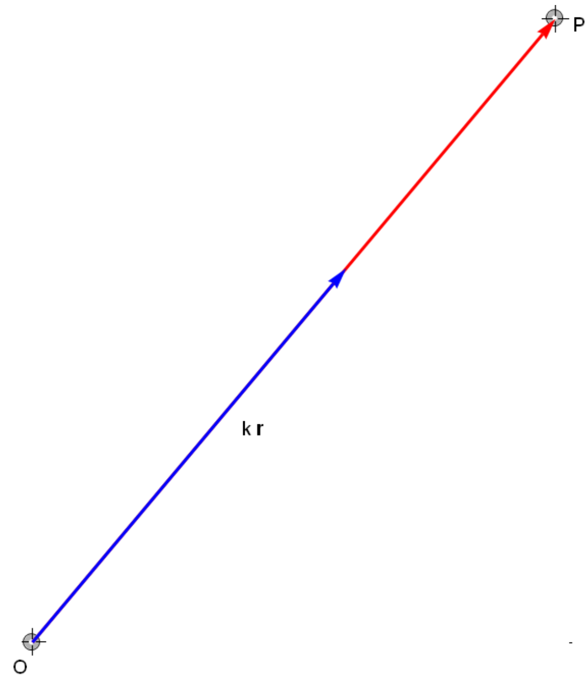


Fig. The vector $k\boldsymbol{r}$ is in the direction of $\boldsymbol{r}$ and is of magnitude kr, where $k = 0.6$.

(D) A unit vector

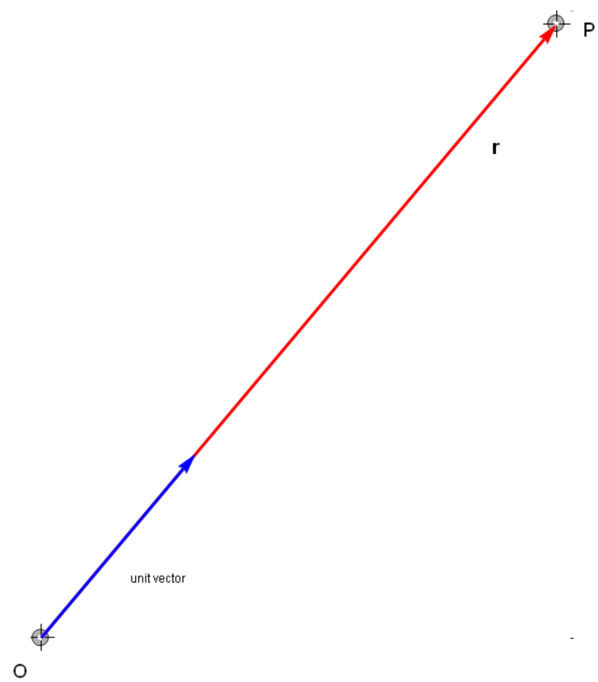
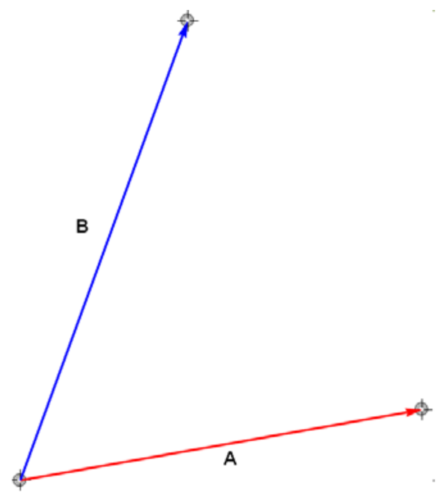


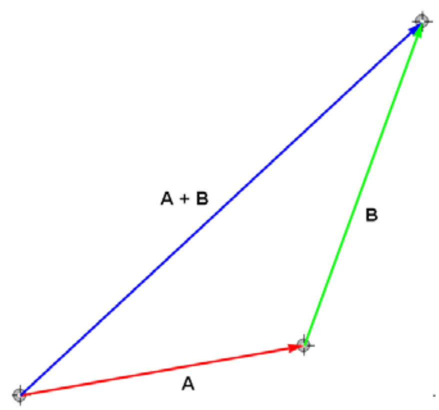
Fig. The vector $\hat{\mathbf{r}}$ is the unit vector in the direction of $\boldsymbol{r}$. Note that $\mathbf{r} = r\hat{\mathbf{r}}$.

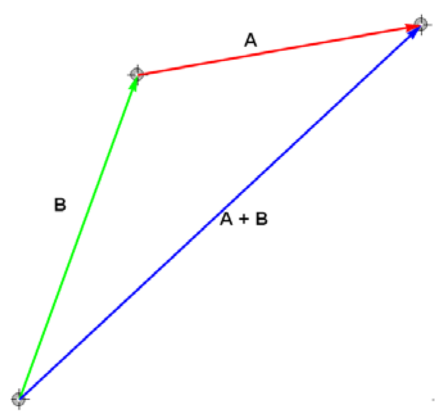
1.1.2. Vector addition

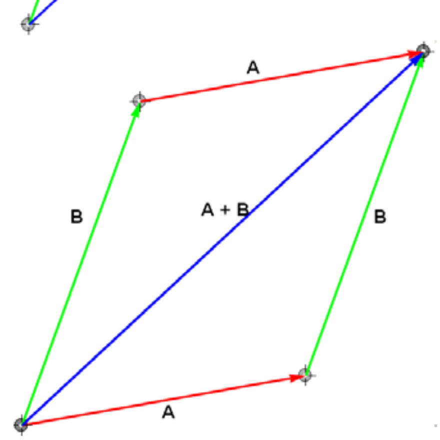
$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ (commutative)

The sum of two vectors is defined by the geometrical construction shown below. This construction is often called the parallelogram of addition of vectors.


B
A


A + B
B
A

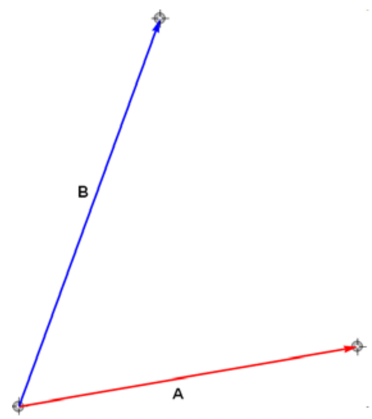

A
B
A + B

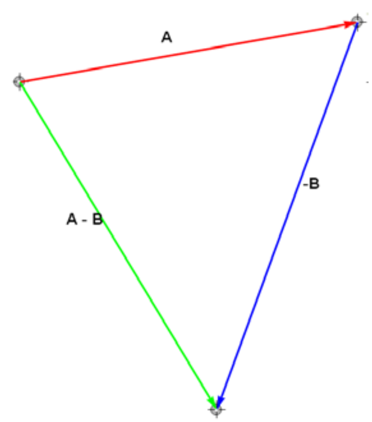

A
B
A + B
B
A

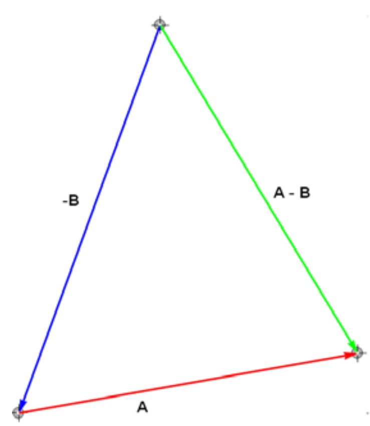
1.1.3. Vector subtraction

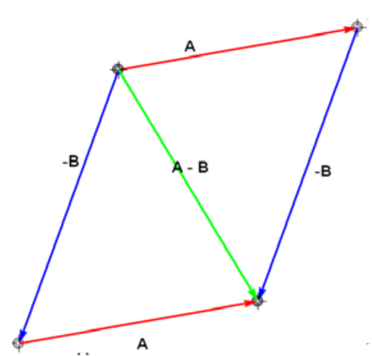
$$\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$$

The subtraction of two vectors is also defined by the geometrical construction shown below.

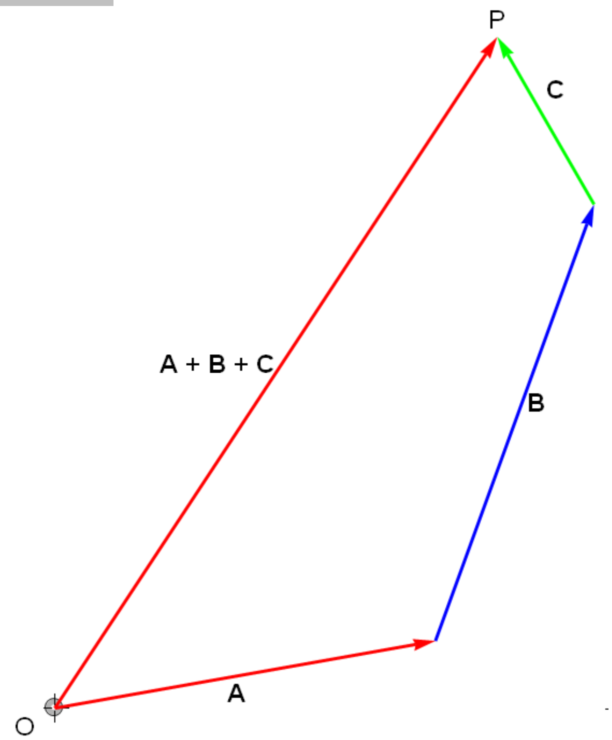

B
A


A
A - B
-B

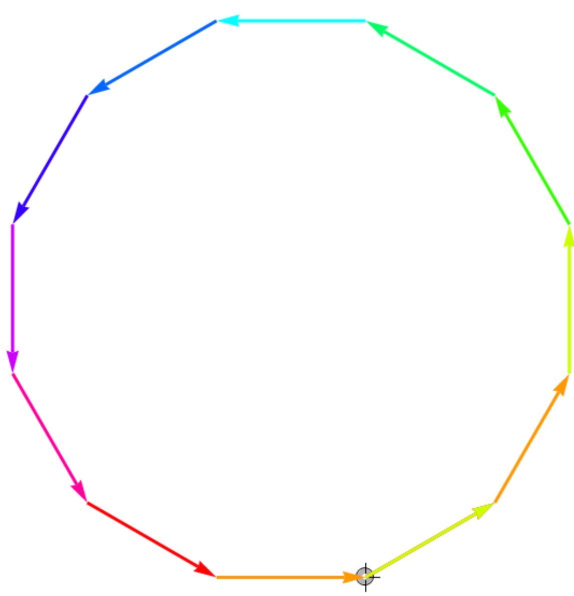

-B
A - B
A

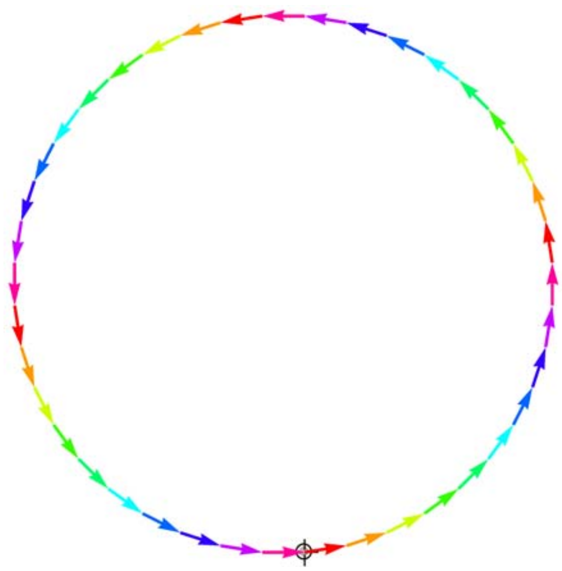

A
-B
A - B
-B
A

1.1.4. Sum of three vectors



1.1.5 Sum of many vectors





1.1.6. Imporrtant theorem for the geometry

(A) Theorem

When the point P is between the point Q and P on the line connecting the two points P and Q, the vector $\overrightarrow{OP}$ is expressed in terms of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ by

$$\overrightarrow{OP} = \alpha\mathbf{A} + \beta\mathbf{B}$$

where $\alpha + \beta = 1$ (α>0 and β>0).

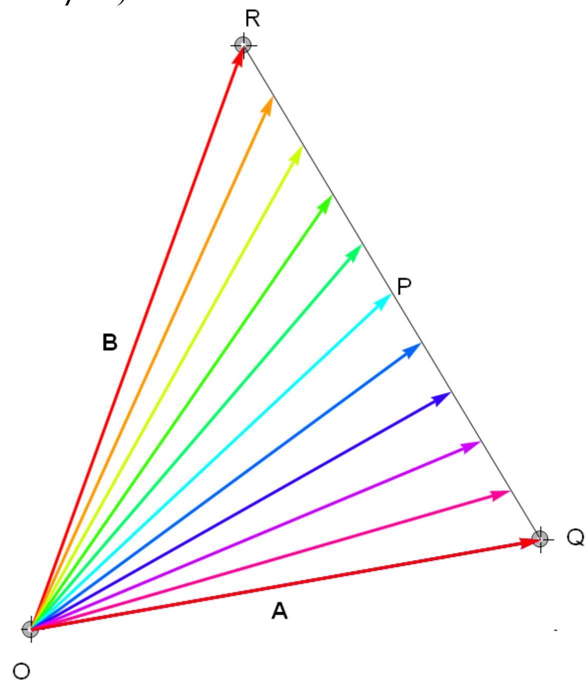
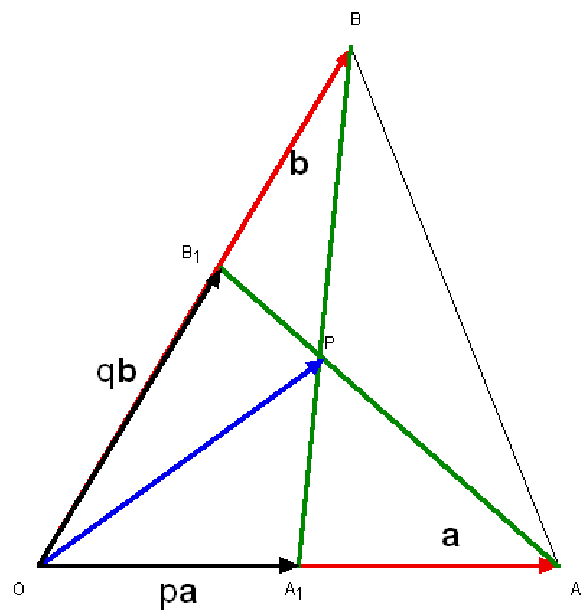


Fig. $\overrightarrow{OP} = \alpha\mathbf{A} + \beta\mathbf{B}$ where $\alpha + \beta = 1$. α is changed between α = 0.1 and 0.9 with $\Delta\alpha = 0.1$.

We now consider the following case.



$$\overrightarrow{OA} = \mathbf{a}$$
$$\overrightarrow{OB} = \mathbf{b}$$
$$\overrightarrow{OA_1} = p\mathbf{a}$$
$$\overrightarrow{OB_1} = q\mathbf{b}$$

where p and q are between 0 and 1. From the above theorem, the vector $\overrightarrow{OP}$ is expressed by

$$\overrightarrow{OP} = \alpha(\mathbf{a}) + \beta(q\mathbf{b}) = \frac{\alpha}{p}(p\mathbf{a}) + \beta q(\mathbf{b})$$

where

$$\alpha + \beta = 1$$
$$\frac{\alpha}{p} + \beta q = 1$$

From these Eqs. we have

$$\alpha = \frac{p(1-p)}{1-pq}$$

$$\beta = \frac{1-p}{1-pq}$$

Then $\overrightarrow{OP}$ is expressed by

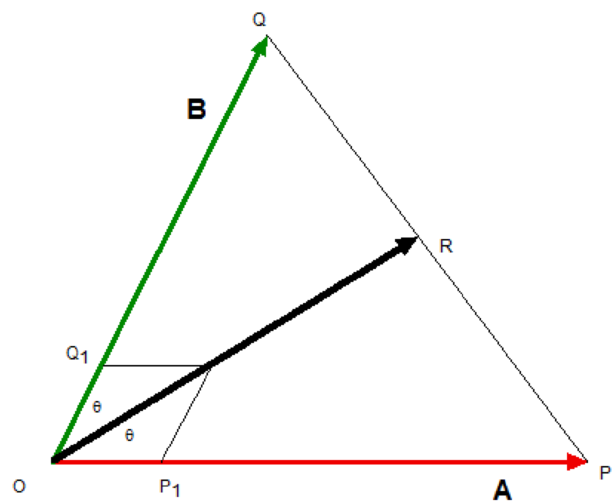
$$\overrightarrow{OP} = \frac{p(1-p)}{1-pq}\mathbf{a} + \frac{q(1-p)}{1-pq}\mathbf{b}$$

(B) Bisecting vector

In a triangle of this figure, the angle POR is equal to the angle QOR. The point R is on the line PQ. What is the expression of $\overrightarrow{OR}$ in terms of the vectors ***A*** and ***B***?. Since R is on the line AB, $\overrightarrow{OR}$ is described by

$$\overrightarrow{OR} = \alpha\mathbf{A} + \beta\mathbf{B} \qquad (1)$$

where $\alpha + \beta = 1$ (α>0 and β>0).



The vector $\overrightarrow{OR}$ is also described by

$$\overrightarrow{OR} = k(\hat{\mathbf{A}} + \hat{\mathbf{B}}) = k(\frac{\mathbf{A}}{A} + \frac{\mathbf{B}}{B}) \qquad (2)$$

where *A* and *B* are the magnitudes of ***A*** and ***B***., $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are the unit vectors for ***A*** and ***B***. From Eqs.(1) and (2), we have

$$\alpha = \frac{k}{A}$$
$$\beta = \frac{k}{B}$$

or

$$\beta = \frac{A}{B}\alpha \qquad (3)$$

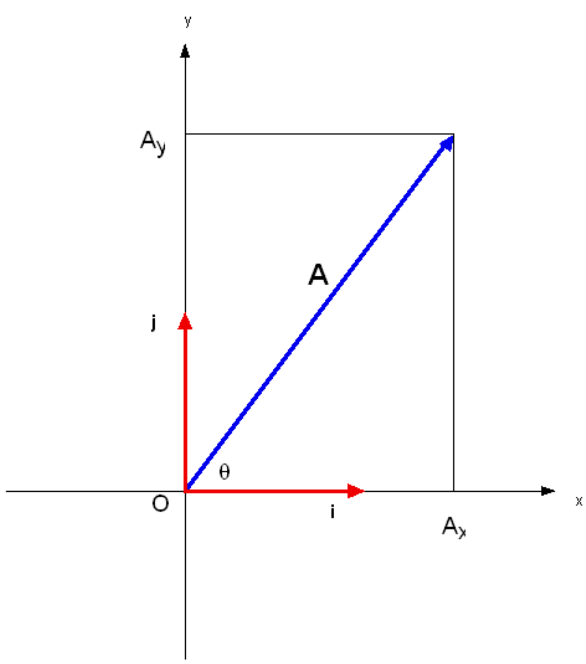
Then we get

$$\boxed{\begin{aligned} \alpha &= \frac{B}{A+B} \\ \beta &= \frac{A}{A+B} \end{aligned}}$$

1.1.7. Cartesian components of vectors

(A) 2D system

Let ***I*** and ***j***, and ***k*** denote a set of mutually perpendicular unit vectors. Let ***i*** and ***j*** drawn from a common origin O, give the positive directions along the system of rectangular axes O*xy*.



We consider a vector A lying in the *xy* plane and making an angle θ with the positive *x* axis. The vector ***A*** can be expressed by

$$\mathbf{A} = (A_x, A_y) = A_x\mathbf{i} + A_y\mathbf{j} = A(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

where

$$A = |\mathbf{A}| = \sqrt{A_x^{\ 2} + A_y^{\ 2}} \qquad \text{and} \qquad \tan\theta = \frac{A_y}{A_x}$$

When the vector ***B*** is expressed by

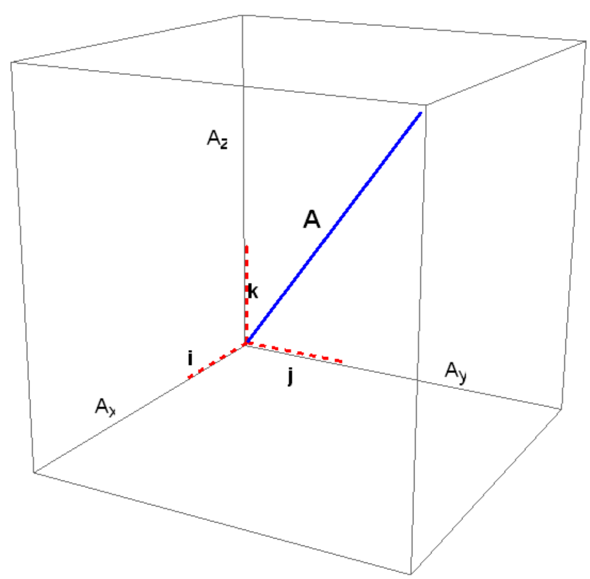
$$\mathbf{B} = (B_x, B_{y_z}) = B_x\mathbf{i} + B_y\mathbf{j}$$

the sum of ***A*** and ***B*** is

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j}$$

(B) 3D system

Let ***i***, ***j***, and ***k*** denote a set of mutually perpendicular unit vectors. Let ***i***, ***j***, and ***k*** drawn from a common origin O, give the positive directions along the system of rectangular axes O*xyz*.



An arbitrary vector ***A*** can be expressed by

$$\mathbf{A} = (A_x, A_y, A_z) = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

where A_x, A_y, and A_z are called the Cartesian components of the vector ***A***. When the vector ***B*** is expressed by

$$\mathbf{B} = (B_x, B_y, B_z) = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

the sum of ***A*** and ***B*** is

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

1.1.8. Scalar product of vectors

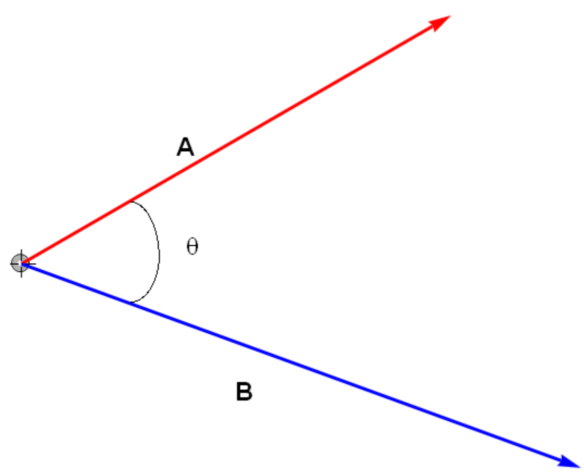
(A) Definition

The scalar product (or dot product) of the vectors **A** and **B** is defined as

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta = AB\cos\theta$$

where θ is the angle between ***A*** and ***B*** and is between 0 and π. The scalar product is a scalar and is commutative,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$



(B) Magnitude:

When ***B*** = ***A***, we have

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2$$

since $\theta = 0$.

(C) Orthogonal (*A* ⊥ *B*):

If

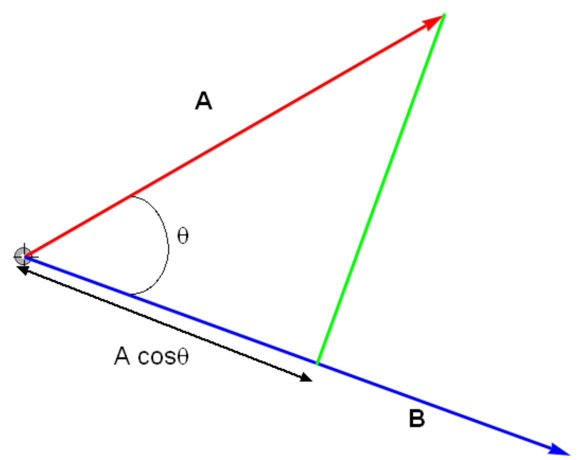
$$\mathbf{A}\cdot\mathbf{B} = 0 \quad (A \neq 0 \text{ and } B \neq 0),$$

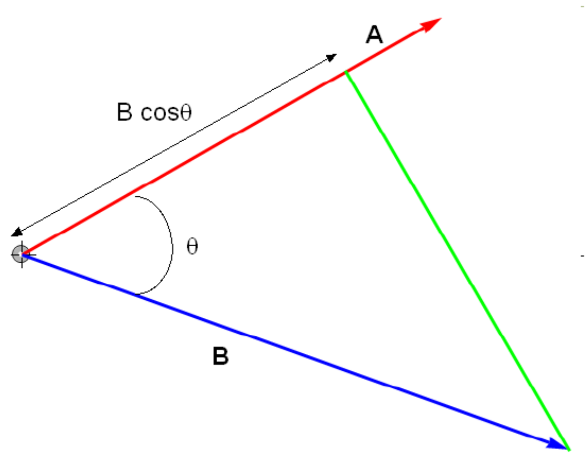
we say that ***A*** is orthogonal to *B* or perpendicular to ***B***.

(C) **Projection:**

The magnitude of the projection of ***A*** on ***B*** is *A* cos*θ*. So $\mathbf{A}\cdot\mathbf{B}$ is the product of the projection of ***A*** on ***B*** with the magnitude of ***A***. We also consider that the magnitude of $\mathbf{A}\cdot\mathbf{B}$ is the product of the projection of on ***B*** on ***A*** with the magnitude of ***B***.

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta = B(A\cos\theta) = A(B\cos\theta)$$





(D) **The expression of the scalar product using Cartesian components of vectors Inner product of *A* and *B***

We now consider two vectors given by

$$\mathbf{A} = (A_x, A_y, A_z) = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$
$$\mathbf{B} = (B_x, B_y, B_z) = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

The scalar product of these two vectors **A** and **B** can be expressed in terms of the components

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ &= (A_xB_x\mathbf{i}\cdot\mathbf{i} + A_xB_y\mathbf{i}\cdot\mathbf{j} + A_xB_z\mathbf{i}\cdot\mathbf{k}) + (A_yB_x\mathbf{j}\cdot\mathbf{i} + A_yB_y\mathbf{j}\cdot\mathbf{j} + A_yB_z\mathbf{j}\cdot\mathbf{k}) \\ &\quad + (A_zB_x\mathbf{k}\cdot\mathbf{i} + A_zB_y\mathbf{k}\cdot\mathbf{j} + A_zB_z\mathbf{k}\cdot\mathbf{k})\end{aligned}$$

or

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

Here we use the above relations for the inner products of the unit vectors.
where

$\mathbf{i}\cdot\mathbf{i} = 1$	$\mathbf{j}\cdot\mathbf{i} = 0$	$\mathbf{k}\cdot\mathbf{i} = 0$
$\mathbf{i}\cdot\mathbf{j} = 0$	$\mathbf{j}\cdot\mathbf{j} = 1$	$\mathbf{k}\cdot\mathbf{j} = 0$
$\mathbf{i}\cdot\mathbf{k} = 0$	$\mathbf{j}\cdot\mathbf{k} = 0$	$\mathbf{k}\cdot\mathbf{k} = 1$

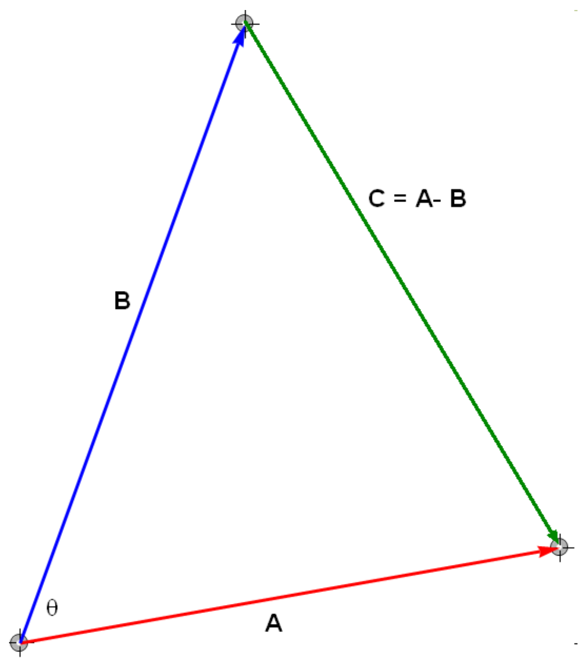
In special cases, the components of ***A*** are given by

$$\mathbf{A}\cdot\mathbf{i} = A_x\mathbf{i}\cdot\mathbf{i} + A_y\mathbf{j}\cdot\mathbf{i} + A_z\mathbf{k}\cdot\mathbf{i} = A_x$$
$$\mathbf{A}\cdot\mathbf{j} = A_x\mathbf{i}\cdot\mathbf{j} + A_y\mathbf{j}\cdot\mathbf{j} + A_z\mathbf{k}\cdot\mathbf{j} = A_y$$
$$\mathbf{A}\cdot\mathbf{i} = A_x\mathbf{i}\cdot\mathbf{k} + A_y\mathbf{j}\cdot\mathbf{k} + A_z\mathbf{k}\cdot\mathbf{k} = A_z$$

The unit vector $\hat{\mathbf{A}}$ of the vector A is expressed by

$$\begin{aligned}\hat{\mathbf{A}} &= \frac{1}{A}(A_x, A_y, A_z) = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k} \\ &= \frac{\mathbf{A}\cdot\mathbf{i}}{A}\mathbf{i} + \frac{\mathbf{A}\cdot\mathbf{j}}{A}\mathbf{j} + \frac{\mathbf{A}\cdot\mathbf{k}}{A}\mathbf{k}\end{aligned}$$

(E) Law of cosine



$$C^2 = \mathbf{C}\cdot\mathbf{C} = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = A^2 + B^2 - 2(\mathbf{A}\cdot\mathbf{B})$$
$$= A^2 + B^2 - 2AB\cos\theta$$

This is the famous trigonometric relation (law of cosine).

1.1.9 Vector product

(A) Definition

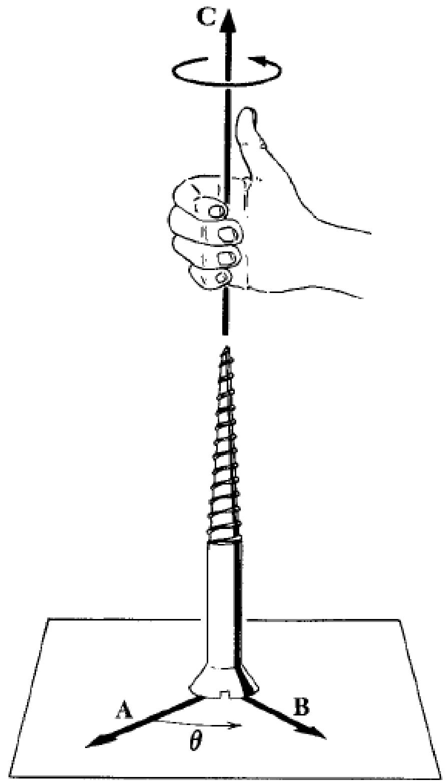
This product is a vector rather than scalar in character, but it is a vector in a somewhat restricted sense. The vector product of ***A*** and ***B*** is defined as

$$\mathbf{C} = \mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\hat{\mathbf{n}} = AB\sin\theta\hat{\mathbf{n}}$$

where $|\mathbf{A}|$ is the magnitude of ***A***. $|\mathbf{B}|$ is the magnitude of ***B***. θ is the angle between ***A*** and ***B***. $\hat{\mathbf{n}}$ is a unit vector, perpendicular to both ***A*** and ***B*** in a sense defined by the right hand thread rule.

We read ***A*** x ***B*** as "***A*** cross ***B***."

The vector ***A*** is rotated by the smallest angle that will bring it into coincidence with the direction of ***B***. The sense of ***C*** is that of the direction of motion of a screw with a right-hand thread when the screw is rotated in the same as was the vector ***A***.



Right-hand-thread rule.

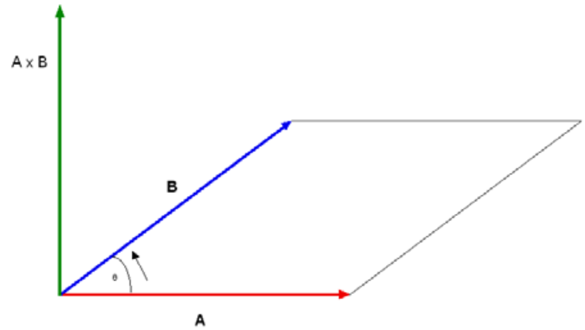
((Note))

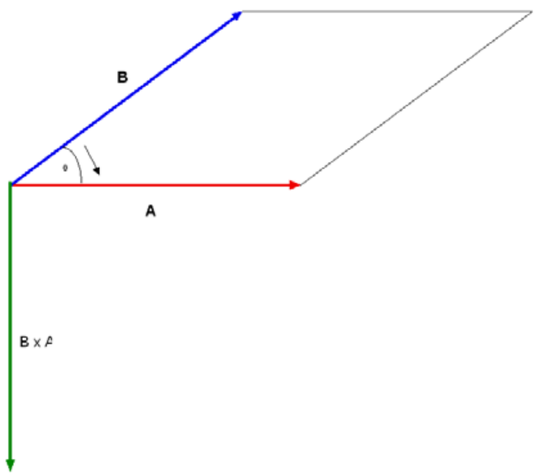
The vector ***C*** is perpendicular to both A and B. Rotate ***A*** into ***B*** through the lesser of the two possible angles – curl the fingers of the right hand in the direction in which ***A*** is rotated, and the thumb will point in the direction of ***C*** = ***A*** x ***B***.

(B)

Because of the sign convention, ***B*** x ***A*** is a vector opposite sign to ***A*** x ***B***. In other words, the vector product is not commutative,

$$\mathbf{B}\times\mathbf{A} = -\mathbf{A}\times\mathbf{B}.$$





(C)
It follows from the definition of the vector product that

$$\mathbf{A} \times \mathbf{A} = 0$$

(D)
The vector product obey the distributive law.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

(E) Cartesian components.
The vectors ***A*** and **B** are expressed by

$$\mathbf{A} = (A_x, A_y, A_z) = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$
$$\mathbf{B} = (B_x, B_y, B_z) = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

Then the vector product ***A*** x ***B*** is expressed in terms of the Cartesian components

$$
\begin{aligned}
\mathbf{A}\times\mathbf{B} &= (A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k})\times(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}) \\
&= (\mathbf{i}\times\mathbf{i})A_xB_x+(\mathbf{i}\times\mathbf{j})A_xB_y+(\mathbf{i}\times\mathbf{k})A_xB_z \\
&\quad +(\mathbf{j}\times\mathbf{i})A_yB_x+(\mathbf{j}\times\mathbf{j})A_yB_y+(\mathbf{j}\times\mathbf{k})A_yB_z \\
&\quad +(\mathbf{k}\times\mathbf{i})A_zB_x+(\mathbf{k}\times\mathbf{j})A_zB_y+(\mathbf{k}\times\mathbf{k})A_zB_z
\end{aligned}
$$

Here we use the relations,

$\mathbf{i}\times\mathbf{i}=0$	$\mathbf{j}\times\mathbf{i}=-\mathbf{k}$	$\mathbf{k}\times\mathbf{i}=\mathbf{j}$
$\mathbf{i}\times\mathbf{j}=\mathbf{k}$	$\mathbf{j}\times\mathbf{j}=0$	$\mathbf{k}\times\mathbf{j}=-\mathbf{i}$
$\mathbf{i}\times\mathbf{k}=-\mathbf{j}$	$\mathbf{j}\times\mathbf{k}=\mathbf{i}$	$\mathbf{k}\times\mathbf{k}=0$

$$
\begin{aligned}
\mathbf{A}\times\mathbf{B} &= \mathbf{k}A_xB_y-\mathbf{j}A_xB_z-\mathbf{k}A_yB_x+\mathbf{i}A_yB_z+\mathbf{j}A_zB_x-\mathbf{i}A_zB_y \\
&= \mathbf{i}(A_yB_z-A_zB_y)+\mathbf{j}(A_zB_x-A_xB_z)+\mathbf{k}(A_xB_y-A_yB_x) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}
\end{aligned}
$$

It is easier for one to remember if the determinant is used.

Using the cofactor, $\mathbf{A}\times\mathbf{B}$ can be simplified as

$$
\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} Ax & Ay \\ B_x & B_y \end{vmatrix}.
$$

where a 2x2 determinant is given by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$.

Note

$$
\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}
$$

Note that

$$
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A} = (\mathbf{C}\times\mathbf{A})\cdot\mathbf{B} = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C},
$$

where the order of ***A***, ***B***, and ***C*** is cyclic.

((**Mathematica**))

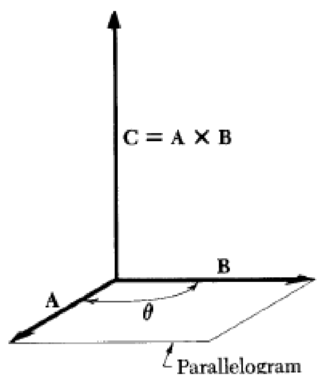
```
A = {A1, A2, A3}
{A1, A2, A3}

B = {B1, B2, B3}
{B1, B2, B3}

CC = {C1, C2, C3}
{C1, C2, C3}

DD = {D1, D2, D3}
{D1, D2, D3}

Cross[A, B]
{-A3 B2 + A2 B3, A3 B1 - A1 B3, -A2 B1 + A1 B2}

A.B
A1 B1 + A2 B2 + A3 B3

Cross[CC, Cross[A, B]] // Simplify
{-A2 B1 C2 + A1 B2 C2 - A3 B1 C3 + A1 B3 C3,
 -B2 (A1 C1 + A3 C3) + A2 (B1 C1 + B3 C3),
 -B3 (A1 C1 + A2 C2) + A3 (B1 C1 + B2 C2)}

Cross[Cross[CC, DD], Cross[A, B]] //
 Simplify
{A3 B1 (C2 D1 - C1 D2) +
   A2 B1 (-C3 D1 + C1 D3) + A1
     (-B3 C2 D1 + B2 C3 D1 + B3 C1 D2 - B2 C1 D3),
 A3 B2 (C2 D1 - C1 D2) +
   A1 B2 (C3 D2 - C2 D3) + A2
     (-B3 C2 D1 + B3 C1 D2 - B1 C3 D2 + B1 C2 D3),
 B3 (-A2 C3 D1 + A1 C3 D2 + A2 C1 D3 - A1 C2 D3) +
   A3 (B2 C3 D1 - B1 C3 D2 - B2 C1 D3 + B1 C2 D3)}
```

1.9.3 Application of the vector product

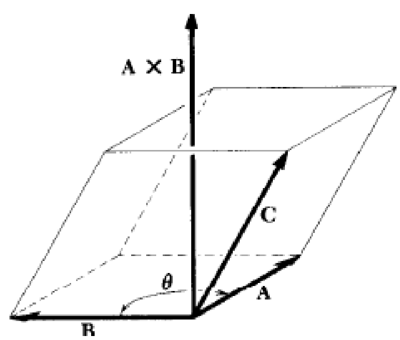
(A) Area of parallelogram

The magnitude of ***A*** x ***B*** is the area of the parallelogram.

$$|\mathbf{A} \times \mathbf{B}| = AB|\sin\theta|$$



(B) Volume of a parallelepiped

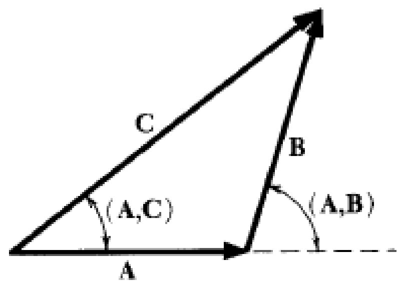


The scalar given by

$$|(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}| = V$$

is the volume of parallelepiped

(C) Law of sine



We consider the triangle defined by $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$, and take the vector product

$$\mathbf{A}\times\mathbf{C} = \mathbf{A}\times(\mathbf{A}+\mathbf{B}) = \mathbf{A}\times\mathbf{A} + \mathbf{A}\times\mathbf{B} = \mathbf{A}\times\mathbf{B}$$

The magnitude of both sides must be equal so that

$$AC\sin(A,C) = AB\sin(A,B) = AB\sin(\pi-(A,B))$$

or

$$\frac{\sin(A,C)}{B} = \frac{\sin[\pi-(A,B)]}{C} \qquad \text{(Law of sine).}$$

where $\sin(\boldsymbol{A}, \boldsymbol{B})$ denotes the sine of the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.

1.10 BAC-CAB rule

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Similarly the following two identities are also very important.

(A)

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{C}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{D})\} - \mathbf{D}\{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\}$$

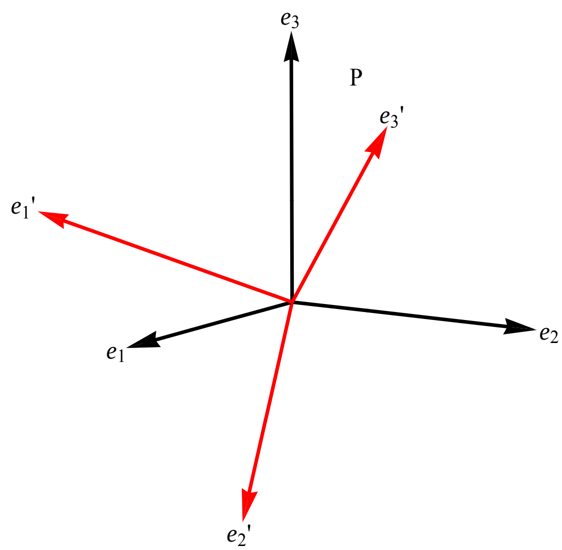
(B)

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

1.2. Advanced topics

See Chapter 1S for more detail in the rotation.

1.2.1. Directional cosine a_{ij}



The vector field is defined in terms of the behavior of its components under the rotation of the co-ordinate axes. Here we use the following notation.

$$\hat{x} = \mathbf{e}_1, \qquad \hat{y} = \mathbf{e}_2, \qquad \hat{z} = \mathbf{e}_3$$

By the rotation of the co-ordinate system, we have the new co-ordinate system, such as

$$\hat{x}' = \mathbf{e}_1', \qquad \hat{y}' = \mathbf{e}_2', \qquad \hat{z}' = \mathbf{e}_3'$$

The new vectors $\mathbf{e}_i'$ is related to the old vectors $\mathbf{e}_j$ through the following relationship.

$$\begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = a \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

where *A* is the 3 x 3 matrix, and

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \qquad \mathbf{e}_i' \cdot \mathbf{e}_j' = \delta_{ij}$$

The matrix element a_{ij} is called the directional cosine. The symbol δ_{ij} is the Kronecker delta, and is defined by

$$\delta_{ij} = 1 \text{ for i = j, and 0 for i} \neq \text{j}.$$

Then we have

$$a_{ij} = (\mathbf{e}_j \cdot \mathbf{e}_i').$$

The vector $\{\boldsymbol{e}_i\}$ is also expressed by using $\{\boldsymbol{e}_j'\}$

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = a^{-1} \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = a^T \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1' \\ \mathbf{e}_2' \\ \mathbf{e}_3' \end{pmatrix},$$

where a^T is the transpose of the matrix a. For simplicity, we can write down

$$\mathbf{e}_i' = \sum_j a_{ij} \mathbf{e}_j, \qquad \mathbf{e}_i = \sum_j (a^T)_{ij} \mathbf{e}_j' = \sum_j a_{ji} \mathbf{e}_j'.$$

((**Note**))

$$a^T = a^{-1}$$

((Proof))
From

$$\mathbf{e}_i' \cdot \mathbf{e}_j' = \delta_{ij}$$

we have

$$\mathbf{e}_i' \cdot \mathbf{e}_j' = (\sum_k a_{ik} \mathbf{e}_k)(\sum_l a_{jl} \mathbf{e}_l) = \sum_{k,l} a_{ik} a_{jl} (\mathbf{e}_k \cdot \mathbf{e}_l) = \sum_{k,l} a_{ik} a_{jl} \delta_{kl} = \sum_k a_{ik} a_{jk} = \delta_{ij}$$

or

$$\sum_k a_{ik} a_{jk} = \delta_{ij},$$

where a^T is the transpose of the matrix a;

$$aa^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In other words, we have

$$aa^T = a^T a = I$$

or

$$a^T = a^{-1}$$

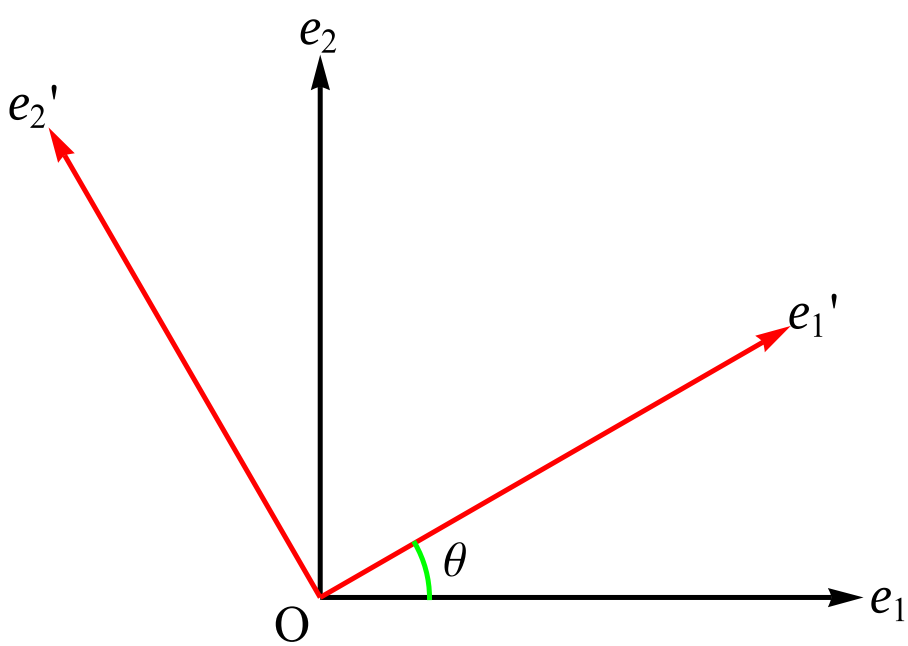
Note that

$$(a^T a)_{ij} = \delta_{ij} = \sum_k (a^T)_{ik} (a)_{kj} = \sum_k a_{ki} a_{kj}$$

or

$$\sum_k a_{ki} a_{kj} = \delta_{ij}.$$

1.2.2 Two dimensional rotation



$$\mathbf{e}_1' = a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2$$
$$\mathbf{e}_2' = a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2$$

with

$$a_{11} = (\mathbf{e}_1.\mathbf{e}_1') = \cos\theta$$
$$a_{12} = (\mathbf{e}_2.\mathbf{e}_1') = \sin\theta$$
$$a_{21} = (\mathbf{e}_1.\mathbf{e}_2') = -\sin\theta$$
$$a_{22} = (\mathbf{e}_1.\mathbf{e}_2') = \cos\theta$$

or

$$a=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

((**Note**)) Mathematica

```
R = RotationMatrix[-θ] // Simplify
```

```
{{Cos[θ], Sin[θ]}, {-Sin[θ], Cos[θ]}}
```

```
R // MatrixForm
```

$$\begin{pmatrix} \mathrm{Cos}[\theta] & \mathrm{Sin}[\theta] \\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix}$$

1.2.3 Three dimensional rotation

Rotations of the body frame are defined to have a countercloskwise sense, with the rotations carried out in the following order.

1. First, make a rotation by an angle ϕ about the initial z axis. ($\xi-\eta-\zeta$).
2. Then, make a second rotation by an angle θ about the body ξ (= ξ') axis, called the line of nodes. ($\xi'-\eta'-\zeta'$).
3. Finally, make a third rotation by an angle ψ about the body ζ' (= z') axis. (x' - y' - z').

$$R_\phi=\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_\theta=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$R_\psi=\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The net result in the body frame is

$$R = R_\psi R_\theta R_\phi = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & \sin\theta\sin\psi \\ -\cos\theta\sin\phi\cos\psi - \cos\phi\sin\psi & \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\cos\phi\sin\theta & \cos\theta \end{pmatrix}$$

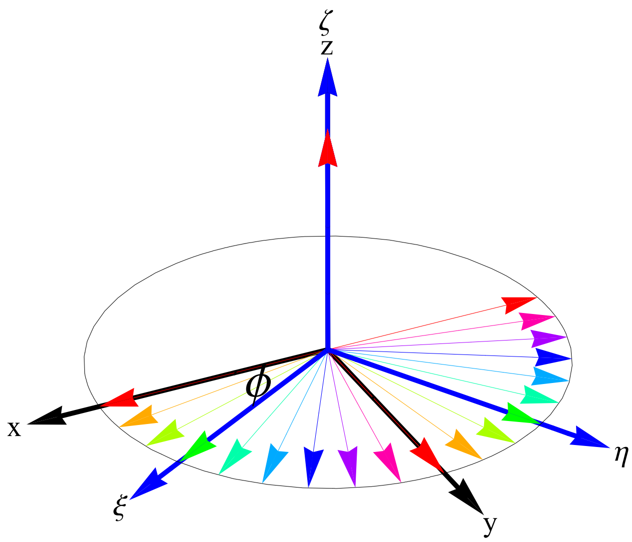


Fig.1 Rotation by an angle ϕ about the initial z axis. ($\xi - \eta - \zeta$).

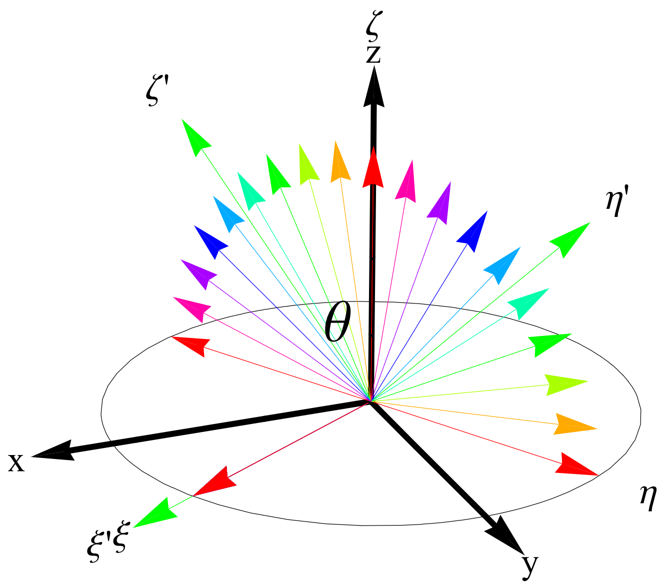


Fig.2 A second rotation by an angle θ about the body ξ (= ξ') axis, called the line of nodes. ($\xi' - \eta' - \zeta'$).

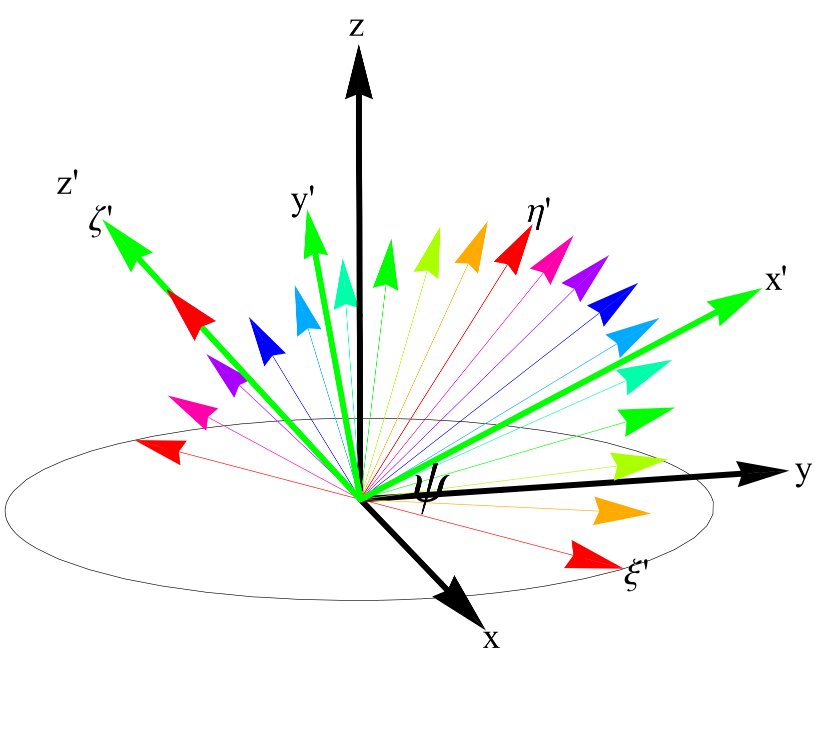


Fig.3 A third rotation by an angle ψ about the body ζ' (= z') axis. (x' - y' - z').

((**Mathematica**))

```
Rϕ = RotationMatrix[-ϕ, {0, 0, 1}] // Simplify;

Rϕ // MatrixForm
```

$$\begin{pmatrix} \mathrm{Cos}[\phi] & \mathrm{Sin}[\phi] & 0 \\ -\mathrm{Sin}[\phi] & \mathrm{Cos}[\phi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
Rθ = RotationMatrix[-θ, {1, 0, 0}] // Simplify;

Rθ // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Cos}[\theta] & \mathrm{Sin}[\theta] \\ 0 & -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix}$$

```
Rψ = RotationMatrix[-ψ, {0, 0, 1}] // Simplify;

Rψ // MatrixForm
```

$$\begin{pmatrix} \mathrm{Cos}[\psi] & \mathrm{Sin}[\psi] & 0 \\ -\mathrm{Sin}[\psi] & \mathrm{Cos}[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
S = Rψ.Rθ.Rϕ // Simplify;

S // MatrixForm
```

$$\begin{pmatrix} \mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] - \mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] + \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] \\ -\mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] - \mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] - \mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] \\ \mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] & -\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix}$$

1.2.4 Definition of vector

Suppose that the vector ***r*** can be expressed by

$$\mathbf{r} = \sum_i x_i \mathbf{e}_i = \sum_i x_i' \mathbf{e}_i'$$

for the old and new co-ordinate systems, respectively. Then we have

$$\mathbf{r} = \sum_j x_j \mathbf{e}_j = \sum_j x_j (\sum_i a_{ij} \mathbf{e}_i') = \sum_{i,j} x_j a_{ij} \mathbf{e}_i' = \sum_i (\sum_j a_{ij} x_j) \mathbf{e}_i' = \mathbf{r}' = \sum_i x_i' \mathbf{e}_i'$$

Then we have

$$x_i' = \sum_j a_{ij} x_j ,$$

or

$$x' = ax$$

We may write (Cartesian co-ordinate)

$$a_{ij} = \frac{\partial x_i'}{\partial x_j}$$

Note that we also have

$$x = a^{-1}x' = a^T x'$$

or

$$x_i = \sum_j (a^T)_{ij} x'_j = \sum_j a_{ji} x_j'.$$

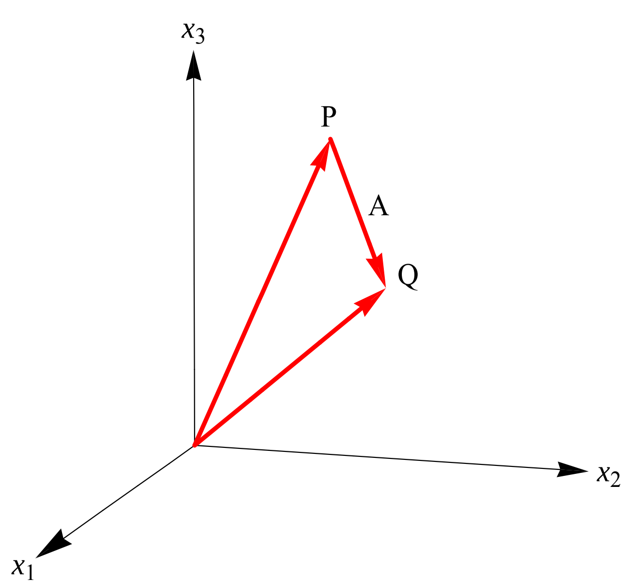
Also we may write

$$a_{ji} = \frac{\partial x_i}{\partial x_j'}, \qquad \text{or} \qquad a_{ij} = \frac{\partial x_j}{\partial x_i'}$$

Using the above notations, we get the original definition;

$$\sum_i x_i \mathbf{e}_i = \sum_i \left(\sum_l a_{li} x_l' \right)\left(\sum_k a_{ki} \mathbf{e}_k' \right) = \sum_{i,k,l} a_{li} a_{ki} x_l' \mathbf{e}_k' = \sum_{k,l} x_l' \mathbf{e}_k' \sum_i a_{li} a_{ki} = \sum_{k,l} x_l' \mathbf{e}_k' \delta_{kl} = \sum_k x_k' \mathbf{e}_k'$$

Now we consider more general case in order to get the definition of vector.



Suppose that

$$\overrightarrow{OP} = \sum_i y_i \mathbf{e}_i = \sum_i y_i' \mathbf{e}_i'$$

$$\overrightarrow{OQ} = \sum_i z_i \mathbf{e}_i = \sum_i z_i' \mathbf{e}_i'$$

where

$$y_i' = \sum_j a_{ij} y_j \,, \qquad z_i' = \sum_j a_{ij} z_j$$

Then we have

$$\mathbf{A} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \sum_i (z_i - y_i)\mathbf{e}_i = \sum_i (z_i' - y_i')\mathbf{e}_i'$$

Since

$$\mathbf{e}_i = \sum_j a_{ji} \mathbf{e}_j' \,,$$

the expression of ***A*** can be rewritten as

$$\sum_{i,j} (z_i - y_i) a_{ji} \mathbf{e}_j' = \sum_i (z_i' - y_i')\mathbf{e}_i' \,.$$

By the interchange between i and j in the left-hand side,

$$\sum_{i,j} (z_i - y_i) a_{ji} \mathbf{e}_j' = \sum_i \sum_j (z_j - y_j) a_{ij} \mathbf{e}_i' = \sum_i (z_i' - y_i')\mathbf{e}_i' \,.$$

Therefore we get

$$z_i' - y_i' = \sum_j a_{ij} (z_j - y_j) \,.$$

Since the component of ***A*** is given by

$$A_i = z_i - y_i \,, \qquad \text{and} \qquad A_i' = z_i' - y_i'$$

in the old and new co-ordinate systems, we can write

$$A_i' = \sum_j a_{ij} A_j \,.$$

In summary, under the rotation of the co-ordinate system,

$$\mathbf{e}_i' = \sum_j a_{ij}\mathbf{e}_j\,, \quad \text{or} \qquad \mathbf{e}_i = \sum_j a^T{}_{ij}\mathbf{e}_j' = \sum_j a_{iji}\mathbf{e}_j'$$

the components of the vector are transformed through

$$A_i' = \sum_j a_{ij}A_j$$

((**Example**))

(1) Newton's second law

We consider how the Newton's second law transforms under the rotation of the coordinate by the angle θ around the z axis. From the definition of the vector for $\boldsymbol{r}$, we have
a relation between the old coordinates and new coordinates,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

since $\boldsymbol{r}$ is a real vector. In the old system, the Newton's second law states that

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = m\begin{pmatrix} \dfrac{d^2x}{dt^2} \\ \dfrac{d^2y}{dt^2} \\ \dfrac{d^2z}{dt^2} \end{pmatrix}$$

In the new system, the Newton's second law should be written as

$$\begin{pmatrix} F_{x'} \\ F_{y'} \\ F_{z'} \end{pmatrix} = m\frac{d^2}{dt^2}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Then we have

$$\begin{pmatrix} F_{x'} \\ F_{y'} \\ F_{z'} \end{pmatrix} = m\frac{d^2}{dt^2}\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} m\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

This means that the force is a real vector. In other words, if Newton's second law is correct one set of axes, they are also valid on any other set of axes.

(2) Angular momentum

The angular momentum is defined by

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = L_x\mathbf{e}_x + L_y\mathbf{e}_y + L_z\mathbf{e}_z$$

where

$$L_x = yp_z - zp_y$$
$$L_y = zp_x - xp_z$$
$$L_z = xp_y - yp_x$$

Now we consider how the angular momentum transforms under the rotation of the coordinate by the angle θ around the z axis. The angular momentum in the new coordinate is

$$\mathbf{L}' = \mathbf{r}'\times\mathbf{p}' = \begin{vmatrix} \mathbf{e}_x' & \mathbf{e}_y' & \mathbf{e}_z' \\ x' & y' & z' \\ p_{x'} & p_{y'} & p_{z'} \end{vmatrix} = L_{x'}\mathbf{e}_x' + L_{y'}\mathbf{e}_y' + L_{z'}\mathbf{e}_z'$$

with

$$L_{x'} = y'p_{z'} - z'p_{y'}$$
$$L_{y'} = z'p_{x'} - x'p_{z'}$$
$$L_{z'} = x'p_{y'} - y'p_{x'}$$

Using

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} p_{x'} \\ p_{y'} \\ p_{z'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

We can show that

$$L_{x'} = y' p_{z'} - z' p_{y'} = L_x \cos\theta + L_y \sin\theta$$
$$L_{y'} = z' p_{x'} - x' p_{z'} = -L_x \sin\theta + L_y \cos\theta$$
$$L_{z'} = x' p_{y'} - y' p_{x'} = L_z$$

or

$$\begin{pmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_x \\ L_y \\ L_{z'} \end{pmatrix}$$

This means that the angular momentum is a vector.

((**Mathematica**))

```
R = {x, y, z}; P = {px, py, pz}; L = Cross[R, P];
A = RotationMatrix[-θ, {0, 0, 1}];

RN = A.R

{x Cos[θ] + y Sin[θ], y Cos[θ] - x Sin[θ], z}

PN = A.P

{px Cos[θ] + py Sin[θ], py Cos[θ] - px Sin[θ], pz}

LN = Cross[RN, PN] // Simplify

{(pz y - py z) Cos[θ] + (-pz x + px z) Sin[θ],
 (-pz x + px z) Cos[θ] + (-pz y + py z) Sin[θ],
 py x - px y}

LN - A.L // Simplify

{0, 0, 0}
```

1.2.5 Scalar product

The scalar is invariant under the rotation of the co-ordinate system. We show that the scalar product $\mathbf{A}\cdot\mathbf{B}$ is scalar;

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{A}'\cdot\mathbf{B}'$$

We start with the definition of the vectors,

$$A_i' = \sum_j a_{ij} A_j \,. \qquad B_i' = \sum_j a_{ik} A_k \,.$$

Then we have

$$\mathbf{A}'\cdot\mathbf{B}' = \sum_i A_i' B_i' = \sum_i (\sum_j a_{ij} A_j)(\sum_k a_{ik} B_k) = \sum_{j,k} A_j B_k \sum_i a_{ij} a_{ik} = \sum_{j,k} A_j B_k \delta_{jk} = \sum_j A_j B_j$$

Then we have

$$\mathbf{A}'\cdot\mathbf{B}' = \sum_i A_i B_i = \mathbf{A}\cdot\mathbf{B}$$

Thus $\mathbf{A}\cdot\mathbf{B}$ is a scalar.

1.2.6. Vector product

Here there still remains the problem of verifying that

$$\mathbf{C} = \mathbf{A}\times\mathbf{B}$$

is indeed a vector.

((**Proof**))

Under the rotation of the co-ordinate system,

$$A_j \to A_j' = \sum_l a_{jl} A_l$$
$$B_k \to B_k' = \sum_m a_{km} B_m$$
$$C_i \to C_i' = A_j' B_k' - A_k' B_j'$$

where i, j, and k are in cyclic order.

$$\begin{aligned}
C_1' &= A_2' B_3' - A_3' B_2' \\
&= \sum_{l,m} (a_{2l} a_{3m} - a_{3l} a_{2m}) A_l B_m \\
&= (a_{21}a_{32} - a_{22}a_{31}) A_1 B_2 + (a_{22}a_{31} - a_{21}a_{32}) A_2 B_1 + (a_{22}a_{33} - a_{23}a_{32}) A_2 B_3 \\
&\quad + (a_{23}a_{32} - a_{22}a_{33}) A_3 B_2 + (a_{23}a_{31} - a_{21}a_{33}) A_3 B_1 + (a_{21}a_{33} - a_{23}a_{31}) A_1 B_3 \\
&= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} (A_2 B_3 - A_3 B_2) + \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} (A_3 B_1 - A_1 B_3) + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} (A_1 B_2 - A_2 B_1) \\
&= a_{11}(A\times B)_1 + a_{12}(A\times B)_2 + a_{13}(A\times B)_3 \\
&= a_{11}C_1 + a_{12}C_2 + a_{13}C_3
\end{aligned}$$

$$
\begin{aligned}
C_2' &= A_3' B_1' - A_1' B_3' \\
&= \sum_{l,m} (a_{3l} a_{1m} - a_{1l} a_{3m}) A_l B_m \\
&= (a_{31}a_{12} - a_{32}a_{11}) A_1 B_2 + (a_{32}a_{11} - a_{31}a_{12}) A_2 B_1 + (a_{32}a_{13} - a_{33}a_{12}) A_2 B_3 \\
&\quad + (a_{33}a_{12} - a_{32}a_{13}) A_3 B_2 + (a_{33}a_{11} - a_{31}a_{13}) A_3 B_1 + (a_{31}a_{13} - a_{33}a_{11}) A_1 B_3 \\
&= \begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} (A_2 B_3 - A_3 B_2) + \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix} (A_3 B_1 - A_1 B_3) + \begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix} (A_1 B_2 - A_2 B_1) \\
&= a_{21} (A \times B)_1 + a_{22} (A \times B)_2 + a_{23} (A \times B)_3 \\
&= a_{21} C_1 + a_{22} C_2 + a_{23} C_3
\end{aligned}
$$

$$
\begin{aligned}
C_3' &= A_1' B_2' - A_2' B_1' \\
&= \sum_{l,m} (a_{1l} a_{2m} - a_{2l} a_{1m}) A_l B_m \\
&= (a_{11}a_{22} - a_{12}a_{21}) A_1 B_2 + (a_{12}a_{21} - a_{11}a_{22}) A_2 B_1 + (a_{12}a_{23} - a_{13}a_{22}) A_2 B_3 \\
&\quad + (a_{13}a_{22} - a_{12}a_{23}) A_3 B_2 + (a_{13}a_{21} - a_{11}a_{23}) A_3 B_1 + (a_{11}a_{23} - a_{13}a_{21}) A_1 B_3 \\
&= \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} (A_2 B_3 - A_3 B_2) + \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} (A_3 B_1 - A_1 B_3) + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} (A_1 B_2 - A_2 B_1) \\
&= a_{31} (A \times B)_1 + a_{32} (A \times B)_2 + a_{33} (A \times B)_3 \\
&= a_{31} C_1 + a_{32} C_2 + a_{33} C_3
\end{aligned}
$$

Thus $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is a real vector. We note that

$$
a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \qquad a_{12} = \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix}, \qquad a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},
$$

$$
a_{21} = \begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix}, \qquad a_{22} = \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix}, \qquad a_{23} = \begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix}
$$

$$
a_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \qquad a_{32} = \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix}, \qquad a_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}
$$

((**Note**))
The above relations among $\{a_{ij}\}$ can be derived in the following way.

$$\mathbf{e}_i = \sum_j a_{ji}\mathbf{e}_j'$$

$$\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3$$

Now we consider about the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ x $\boldsymbol{e}_3$, respectively.

$$\mathbf{e}_1 = \sum_j a_{j1}\mathbf{e}_j' = a_{11}\mathbf{e}_1' + a_{21}\mathbf{e}_2' + a_{31}\mathbf{e}_3'$$

$$\begin{aligned}
\mathbf{e}_2 \times \mathbf{e}_3 &= (\sum_l a_{l2}\mathbf{e}_l') \times (\sum_m a_{m3}\mathbf{e}_m') \\
&= (a_{12}\mathbf{e}_1' + a_{22}\mathbf{e}_2' + a_{32}\mathbf{e}_3') \times (a_{13}\mathbf{e}_1' + a_{23}\mathbf{e}_2' + a_{33}\mathbf{e}_3') \\
&= (a_{22}a_{33} - a_{32}a_{23})\mathbf{e}_1' + (a_{32}a_{13} - a_{12}a_{33})\mathbf{e}_2' + (a_{12}a_{23} - a_{22}a_{13})\mathbf{e}_3'
\end{aligned}$$

Thus we find

$$a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \qquad a_{21} = \begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix}, \qquad a_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix},$$

Similarly we have

$$a_{12} = \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix}, \qquad a_{22} = \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix}, \qquad a_{32} = \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix},$$

$$a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \qquad a_{23} = \begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix} \qquad a_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

1.2.7. Tensor

Ohm's law;

$$J_i = \sum_k \sigma_{ik} E_k$$

σ is the tensor of second rank.

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

scalar: tensor of rank zero
vector: tensor of first rank

((**Definition of tensor of the second rank**))

$$\sigma_{ij}' = \sum_{k,l} a_{ik} a_{jl} \sigma_{kl}$$

where
Only in Cartesian coordinates we have

$$a_{ij} = \frac{\partial x_i'}{\partial x_j} = \frac{\partial x_j}{\partial x_i'}.$$

So there is no difference between contravariant and covariant transformation. In other systems, this in general does not apply, so the distinction between contavariant and covariant is real.

$$C_{ij}' = \sum_{k,l} \frac{\partial x_k}{\partial x_i'} \frac{\partial x_l}{\partial x_j'} C_{kl}$$ Covariant wrt i, j

$$A'^{ij} = \sum_{k,l} \frac{\partial x_i'}{\partial x_k} \frac{\partial x_j'}{\partial x_l} A^{kl}$$ Contravariant wrt i, j

$$B'^{i}{}_{j} = \sum_{k,l} \frac{\partial x_i'}{\partial x_k} \frac{\partial x_l}{\partial x_j'} B_l^k$$ Contravariant wrt i, covariant wrt j.

Summation convention:

$$B'^{i}{}_{j} = \frac{\partial x_i'}{\partial x_k} \frac{\partial x_l}{\partial x_j'} B_l^k$$

The Kronecker delta δ_{ij} is really a mixed tensor of second rank δ_j^i.
We have, using the summation convention

$$\delta_l^k \frac{\partial x_i'}{\partial x_k} \frac{\partial x_l}{\partial x_j'} = \frac{\partial x_i'}{\partial x_k} \frac{\partial x_k}{\partial x_j'}$$

by definition of the Kronecker delta. Now we have

$$\frac{\partial x_i'}{\partial x_k} \frac{\partial x_k}{\partial x_j'} = \frac{\partial x_i'}{\partial x_j'} = \delta'^{i}{}_{j}$$

by direct partial differentiation of the right-hand side (chain rule). Hence,

$$\delta'^{i}_{j} = \frac{\partial x_i'}{\partial x_k}\frac{\partial x_l}{\partial x_j'}\delta^k_l$$

1.2.8. Gradient ∇

$\nabla\varphi$ (φ; scalar) Nabra, gradient, del

The gradient φ is defined as

$$\nabla\varphi = \hat{x}\frac{\partial\varphi}{\partial x} + \hat{y}\frac{\partial\varphi}{\partial y} + \hat{z}\frac{\partial\varphi}{\partial z} = (\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}); \quad \text{gradient of the scalar } \varphi$$

((**Example**))

$f = f(r)$

with $r = \sqrt{x^2 + y^2 + z^2}$.

$$\begin{aligned}
\nabla f(r) &= \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} \\
&= \hat{x}\frac{df}{dr}\frac{\partial r}{\partial x} + \hat{y}\frac{df}{dr}\frac{\partial r}{\partial y} + \hat{z}\frac{df}{dr}\frac{\partial r}{\partial z} \\
&= \frac{1}{r}\frac{df}{dr}(x\hat{x} + y\hat{y} + z\hat{z}) \\
&= \frac{\mathbf{r}}{r}\frac{df}{dr} = \hat{\mathbf{r}}\frac{df}{dr}
\end{aligned}$$

where

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(A) Geometrical interpretation

Let us give a geometrical interpretation of $\nabla\varphi$.

$$d\mathbf{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

From the definition, we have

$$d\varphi = \nabla\varphi \cdot d\mathbf{r} = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

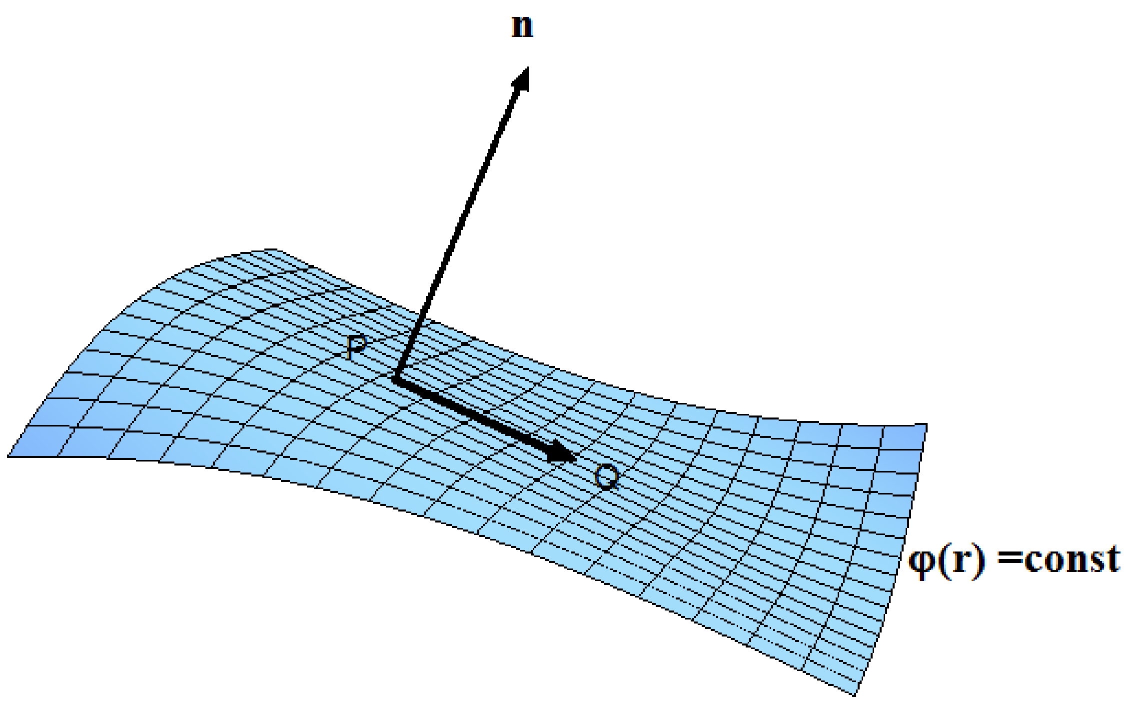


Fig. The normal vector ***n*** which is perpendicular to the line PQ on the surface. $\nabla\varphi$ is perpendicular to the surface (φ = constant).

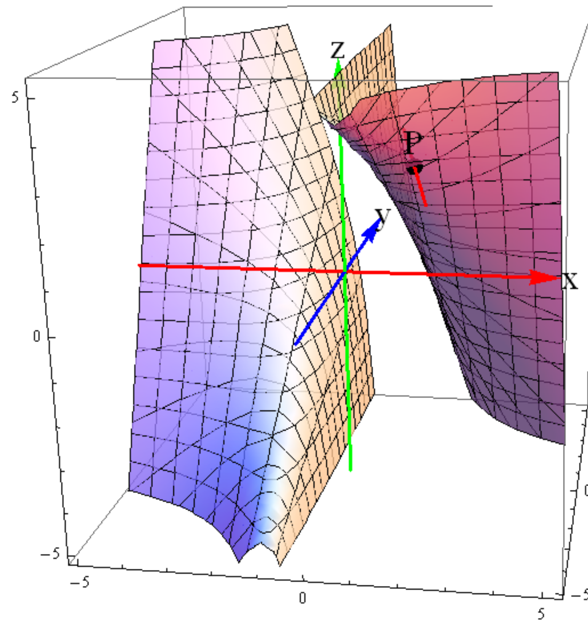
If we choose two points P and Q on the surface $\varphi(\mathbf{r}) = const$, where $\overrightarrow{PQ} = d\mathbf{r}$ in the limit of $d\mathbf{r} \to 0$.

Since

$$d\varphi = \nabla\varphi \cdot d\mathbf{r} = 0,$$

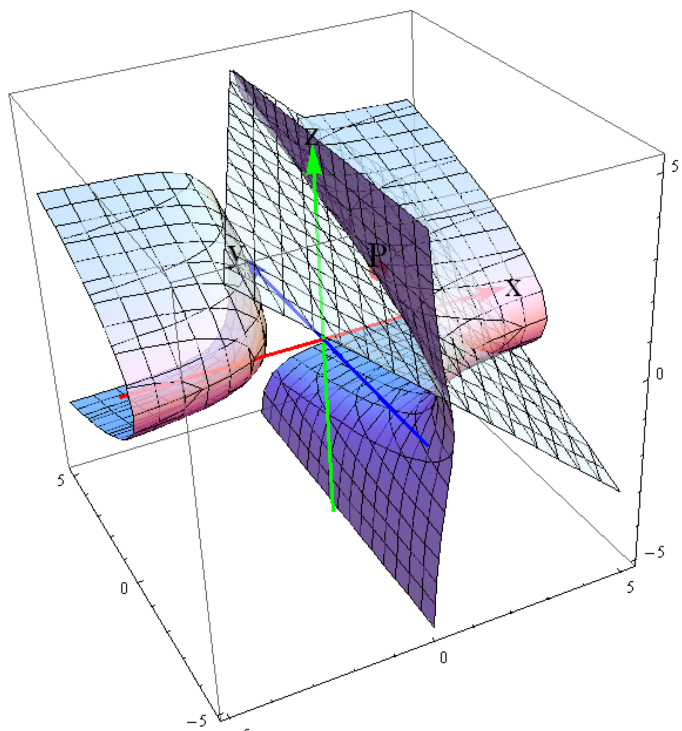
we find that $\nabla\varphi$ is perpendicular to the surface (φ = constant). It is called the normal vector.

((**Example**-1))
Find a unit normal to the surface $x^2 y + 2xz = 4$ at the point P (2, -2, 3).
$\mathbf{A} = \nabla(x^2 y + 2xz) = (2xy + 2z, x^2, 2x)$. $\boldsymbol{A}$ = (-2, 4, 4) at the point P. Then a unit normal to the surface is (-1/3, 2/3, 2/3). Another unit normal is (1/3, -2/3, -2/3).



((**Example-2**)) Find an equation for the tangent plane to the surface $2xz^2 - 3xy - 4x - 7 = 0$ at the point P $\boldsymbol{r}_0 = (1, -1, 2)$.

$\mathbf{A} = \nabla(2xz^2 - 3xy - 4x) = (2z^2 - 3y - 4, -3x, 4xz)$. Then a normal to the surface at the point P is $\boldsymbol{A} = (7, -3, 8)$ at the point P. The equation of a plane passing through a point P, which is perpendicular to A is $(\boldsymbol{r} - r_0).\mathbf{A} = 0$; $7(x-1) - 3(y+1) + 8(z-2) = 0$



(B) Vector $\nabla\varphi$

Next we will verify that $\nabla\varphi$ is a vector. φ is scalar, which means the invariant under the rotation of the coordinate system.

$$\varphi'(x_i') = \varphi(x_i).$$

$$\frac{\partial\varphi'}{\partial x_i'} = \frac{\partial\varphi}{\partial x_i'} = \sum_j \frac{\partial\varphi}{\partial x_j}\frac{\partial x_j}{\partial x_i'} = \sum_j a_{ij}\frac{\partial\varphi}{\partial x_j},$$

since

$$a_{ij} = \frac{\partial x_j}{\partial x_i'}$$

Thus $\nabla\varphi$ is a real vector (contravariant vector)

(C) Plotting of equi-potential lines and vector fields

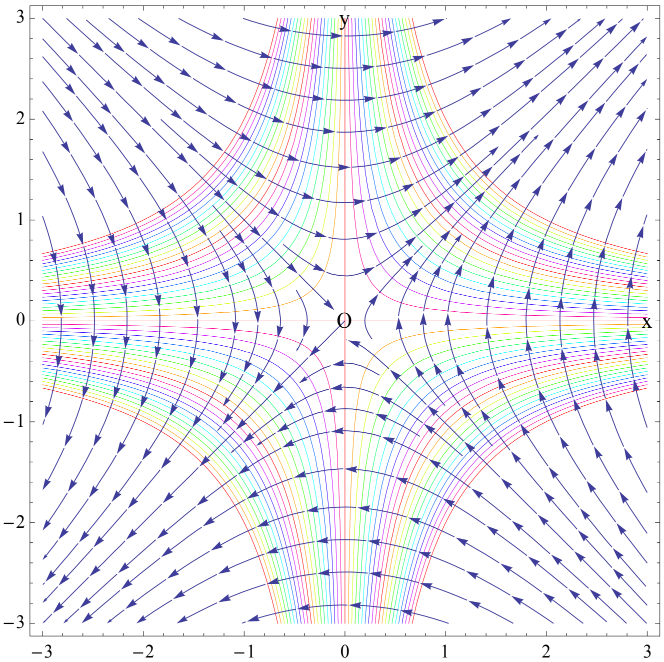
Now we consider a rather simple 2D function,

$$\varphi = -xy$$

The gradient operating on this function generate the vector field

$$F = -\nabla\varphi = (y, x)$$

Using the Matematica, we make a plot of the equipotential lines of φ in the x-y plane (ContourPlot) and a plot of the field lines of ***F*** in the x-y plane (StreamPlot). The field lines are perpendicular to the equipotential lines of constant φ.



1.2.9 Divergence

Now we define the divergence of the vector as

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

(A) $\nabla \cdot \mathbf{F}$ is a real scalar.

Under the rotation of the co-ordinate system

$$F_i' = \sum_j a_{ij} F_j$$

Then we have

$$\frac{\partial F_i'}{\partial x_k'} = \sum_j \frac{\partial}{\partial x_k'}(a_{ij} F_j) = \sum_{j,l} a_{ij} \frac{\partial x_l}{\partial x_k'} \frac{\partial F_j}{\partial x_l} = \sum_{j,l} a_{ij} a_{kl} \frac{\partial F_j}{\partial x_l}$$

where

$$a_{ij} = \frac{\partial x_i'}{\partial x_j}.$$

Then we have

$$\sum_i \frac{\partial F_i'}{\partial x_i'} = \sum_{i,j,l} a_{ij} a_{il} \frac{\partial F_j}{\partial x_l} = \sum_{j,l} (\sum_i a_{ij} a_{il}) \frac{\partial F_j}{\partial x_l} = \sum_{j,l} \delta_{jl} \frac{\partial F_j}{\partial x_l} = \sum_j \frac{\partial F_j}{\partial x_j}$$

Thus $\nabla \cdot \mathbf{F}$ is a scalar.

(B) Definition of solenoid

$\nabla \cdot \mathbf{B} = 0 \qquad \leftrightarrow \qquad$ ***B*** is said to be soloenoid.

1.2.10 $\nabla \times \mathbf{F}$

Now we define the rotation of the vector as

$$\mathbf{W} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

We show that $\nabla \times \mathbf{F}$ is a real vector.

((Proof))

Now we put

$$W_1' = \frac{\partial F_3'}{\partial x_2'} - \frac{\partial F_2'}{\partial x_3'}$$

(For simplicity we use $x_1 = x$, $x_2 = y$, $x_3 = z$)

$$\frac{\partial F_3'}{\partial x_2'} = \sum_j a_{3j} \frac{\partial F_j}{\partial x_2'} = \sum_j a_{3j} \frac{\partial x_l}{\partial x_2'} \frac{\partial F_j}{\partial x_l} = \sum_{j,l} a_{3j} a_{2l} \frac{\partial F_j}{\partial x_l}$$

Similarly, we have

$$\frac{\partial F_2'}{\partial x_3'} = \sum_j a_{2j} \frac{\partial F_j}{\partial x_3'} = \sum_j a_{2j} \frac{\partial x_l}{\partial x_3'} \frac{\partial F_j}{\partial x_l} = \sum_{j,l} a_{2j} a_{3l} \frac{\partial F_j}{\partial x_l}$$

Then we have

$$W_1' = \frac{\partial F_3'}{\partial x_2'} - \frac{\partial F_2'}{\partial x_3'} = \sum_{j,l} (a_{3j} a_{2l} - a_{2j} a_{3l}) \frac{\partial F_j}{\partial x_l}$$

or

$$W_1' = (a_{31}a_{22} - a_{21}a_{32})\frac{\partial F_1}{\partial x_2} + (a_{31}a_{23} - a_{21}a_{33})\frac{\partial F_1}{\partial x_3}$$
$$+ (a_{32}a_{21} - a_{22}a_{31})\frac{\partial F_2}{\partial x_1} + (a_{32}a_{23} - a_{22}a_{33})\frac{\partial F_2}{\partial x_3}$$
$$+ (a_{33}a_{21} - a_{23}a_{31})\frac{\partial F_3}{\partial x_1} + (a_{33}a_{22} - a_{23}a_{32})\frac{\partial F_3}{\partial x_2}$$

or

$$W_1' = -a_{13}\frac{\partial F_1}{\partial x_2} + a_{12}\frac{\partial F_1}{\partial x_3} + a_{13}\frac{\partial F_2}{\partial x_1} - a_{11}\frac{\partial F_2}{\partial x_3} - a_{12}\frac{\partial F_3}{\partial x_1} + a_{11}\frac{\partial F_3}{\partial x_2}$$
$$= a_{11}(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}) + a_{12}(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}) + a_{13}(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2})$$
$$= a_{11}W_1 + a_{12}W_2 + a_{13}W_3$$

Therefore,

$$Wi' = \sum_j a_{ij}W_j$$

which means that $\nabla \times \mathbf{F}$ is a real vector.

1.2.11 Successive application of ∇

(A) $\nabla \cdot (\nabla \varphi)$

This is defined by a Laplacian,

$$\nabla \cdot \nabla \varphi = \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

The equation $\nabla^2 \varphi = 0$ is called as the *Laplace equation*.

(B) $\nabla \varphi$ **is irrotational.**

$$\nabla \times (\nabla \varphi) = 0$$

since

$$\nabla \times (\nabla \varphi) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} = 0$$

Thus $\nabla \varphi$ is irrotational.

(C) $(\nabla \times \mathbf{F})$ is solenoid.

$$\nabla \cdot (\nabla \times \mathbf{F}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

(D) Formula

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

((Proof))

We use the formula given by

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

with $\mathbf{A} = \nabla$**,** $\mathbf{B} = \nabla$**,** and ***C = F***. Then we find

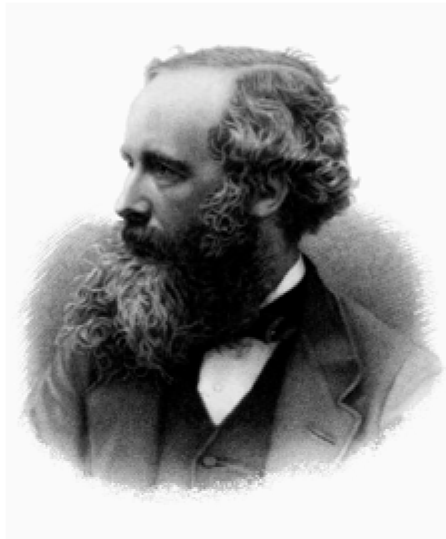
$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

1.2.12 Examples

(A) Electromagnetic wave equation

Derivation of electromagnetic wave equation from Maxwell’s equation.

James Clerk Maxwell (13 June 1831 – 5 November 1879) was a Scottish theoretical physicist and mathematician. His most important achievement was classical electromagnetic theory, synthesizing all previously unrelated observations, experiments and equations of electricity, magnetism and even optics into a consistent theory. His set of equations—Maxwell's equations—demonstrated that electricity, magnetism and even light are all manifestations of the same phenomenon: the electromagnetic field. From that moment on, all other classic laws or equations of these disciplines became simplified cases of Maxwell's equations. Maxwell's work in electromagnetism has been called the "second great unification in physics", after the first one carried out by Isaac Newton.



http://en.wikipedia.org/wiki/James_Clerk_Maxwell

Maxwell's equations in vacuum (in SI units);

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \varepsilon_0 \frac{\partial}{\partial t} \mathbf{E})$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}$$

Suppose that $\rho = 0$ and $\boldsymbol{J} = 0$.
Then we have

$$\frac{\partial}{\partial t} \nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} = \nabla \times \frac{\partial}{\partial t} \mathbf{B} = (-) \nabla \times (\nabla \times \mathbf{E})$$

or

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} .$$

where $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ is the velocity of light.

Since

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$
$$\nabla \cdot \mathbf{E} = 0$$

we have

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}.$$

This equation is called as electromagnetic wave equation. Similarly we have

$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B}.$$

(B) Calculations

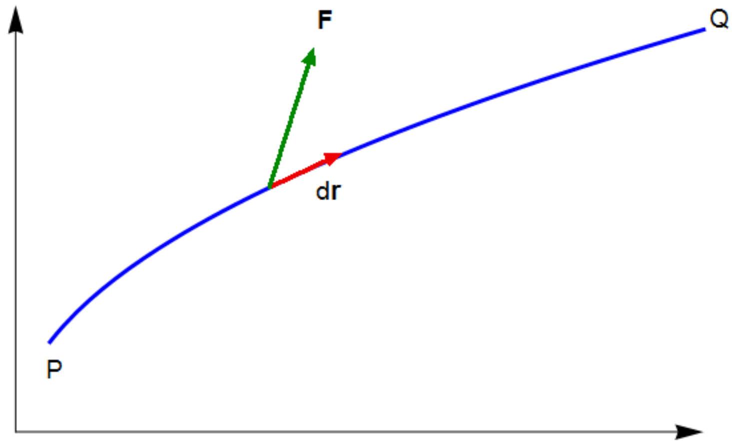
If $\mathbf{A} = (x^2 y, -2xz, 2yz)$, find $\nabla \times \mathbf{A}$, $\nabla \times (\nabla \times \mathbf{A})$, and $\nabla \times (\nabla \times (\nabla \times \mathbf{A}))$.

We use the Mathematica.

```
Clear["Gobal`"];

Needs["VectorAnalysis`"]

SetCoordinates[Cartesian[x, y, z]];

A1 = {x^2 y, -2 x z , 2 y z};

Curl[A1]

{2 x + 2 z, 0, -x^2 - 2 z}

Curl[Curl[A1]]

{0, 2 + 2 x, 0}

Curl[Curl[Curl[A1]]]

{0, 0, 2}
```

1.2.13 Line and surface integral

We consider about the line integral

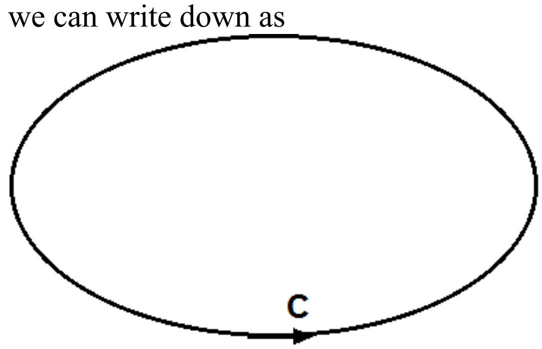


$$I = \int_{PQ} \mathbf{A} \cdot d\mathbf{r}$$

where $|d\mathbf{r}| = ds$ and the tangential component is assumed to be A_S'

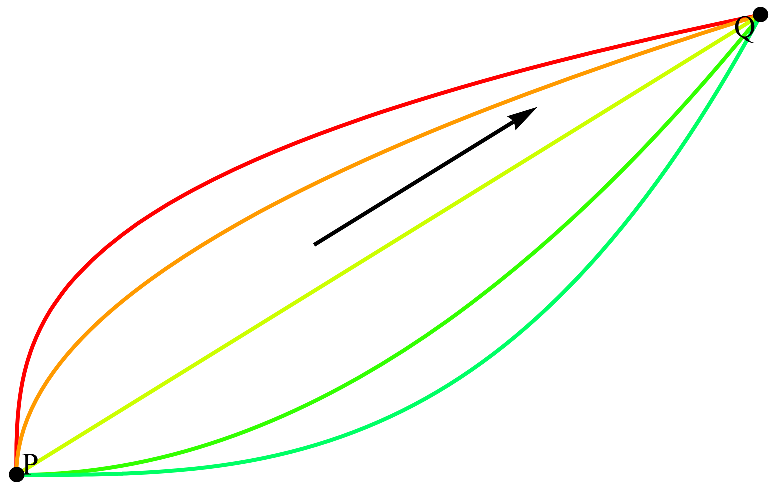
$$I = \int_{PQ} A_s ds$$

If the contour is closed, we can write down as



$$\oint \mathbf{A} \cdot d\mathbf{r}$$

In general the line integral depends on the choice of path. If $\mathbf{F} = \nabla \varphi$ (φ; scalar)



$$I = \int_{PQ} \mathbf{F} \cdot d\mathbf{r} = \int_{PQ} \nabla \varphi \cdot d\mathbf{r} = \varphi(Q) - \varphi(P)\,.$$

This value does not depend on the path of integral.

1.2.14 Surface Integral

$\hat{n}$ normal vector to the surface

$$d\mathbf{a} = \mathbf{n}da \qquad \text{(d}a\text{; area element)}$$

Then the surface integral is defined by

$$\int_S \mathbf{F} \cdot d\mathbf{a} = \int_S \mathbf{F} \cdot \mathbf{n}da$$

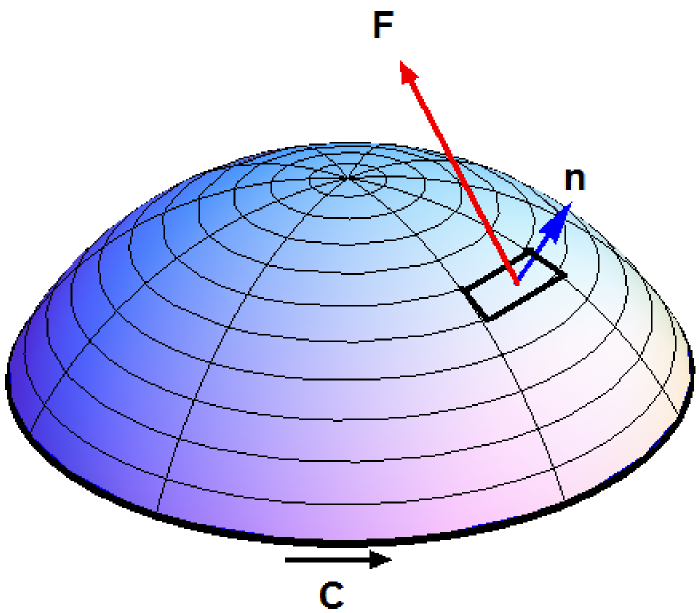
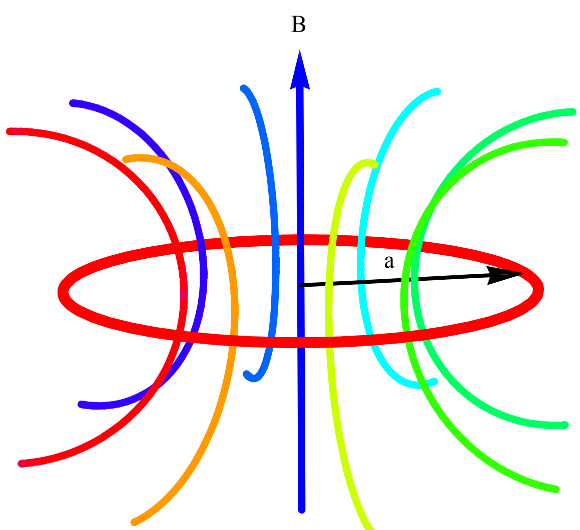


Fig. Right-hand rule for the positive normal.

If $\boldsymbol{F}$ corresponds to the magnetic field; $\boldsymbol{F} = \boldsymbol{B}$,

$\Phi = \int_S \mathbf{B} \cdot d\mathbf{a}$ is a magnetic flux through the area element S.



1.2.15 Gauss's theorem

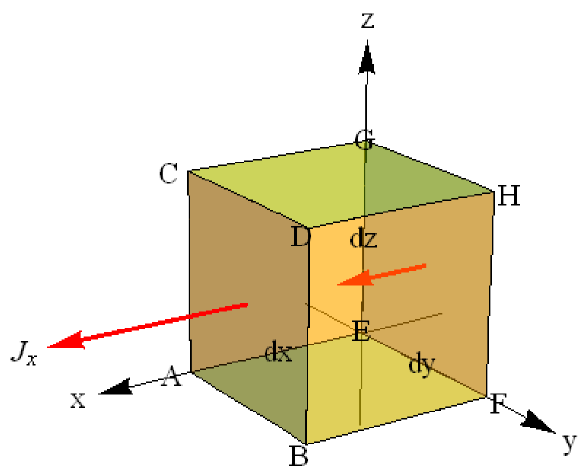
Here we define the volume integral as

$$\int_V \phi d\tau$$

where ϕ is a scalar.

(A) Gauss's theorem

$$\int_V \nabla \cdot \mathbf{F} d\tau = \int_S \mathbf{F} \cdot d\mathbf{a}$$



First we consider the physical interpretation of $\nabla \cdot \mathbf{F}$. Suppose that $\boldsymbol{F}$ = $\boldsymbol{J}$ (current density). The current coming out through ABCD is

$$J_x |_{x=dx} dydz = (J_x |_{x=0} + \frac{\partial J_x}{\partial x} dx) dydz$$

The current coming in through EFGH is equal to

$$J_x |_{x=0} dydz$$

Thus the net current through EFGH is equal to

$$\frac{\partial J_x}{\partial x} dxdydz$$

Thus the net current along the x direction through this small region is

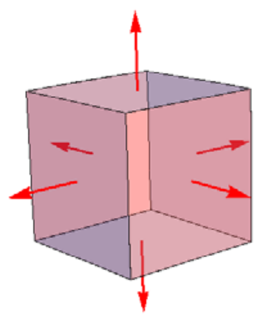
$$\frac{\partial J_x}{\partial x} dxdydz$$

Similarly for the y and z components, we have the net current along the y-direction and z-direction through the small region as

$$\frac{\partial J_y}{\partial y} dxdydz, \qquad \frac{\partial J_z}{\partial z} dxdydz$$

respectively. Therefore the net current coming out through the volume element $d\tau = dxdydz$ can be expressed by

$$\sum_{\substack{Six \\ surface}} \mathbf{J} \cdot d\mathbf{a} = (\nabla \cdot \mathbf{J}) d\tau$$

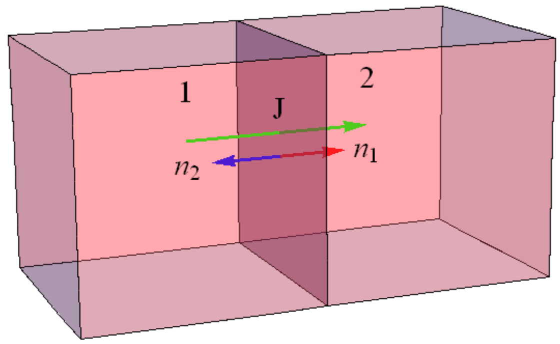


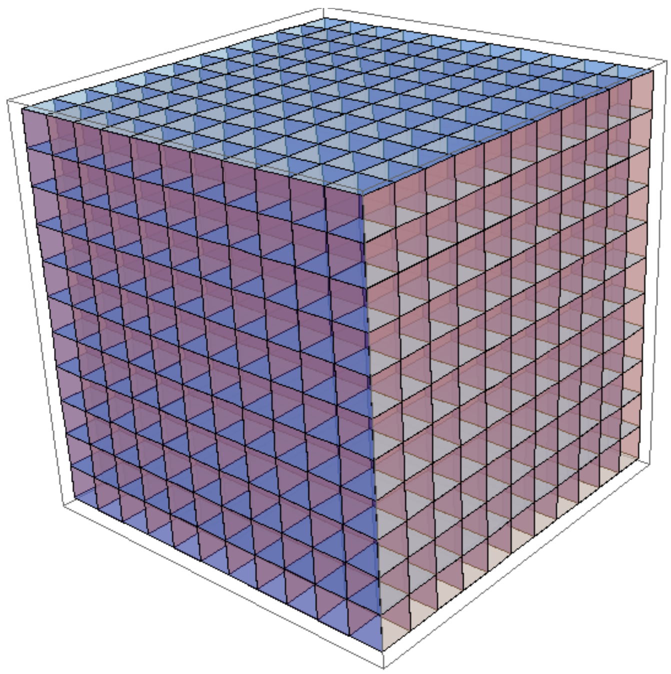
Summing over all parallel-pipes, we find that $\mathbf{J} \cdot d\mathbf{a}$ terms cancel out for all interior faces. Only the contributions of the exterior surface survive.

$$\sum_{\substack{exterior \\ surface}} \mathbf{J} \cdot d\mathbf{a} = \sum_{volume} (\nabla \cdot \mathbf{J}) d\tau$$

or

$$\boxed{\int_A \mathbf{J} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{J} d\tau} \qquad \text{(Gauss' theorem)}$$





(B) **Gauss' theorem**

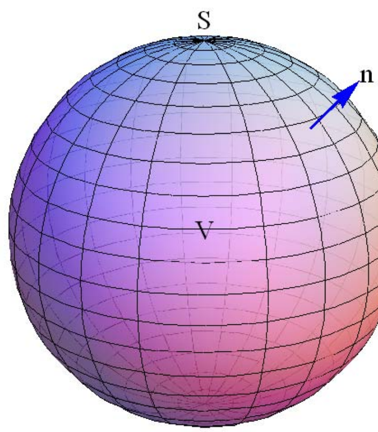
Let ***F*** be a continuous and differentiable vector throughtout a region *V* of the space. Then

$$\int_S \mathbf{F} \cdot d\mathbf{a} = \int_S \mathbf{F} \cdot \mathbf{n} da = \int_V \nabla \cdot \mathbf{F} d\tau$$

where the surface integral is taken over the entire surface that encloses *V*.

((**Example-1**))

In the maxwell's equation, we have



$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

where ρ is the charge density. From the Gauss's law, we have

$$\int_V \nabla \cdot \mathbf{E} d\tau = \int_V \frac{\rho}{\varepsilon_0} d\tau = \int_S \mathbf{E} \cdot d\mathbf{a} .$$

We assume that the volume *V* is formed of sphere with radius *r*. From the symmetry, ***E*** is perpendicular to the sphere surfaces,

$$\mathbf{E} = E_r \mathbf{e}_r .$$

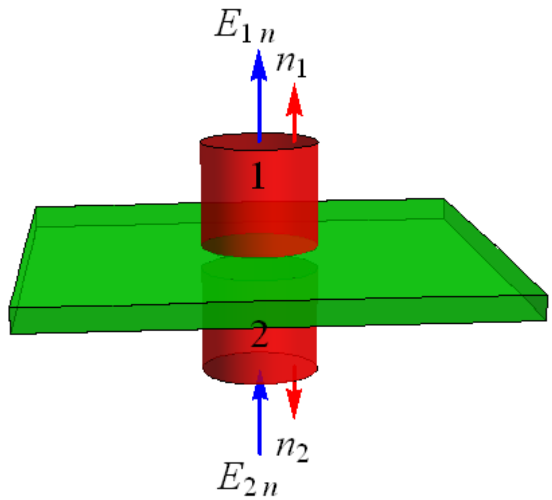
Thus we have

$$\int_V \frac{\rho}{\varepsilon_0} d\tau = \int_S E_r \mathbf{e}_r \cdot d\mathbf{a}$$

Since $Q = \int_V \rho d\tau$, we get

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad \text{(Coulomb's law)}$$

((**Example-2**))



$$\nabla \cdot \mathbf{E} = 0$$

if $\rho = 0$.

From the Gauss's law, we have

$$\int_V \nabla \cdot \mathbf{E} d\tau = \int_V \frac{\rho}{\varepsilon_0} d\tau = 0 = \int_S \mathbf{E} \cdot d\mathbf{a}$$

E_{1n} and E_{2n} are the normal components of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$. Then we have

$$(E_{1n} - E_{2n}) \Delta a = 0 .$$

Therefore we have the boundary condition for $\boldsymbol{E}$ as

$$E_{1n} = E_{2n}$$

1.2.16 Green's theorem

$$\int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi) d\tau = \int_S (\psi \nabla \phi - \phi \nabla \psi) \cdot d\mathbf{a}$$

((Proof)) In the Gauss's theorem, we put

$$\mathbf{A} = \psi \nabla \phi$$

Then we have

$$I_1 = \int_V \nabla \cdot \mathbf{A}\, d\tau = \int_V \nabla \cdot (\psi \nabla \phi) d\tau = \int_S (\psi \nabla \phi) \cdot d\mathbf{a}$$

Noting that

$$\nabla \cdot (\psi \nabla \phi) = \psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi$$

we have

$$I_1 = \int_V (\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi) d\tau = \int_S (\psi \nabla \phi) \cdot d\mathbf{a}$$

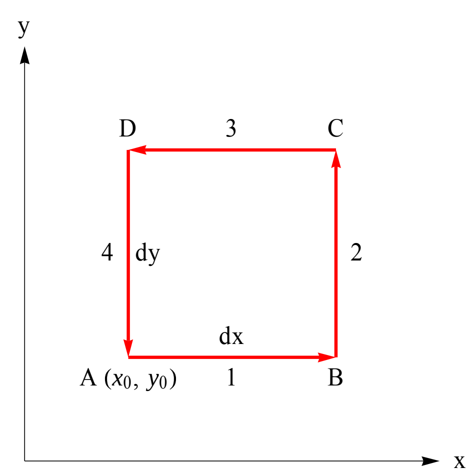
By replacing $\psi \leftrightarrow \phi$, we also have

$$I_1 = \int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d\tau = \int_S (\phi \nabla \psi) \cdot d\mathbf{a}$$

Thus we find the Green's theorem

$$I_1 - I_2 = \int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi) d\tau = \int_S (\psi \nabla \phi - \phi \nabla \psi) \cdot d\mathbf{a}$$

1.2.17 Stokes' theorem



$$\oint \mathbf{F}\cdot d\mathbf{l} = (circulation)_{1234}$$
$$= \int_1 F_x dx + \int_2 F_y dy - \int_3 F_x dx - \int_4 F_y dy$$
$$= \int_1 \{F_x(x,y_0) - F_x(x,y_0+dy)\}dx + \int_2 \{F_y(x_0+dx,y) - F_y(x_0,y)\}dy$$

Note that

$$F_x(x,y_0+dy) - F_x(x,y_0) = \left(\frac{\partial F_x}{\partial y}\right)_{x_0,y_0} dy$$
$$F_y(x_0+dx,y) - F_y(x_0,y) = \left(\frac{\partial F_y}{\partial x}\right)_{x_0,y_0} dx$$

Then we have

$$(circulation)_{1234} = (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})dxdy = (\nabla\times\mathbf{F})_z dxdy$$

We can write down this as

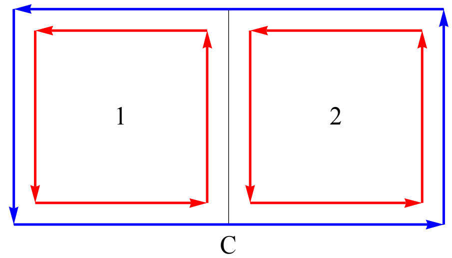
$$\sum_{\substack{Four\\ sides}} \mathbf{F}\cdot d\mathbf{l} = (\nabla\times\mathbf{F})\cdot\mathbf{e}_z dxdy = (\nabla\times\mathbf{F})\cdot d\mathbf{a}_1$$

where

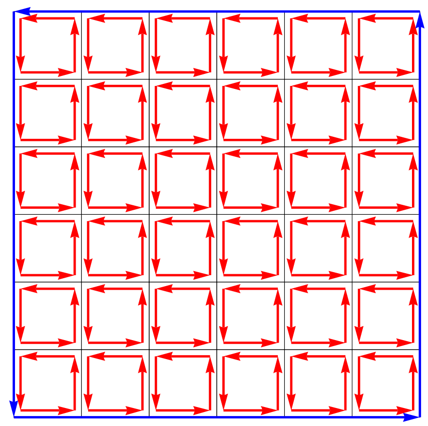
$$d\mathbf{a}_1 = \mathbf{e}_z dxdy$$

Imagine that paths 1 and 2 are expanded out until they coalesce with path C (or path 3). Since the line integrals of $\boldsymbol{F}$ along the potions that 1 and 2 have in common will cancel each other,

$$\oint_C \mathbf{F}\cdot d\mathbf{l}$$
$$= \oint_1 \mathbf{F}\cdot d\mathbf{l} + \oint_2 \mathbf{F}\cdot d\mathbf{l} \qquad .$$
$$= (\nabla\times\mathbf{F})\cdot d\mathbf{a}_1 + (\nabla\times\mathbf{F})\cdot d\mathbf{a}_2 = \int_{1,2} (\nabla\times\mathbf{F})\cdot d\mathbf{a}$$



Now let the surface S be divided up into a large number N of elements.

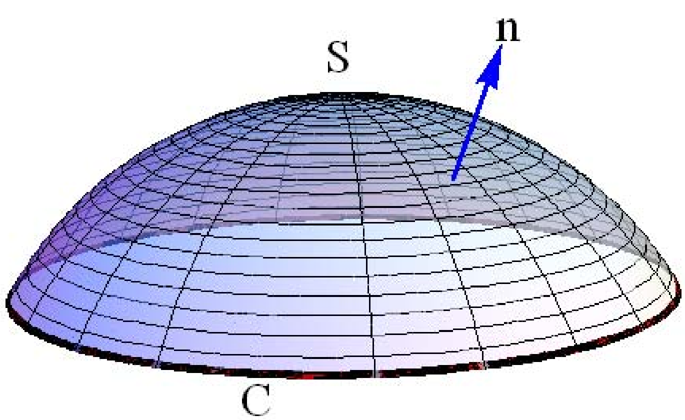


The above idea is extended to arrive at

$$\oint_C \mathbf{F}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{F})\cdot d\mathbf{a}$$

((Stoke's theorem))
Let S be a surface of any shape bounded by a closed curve C. If F is a vector, then

$$\oint_C \mathbf{F}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{F})\cdot d\mathbf{a} = \int_S (\nabla\times\mathbf{F})\cdot \mathbf{n}da\,.$$



1.3 Curvilinear co-ordinates

1.3.1 General definition

We consider that new co-ordinate (q_1, q_2, q_3) are related to (x, y, z) through

$$\begin{aligned} x &= x(q_1, q_2, q_3) \\ y &= y(q_1, q_2, q_3) \\ z &= z(q_1, q_2, q_3) \end{aligned} \qquad \text{or} \qquad \begin{aligned} q_1 &= q_1(x, y, z) \\ q_2 &= q_2(x, y, z) \\ q_3 &= q_3(x, y, z) \end{aligned}$$

Since

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q_1} dq_1 + \frac{\partial \mathbf{r}}{\partial q_2} dq_2 + \frac{\partial \mathbf{r}}{\partial q_3} dq_3 = \sum_j \frac{\partial \mathbf{r}}{\partial q_j} dq_j ,$$

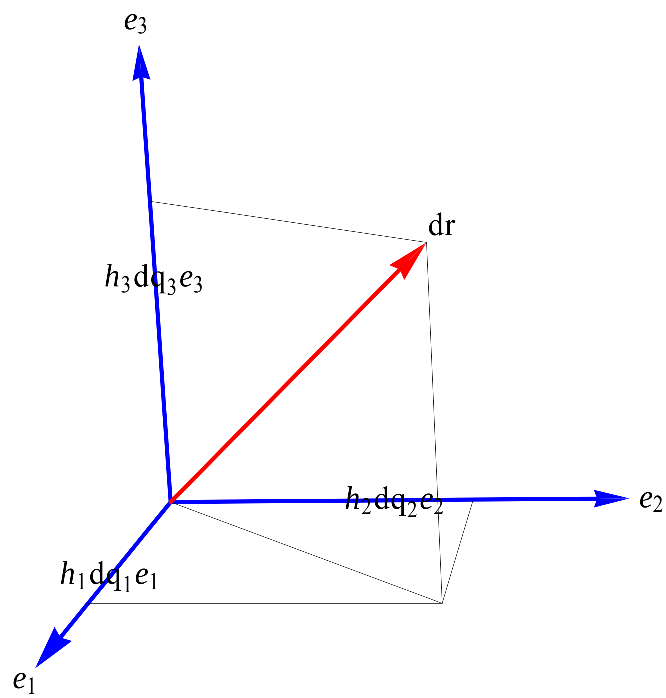
we have

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = \sum_{i,j} \left(\frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j}\right) dq_i dq_j = \sum_{i,j} g_{ij} dq_i dq_j ,$$

where

$$g_{ij} = \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} = \frac{\partial x}{\partial q_i} \cdot \frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i} \cdot \frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i} \cdot \frac{\partial z}{\partial q_j} \text{ (second rank tensor).}$$

We now consider the general coordinate system. The relation between the constants h_1, h_2, and h_3 and the tensor g_{ij} will be discussed later.



$$d\mathbf{r} = ds_1\mathbf{e}_1 + ds_2\mathbf{e}_2 + ds_3\mathbf{e}_3 = h_1 dq_1\mathbf{e}_1 + h_2 dq_2\mathbf{e}_2 + h_3 dq_3\mathbf{e}_3 = \sum_i h_i dq_i \mathbf{e}_i$$

$$ds^2 = d\mathbf{r}\cdot d\mathbf{r} = \sum_{i,j} h_i h_j dq_i dq_j (\mathbf{e}_i \cdot \mathbf{e}_i)$$

or we have

$$g_{ij} = h_i h_j (\mathbf{e}_i \cdot \mathbf{e}_i)$$

or

$$g_{ii} = h_i^2 .$$

Then

$$\frac{g_{ij}}{\sqrt{g_{ii} g_{jj}}} = (\mathbf{e}_i \cdot \mathbf{e}_i)$$

Now we limit ourselves to orthogonal co-ordinate system.

$$g_{ij} \text{ for } i \neq j.$$

In order to simplify the notation, we use $g_{ii} = h_i^2$, so that

$$ds^2 = \sum_i (h_i dq_i)^2$$

$$\begin{aligned} d\mathbf{r} &= ds_1\mathbf{e}_1 + ds_2\mathbf{e}_2 + ds_3\mathbf{e}_3 \\ &= h_1 dq_1\mathbf{e}_1 + h_2 dq_2\mathbf{e}_2 + h_3 dq_3\mathbf{e}_3 \\ &= \sum_i h_i dq_i \mathbf{e}_i \end{aligned}$$

Where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are unit vectors which are perpendicular to each other.

$$\mathbf{e}_1 = \frac{1}{h_1}\frac{\partial \mathbf{r}}{\partial q_1} = \frac{\partial \mathbf{r}}{\partial s_1}$$
$$\mathbf{e}_2 = \frac{1}{h_2}\frac{\partial \mathbf{r}}{\partial q_2} = \frac{\partial \mathbf{r}}{\partial s_2}$$
$$\mathbf{e}_3 = \frac{1}{h_3}\frac{\partial \mathbf{r}}{\partial q_3} = \frac{\partial \mathbf{r}}{\partial s_3}$$

where

$$h_i^2 = g_{ii} = \frac{\partial \mathbf{r}}{\partial q_i}\cdot\frac{\partial \mathbf{r}}{\partial q_j}$$

or

$$h_i = \sqrt{g_{ii}} = \sqrt{\frac{\partial \mathbf{r}}{\partial q_i}\cdot\frac{\partial \mathbf{r}}{\partial q_i}} = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2} \quad \text{(second rank tensor).}$$

The volume element for an orthogonal curvilinear coordinate system is given by

$$dV = h_1 dq_1\mathbf{e}_1 \cdot \{(h_2 dq_2\mathbf{e}_2)\times(h_3 dq_3\mathbf{e}_3)\} = h_1 h_2 h_3 dq_1 dq_2 dq_3$$

1.3.2 Spherical coordinete

(A) Unit vectors

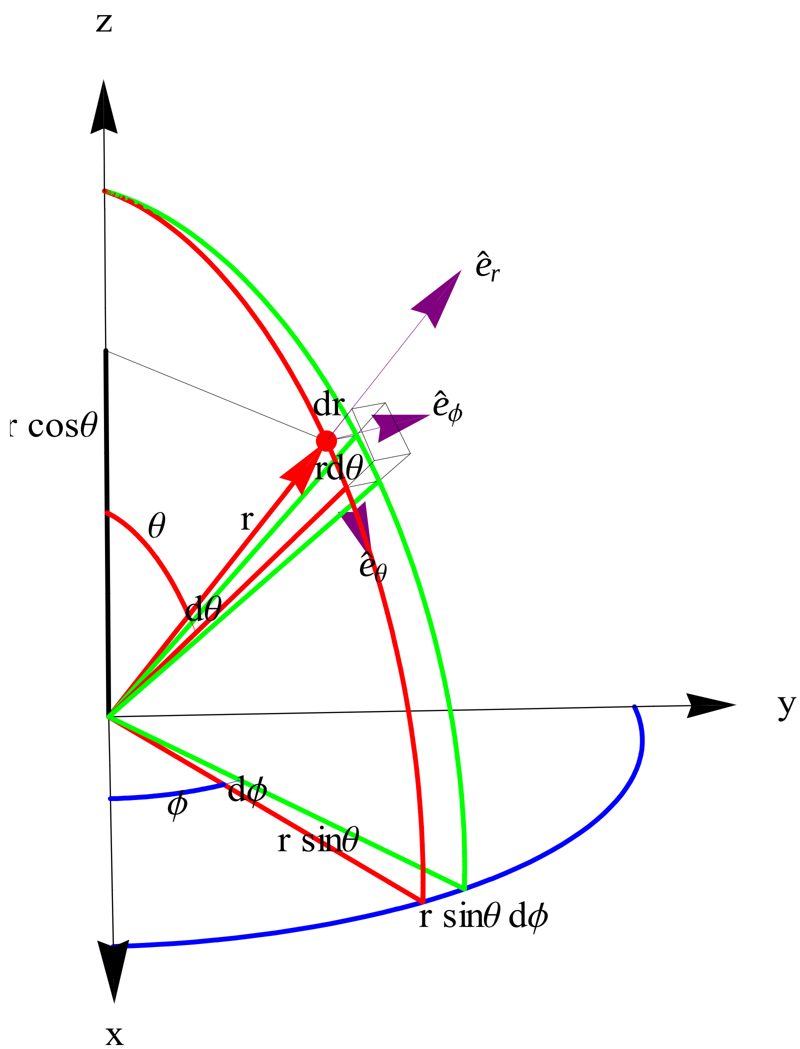
The position of a point P with Cartesian coordinates x, y, and z may be expressed in terms of r, θ, and ϕ of the spherical coordinates;

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

or

$$\mathbf{r} = r\sin\theta\cos\phi\mathbf{e}_\mathbf{x} + r\sin\theta\sin\phi\mathbf{e}_y + r\cos\theta\mathbf{e}_z$$

$$d\mathbf{r}=\sum_{j=1}^{3}\mathbf{e}_j ds_j=\sum_{j=1}^{3}\mathbf{e}_j h_j dq_j$$



$$h_r=\sqrt{\left(\frac{\partial x}{\partial r}\right)^2+\left(\frac{\partial y}{\partial r}\right)^2+\left(\frac{\partial z}{\partial r}\right)^2}=1$$

$$h_\theta=\sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2+\left(\frac{\partial y}{\partial \theta}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2}=r$$

$$h_\phi=\sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2+\left(\frac{\partial y}{\partial \phi}\right)^2+\left(\frac{\partial z}{\partial \phi}\right)^2}=r\sin\theta$$

or

$$d\mathbf{r} = h_r\mathbf{e}_r dr + h_\theta \mathbf{e}_\theta d\theta + h_\phi \mathbf{e}_\phi d\phi ={}_r \mathbf{e}_r dr + r\mathbf{e}_\theta d\theta + r\sin\theta \mathbf{e}_\phi d\phi$$

$$\mathbf{e}_\mathbf{r} = \frac{\partial \mathbf{r}}{\partial r} = \sin\theta\cos\phi \mathbf{e}_\mathbf{x} + \sin\theta\sin\phi \mathbf{e}_y + \cos\theta \mathbf{e}_z$$

$$\mathbf{e}_\theta = \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \theta} = \cos\theta\cos\phi \mathbf{e}_\mathbf{x} + \cos\theta\sin\phi \mathbf{e}_y - \sin\theta \mathbf{e}_z$$

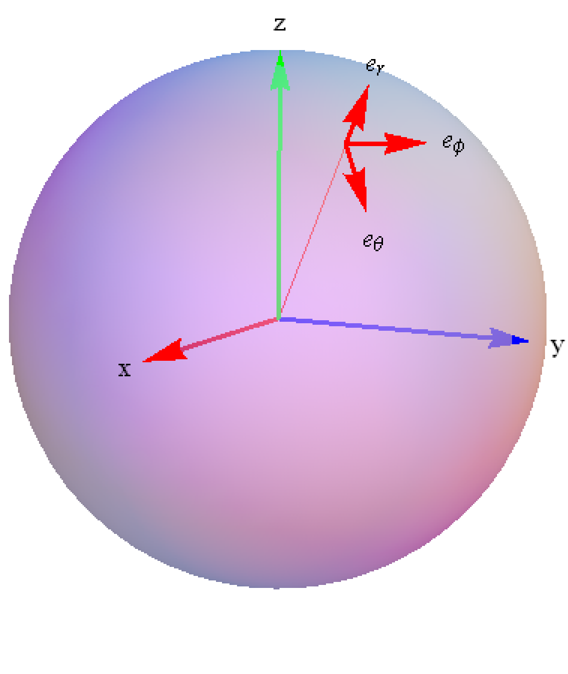
$$\mathbf{e}_\phi = \frac{1}{r\sin\theta}\frac{\partial \mathbf{r}}{\partial \phi} = -\sin\phi \mathbf{e}_\mathbf{x} + \cos\phi \mathbf{e}_y$$

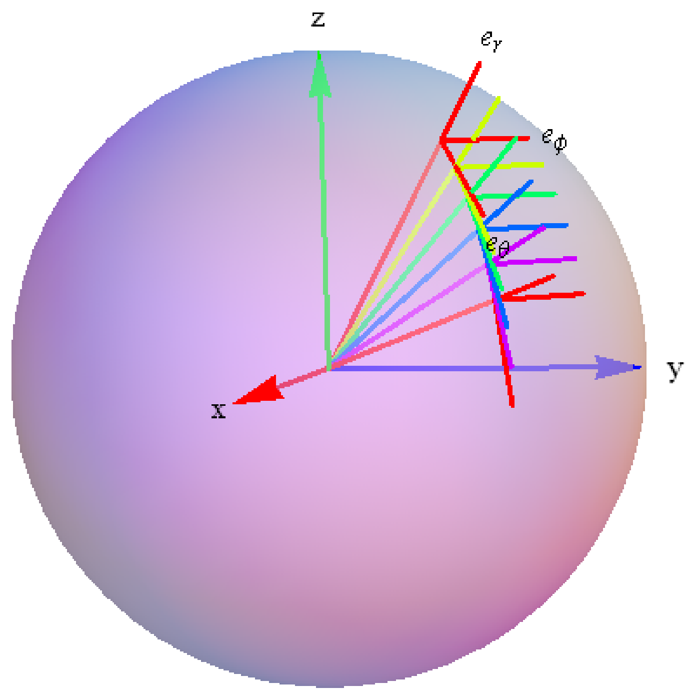
This can be described using a matrix A as

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix} = \mathbf{A}\begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}\begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix}.$$

or by using the inverse matrix A^{-1} as

$$\begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix} = \mathbf{A}^{T}\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix}$$





z
e_γ
e_ϕ
e_θ
y
x

```
A = {{Sin[θ] Cos[ϕ], Sin[θ] Sin[ϕ], Cos[θ]},
   {Cos[θ] Cos[ϕ], Cos[θ] Sin[ϕ], -Sin[θ]},
   {-Sin[ϕ], Cos[ϕ], 0}};
```

```
A // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\phi]\,\text{Sin}[\theta] & \text{Sin}[\theta]\,\text{Sin}[\phi] & \text{Cos}[\theta] \\ \text{Cos}[\theta]\,\text{Cos}[\phi] & \text{Cos}[\theta]\,\text{Sin}[\phi] & -\text{Sin}[\theta] \\ -\text{Sin}[\phi] & \text{Cos}[\phi] & 0 \end{pmatrix}$$

```
Ainv = Inverse[A] // Simplify;
```

```
{{Cos[ϕ] Sin[θ], Cos[θ] Cos[ϕ], -Sin[ϕ]},
 {Sin[θ] Sin[ϕ], Cos[θ] Sin[ϕ], Cos[ϕ]},
 {Cos[θ], -Sin[θ], 0}}
```

```
Ainv // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\phi]\,\text{Sin}[\theta] & \text{Cos}[\theta]\,\text{Cos}[\phi] & -\text{Sin}[\phi] \\ \text{Sin}[\theta]\,\text{Sin}[\phi] & \text{Cos}[\theta]\,\text{Sin}[\phi] & \text{Cos}[\phi] \\ \text{Cos}[\theta] & -\text{Sin}[\theta] & 0 \end{pmatrix}$$

```
A.Ainv // Simplify
```

```
{{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```

The time derivatives $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_\phi$ are obtained as

$$\dot{\mathbf{e}}_{\mathbf{r}} = \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi$$

$$\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r + \dot{\phi}\cos\theta\mathbf{e}_\phi$$

$$\dot{\mathbf{e}}_\phi = -\dot{\phi}(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta)$$

We note that

$$\frac{\partial\mathbf{e}_{\mathbf{r}}}{\partial r} = 0\,, \qquad \frac{\partial\mathbf{e}_{\mathbf{r}}}{\partial\theta} = \mathbf{e}_\theta\,, \qquad \frac{\partial\mathbf{e}_{\mathbf{r}}}{\partial\phi} = \sin\theta\mathbf{e}_\phi$$

$$\frac{\partial\mathbf{e}_\theta}{\partial r} = 0 \qquad \frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r\,, \qquad \frac{\partial\mathbf{e}_\theta}{\partial\phi} = \cos\theta\mathbf{e}_\phi\,.$$

(B) $\nabla\psi$

From the definition of $\nabla\psi$, we have

$$\nabla\psi = \sum_{j=1}^{3} \mathbf{e}_j \frac{\partial\psi}{\partial s_j} = \sum_{j=1}^{3} \mathbf{e}_j \frac{\partial\psi}{h_j \partial q_j}$$

or,

$$\boxed{\nabla\psi = \mathbf{e}_r \frac{\partial\psi}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}}$$

where ψ is a scalar function of r, θ, and ϕ.

(C) $\nabla\cdot\mathbf{A}$

When a vector $\boldsymbol{A}$ is defined by

$$\mathbf{A} = A_r\mathbf{e}_r + A_\theta\mathbf{e}_\theta + A_\phi\mathbf{e}_\phi$$

The divergence is given by

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{1}{h_r h_\theta h_\phi}[\frac{\partial}{\partial r}(h_\theta h_\phi A_r) + \frac{\partial}{\partial\theta}(h_\phi h_r A_\theta) + \frac{\partial}{\partial\phi}(h_r h_\theta A_\phi)] \\ &= \frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta A_r) + \frac{\partial}{\partial\theta}(r\sin\theta A_\theta) + \frac{\partial}{\partial\phi}(rA_\phi)]\end{aligned}$$

or

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi$$

(D) $\nabla\times\mathbf{A}$

$\nabla\times\mathbf{A}$ is given by

$$\nabla\times\mathbf{A} = \frac{1}{h_r h_\theta h_\phi}\begin{vmatrix} h_r\mathbf{e}_r & h_\theta\mathbf{e}_\theta & h_\phi\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ h_r A_r & h_\theta A_\theta & h_\phi A_\phi \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

(E) Laplacian

$$\nabla^2\psi = \frac{1}{h_r h_\theta h_\phi}\left[\frac{\partial}{\partial r}\left(\frac{h_\theta h_\phi}{h_r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{h_\phi h_r}{h_\theta}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{h_r h_\theta}{h_\phi}\frac{\partial\psi}{\partial\phi}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]$$

or

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

We can rewrite the first term of the right hand side as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r^2}(r)$$

which can be useful in shortening calculations.

Note that we also use the expression for the operator

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} =$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{{}^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}$$

((**Mathematica**))

We derive the above formula using the Mathematica.

We use the Spherical co-rdinate.
We need a Vector Analysis Package. We also need SetCordinatinates.In this system the vector is expressed in terms of (Ar, Aθ, Aϕ)

```
Clear["Gobal`"];

Needs["VectorAnalysis`"];

SetCoordinates[Spherical[r, θ, ϕ]];
```

Vector analysis
Grad, Curl, Laplacian which are expressed in terms of the spherical coordin ates

```
eq1 = Laplacian[ψ[r, θ, ϕ]] // Simplify
```

$$\frac{1}{r^2}\left(\text{Csc}[\theta]^2\,\psi^{(0,0,2)}[r,\theta,\phi]+\text{Cot}[\theta]\,\psi^{(0,1,0)}[r,\theta,\phi]+\psi^{(0,2,0)}[r,\theta,\phi]+2\,r\,\psi^{(1,0,0)}[r,\theta,\phi]+r^2\,\psi^{(2,0,0)}[r,\theta,\phi]\right)$$

```
eq2 = Grad[ψ[r, θ, ϕ]]
```

$$\left\{\psi^{(1,0,0)}[r,\theta,\phi],\ \frac{\psi^{(0,1,0)}[r,\theta,\phi]}{r},\ \frac{\text{Csc}[\theta]\,\psi^{(0,0,1)}[r,\theta,\phi]}{r}\right\}$$

```
A = {Ar[r, θ, ϕ], Aθ[r, θ, ϕ], Aϕ[r, θ, ϕ]};

eq3 = Curl[A]
```

$$\left\{\frac{1}{r^2}\text{Csc}[\theta]\left(r\,A\phi[r,\theta,\phi]\,\text{Cos}[\theta]-r\,A\theta^{(0,0,1)}[r,\theta,\phi]+r\,\text{Sin}[\theta]\,A\phi^{(0,1,0)}[r,\theta,\phi]\right),\ \frac{1}{r}\text{Csc}[\theta]\left(-A\phi[r,\theta,\phi]\,\text{Sin}[\theta]+Ar^{(0,0,1)}[r,\theta,\phi]-r\,\text{Sin}[\theta]\,A\phi^{(1,0,0)}[r,\theta,\phi]\right),\ \frac{A\theta[r,\theta,\phi]-Ar^{(0,1,0)}[r,\theta,\phi]+r\,A\theta^{(1,0,0)}[r,\theta,\phi]}{r}\right\}$$

```
eq3 = Div[A]
```

$$\frac{1}{r^2}\text{Csc}[\theta]\left(r\,A\theta[r,\theta,\phi]\,\text{Cos}[\theta]+2\,r\,Ar[r,\theta,\phi]\,\text{Sin}[\theta]+r\,A\phi^{(0,0,1)}[r,\theta,\phi]+r\,\text{Sin}[\theta]\,A\theta^{(0,1,0)}[r,\theta,\phi]+r^2\,\text{Sin}[\theta]\,Ar^{(1,0,0)}[r,\theta,\phi]\right)$$

1.3.3 Velocity and acceleration in the spherical coordinate

The velocity (***v***) and acceleration (***a***) in the spherical co-ordinates are given by

$$v_r = \dot{r} \qquad a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta$$

$$v_\theta = r\dot{\theta} \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta$$

$$v_\phi = r\sin\theta\dot{\phi} \qquad a_\phi = r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta$$

((**Mathematica**))

We drive the above formula using the Mathematica.

Velocity and acceleration in the spherical coordinates

```
Clear["Gobal`"]

<< "VectorAnalysis`"

SetCoordinates[Cartesian[x, y, z]]

Cartesian [x, y, z]

RR[t_] := {r[t] Sin[θ[t]] Cos[ϕ[t]], r[t] Sin[θ[t]] Sin[ϕ[t]], r[t] Cos[θ[t]]}

D[RR[t], t] // FullSimplify

{Cos[ϕ[t]] (Sin[θ[t]] r′[t] + Cos[θ[t]] r[t] θ′[t]) - r[t] Sin[θ[t]] Sin[ϕ[t]] ϕ′[t],
 Sin[ϕ[t]] (Sin[θ[t]] r′[t] + Cos[θ[t]] r[t] θ′[t]) + Cos[ϕ[t]] r[t] Sin[θ[t]] ϕ′[t],
 Cos[θ[t]] r′[t] - r[t] Sin[θ[t]] θ′[t]}

D[RR[t], {t, 1}] // FullSimplify

{Cos[ϕ[t]] (Sin[θ[t]] r′[t] + Cos[θ[t]] r[t] θ′[t]) - r[t] Sin[θ[t]] Sin[ϕ[t]] ϕ′[t],
 Sin[ϕ[t]] (Sin[θ[t]] r′[t] + Cos[θ[t]] r[t] θ′[t]) + Cos[ϕ[t]] r[t] Sin[θ[t]] ϕ′[t],
 Cos[θ[t]] r′[t] - r[t] Sin[θ[t]] θ′[t]}

D[RR[t], {t, 2}] // FullSimplify

{Cos[θ[t]] (2 θ′[t] (Cos[ϕ[t]] r′[t] - r[t] Sin[ϕ[t]] ϕ′[t]) + Cos[ϕ[t]] r[t] θ″[t]) +
  Sin[θ[t]] (Cos[ϕ[t]] (-r[t] (θ′[t]^2 + ϕ′[t]^2) + r″[t]) -
     Sin[ϕ[t]] (2 r′[t] ϕ′[t] + r[t] ϕ″[t])), Sin[ϕ[t]]
   (Sin[θ[t]] (-r[t] (θ′[t]^2 + ϕ′[t]^2) + r″[t]) + Cos[θ[t]] (2 r′[t] θ′[t] + r[t] θ″[t])) +
  Cos[ϕ[t]] (2 (Sin[θ[t]] r′[t] + Cos[θ[t]] r[t] θ′[t]) ϕ′[t] + r[t] Sin[θ[t]] ϕ″[t]),
 Cos[θ[t]] (-r[t] θ′[t]^2 + r″[t]) - Sin[θ[t]] (2 r′[t] θ′[t] + r[t] θ″[t])}

D[RR[t], {t, 3}] // FullSimplify

{Cos[θ[t]] (-3 Sin[ϕ[t]] (ϕ′[t] (2 r′[t] θ′[t] + r[t] θ″[t]) + r[t] θ′[t] ϕ″[t]) + Cos[ϕ[t]]
      (3 θ′[t] r″[t] + 3 r′[t] θ″[t] + r[t] (-θ′[t]^3 - 3 θ′[t] ϕ′[t]^2 + θ^(3)[t]))) + Sin[θ[t]]
   (Cos[ϕ[t]] (-3 r′[t] (θ′[t]^2 + ϕ′[t]^2) - 3 r[t] (θ′[t] θ″[t] + ϕ′[t] ϕ″[t]) + r^(3)[t]) +
     Sin[ϕ[t]] (-3 (ϕ′[t] r″[t] + r′[t] ϕ″[t]) + r[t] (3 θ′[t]^2 ϕ′[t] + ϕ′[t]^3 - ϕ^(3)[t]))),
 Sin[ϕ[t]] (Sin[θ[t]] (-3 r′[t] (θ′[t]^2 + ϕ′[t]^2) -
        3 r[t] (θ′[t] θ″[t] + ϕ′[t] ϕ″[t]) + r^(3)[t]) +
     Cos[θ[t]] (3 θ′[t] r″[t] + 3 r′[t] θ″[t] + r[t] (-θ′[t]^3 - 3 θ′[t] ϕ′[t]^2 + θ^(3)[t]))) +
  Cos[ϕ[t]] (3 Cos[θ[t]] (ϕ′[t] (2 r′[t] θ′[t] + r[t] θ″[t]) + r[t] θ′[t] ϕ″[t]) +
     Sin[θ[t]] (3 ϕ′[t] r″[t] + 3 r′[t] ϕ″[t] + r[t] (-3 θ′[t]^2 ϕ′[t] - ϕ′[t]^3 + ϕ^(3)[t]))),
 Cos[θ[t]] (-3 θ′[t] (r′[t] θ′[t] + r[t] θ″[t]) + r^(3)[t]) +
  Sin[θ[t]] (-3 (θ′[t] r″[t] + r′[t] θ″[t]) + r[t] (θ′[t]^3 - θ^(3)[t]))}
```

Unit vectors along the r, θ, and ϕ directions (Cartesian coordinate)

```
ur = ∂r[t] RR[t] // Simplify
```

{Cos[ϕ[t]] Sin[θ[t]], Sin[θ[t]] Sin[ϕ[t]], Cos[θ[t]]}

```
uθ = ∂θ[t] RR[t] / r[t] // Simplify
```

{Cos[θ[t]] Cos[ϕ[t]], Cos[θ[t]] Sin[ϕ[t]], -Sin[θ[t]]}

```
uϕ = ∂ϕ[t] RR[t] / (r[t] Sin[θ[t]]) // Simplify
```

{-Sin[ϕ[t]], Cos[ϕ[t]], 0}

```
ur.uϕ
```

0

```
ur.uθ // Simplify
```

0

■ Velocity and kinetic energy in the spherical coordinates

```
Vr = D[RR[t], t].ur // Simplify
```

r′[t]

```
Vθ = D[RR[t], t].uθ // Simplify
```

r[t] θ′[t]

```
Vϕ = D[RR[t], t].uϕ // Simplify
```

r[t] Sin[θ[t]] ϕ′[t]

```
K1 = m/2 (Vr^2 + Vθ^2 + Vϕ^2) // Simplify
```

$\frac{1}{2}$ m (r′[t]^2 + r[t]^2 (θ′[t]^2 + Sin[θ[t]]^2 ϕ′[t]^2))

■ Acceleration in the spherical coordinate

```
Ar = D[RR[t], {t, 2}].ur // Simplify
```

-r[t] (θ′[t]^2 + Sin[θ[t]]^2 ϕ′[t]^2) + r″[t]

```
Aθ = D[RR[t], {t, 2}].uθ // Simplify
```

2 r′[t] θ′[t] + r[t] (-Cos[θ[t]] Sin[θ[t]] ϕ′[t]^2 + θ″[t])

```
Aϕ = D[RR[t], {t, 2}].uϕ // Simplify
```

2 Sin[θ[t]] r′[t] ϕ′[t] + r[t] (2 Cos[θ[t]] θ′[t] ϕ′[t] + Sin[θ[t]] ϕ″[t])

■ Some application

```
Sr = D[RR[t], {t, 3}].ur // Simplify
```

$\frac{1}{2}$ (-6 r′[t] (θ′[t]^2 + Sin[θ[t]]^2 ϕ′[t]^2) -
3 r[t] (θ′[t] (Sin[2 θ[t]] ϕ′[t]^2 + 2 θ″[t]) + 2 Sin[θ[t]]^2 ϕ′[t] ϕ″[t]) + 2 r^(3)[t])

```
Sθ = D[RR[t], {t, 3}].uθ // Simplify
```

$\frac{1}{2}$ (6 θ′[t] r″[t] + r′[t] (-3 Sin[2 θ[t]] ϕ′[t]^2 + 6 θ″[t]) -
r[t] (2 θ′[t]^3 + 6 Cos[θ[t]]^2 θ′[t] ϕ′[t]^2 + 3 Sin[2 θ[t]] ϕ′[t] ϕ″[t] - 2 θ^(3)[t]))

```
Sϕ = D[RR[t], {t, 3}].uϕ // Simplify
```

3 Sin[θ[t]] ϕ′[t] r″[t] + 3 r′[t] (2 Cos[θ[t]] θ′[t] ϕ′[t] + Sin[θ[t]] ϕ″[t]) +
r[t] (-3 Sin[θ[t]] θ′[t]^2 ϕ′[t] - Sin[θ[t]] ϕ′[t]^3 +
3 Cos[θ[t]] ϕ′[t] θ″[t] + 3 Cos[θ[t]] θ′[t] ϕ″[t] + Sin[θ[t]] ϕ^(3)[t])

1.3.4 Quantum mechanical orbital angular momentum

The orbital angular momentum in the quantum mechanics is defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = -i\hbar (r \times \nabla)$$

using the expression

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

in the spherical coordinate. Then we have

$$\mathbf{L} = -i\hbar(\mathbf{r}\times\nabla) = -i\hbar\mathbf{e}_r r \times (\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi})$$
$$= i\hbar(-\mathbf{e}_\phi \frac{\partial}{\partial \theta} + \mathbf{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial \phi})$$

.

The angular momentum L_x, L_y, and L_z (Cartesian components) can be described by

$$\mathbf{L} = i\hbar[-(-\sin\phi\mathbf{e}_x + \cos\phi\mathbf{e}_y)\frac{\partial}{\partial \theta} + (\cos\theta\cos\phi\mathbf{e}_x + \cos\theta\sin\phi\mathbf{e}_y - \sin\theta\mathbf{e}_z)\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}].$$

or

$$L_x = i\hbar(\sin\phi\frac{\partial}{\partial \theta} + \cot\theta\cos\phi\frac{\partial}{\partial \phi})$$

$$L_y = i\hbar(-\cos\phi\frac{\partial}{\partial \theta} + \cot\theta\sin\phi\frac{\partial}{\partial \phi})$$

$$L_z = -i\hbar\frac{\partial}{\partial \phi}$$

We define L_+ and L_- as

$$L_+ = L_x + iL_y = -i\hbar e^{i\phi}(i\frac{\partial}{\partial \theta} - \cot\theta\frac{\partial}{\partial \phi})$$

and

$$L_- = L_x - iL_y = -i\hbar e^{-i\phi}(-i\frac{\partial}{\partial \theta} - \cot\theta\frac{\partial}{\partial \phi})$$

We note that the operator ∇ can be expressed using the operator $\boldsymbol{L}$ as

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} - \frac{i}{\hbar}\frac{\mathbf{r}\times\mathbf{L}}{r^2}$$

The proof of this equation is given as follows.

$$\frac{(\mathbf{r}\times\mathbf{L})}{i\hbar} = r\mathbf{e}_r \times (-\mathbf{e}_\phi \frac{\partial}{\partial\theta} + \mathbf{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}) = r(\mathbf{e}_\theta \frac{\partial}{\partial\theta} + \mathbf{e}_\phi \frac{1}{\sin\theta}\frac{\partial}{\partial\phi})$$

or

$$\frac{(\mathbf{r}\times\mathbf{L})}{i\hbar r^2} = \frac{1}{r}\mathbf{e}_\theta \frac{\partial}{\partial\theta} + \mathbf{e}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} = \nabla - \mathbf{e}_r \frac{\partial}{\partial r}$$

or

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} - \frac{i(\mathbf{r}\times\mathbf{L})}{\hbar r^2}$$

From $\mathbf{L}^2 = L_x{}^2 + L_y{}^2 + L_z{}^2$, we have

$$\mathbf{L}^2 = -\hbar^2[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})]$$

where the proof is given by Mathematica. Using

$$\frac{\mathbf{L}^2}{\hbar^2} = -r^2\nabla^2 + \frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$$

we can also prove that

$$r\nabla^2 - \nabla(1 + r\frac{\partial}{\partial r}) = \frac{i}{\hbar}\nabla\times\mathbf{L}$$

((Note))

$$\begin{aligned}
\nabla^2 &= -\frac{1}{\hbar^2 r^2}\mathbf{L}^2 + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) \\
&= -\frac{1}{\hbar^2 r^2}\mathbf{L}^2 + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) \\
&= -\frac{1}{\hbar^2 r^2}\mathbf{L}^2 + \frac{1}{r}\frac{\partial^2}{\partial r^2}(r)
\end{aligned}$$

1.3.5 Mathematica

Arfken 2-5-13

Show that

$$-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)=-i\ \hbar\frac{\partial}{\partial\phi}$$

This is the quantum mchanical operator corresponding to the z-componenet of orbital angular momentum.

Arfken 2-5-14

With the quantum mechanical orbial angular momentum operator defined as $\mathbf{L}=\mathbf{r}\times\mathbf{p}=\mathbf{r}\times(-i\hbar\boldsymbol{\nabla})$, show that

(a) $$L_x+i\ L_y=-\hbar\,e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)$$

(b) $$L_x+i\ L_y=-\hbar\,e^{-i\phi}\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\phi}\right)$$

Arfken 2-5-15

Verify that $\boldsymbol{L}\times\boldsymbol{L}=i\,\boldsymbol{L}$ in spherical polar coordinates. $\boldsymbol{L}=-i\hbar(\mathbf{r}\times\boldsymbol{\nabla})$, the quantum mechanical orbital angular momentm operator

---

Arfken 2-5-16

(a) Show that

$$\boldsymbol{L} = -\,i\,\hbar\,(\mathbf{r}\times\nabla) = i\,\hbar\,(\mathbf{e}\theta\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\phi}\; - \mathbf{e}\phi\,\frac{\partial}{\partial\theta})$$

(b) Resolving $\mathbf{e}\theta$ and $\mathbf{e}\phi$ into Cartesiancomponents, determine L_x, L_y, and L_z in terms of θ, ϕ, and their derivatives.

(c) From $L_x{}^2 + L_y{}^2 + L_z{}^2$, show that

$$\frac{L^2}{\hbar^2} = -\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\,(\sin\theta\,\frac{\partial}{\partial\theta}) - \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\theta^2} = -\,r^2\,\nabla^2 + \frac{\partial}{\partial r}\left(r^2\,\frac{\partial}{\partial r}\right)$$

or

$$\boldsymbol{\nabla}^{\mathbf{2}} = -\,\frac{\boldsymbol{L}^2}{\hbar^2\,r^2} + \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial}{\partial r}\right)$$

This identity is useful in relating orbital angular momentum.

Arfken 2-5-17

With $\boldsymbol{L} = -\,i\,\hbar\,(\boldsymbol{r}\times\nabla)$, verify the operator identities

(a) $\nabla = \mathbf{e}\mathrm{r}\,\frac{\partial}{\partial r} - i\,\frac{\boldsymbol{r}\times\boldsymbol{L}}{r^2}$

(b) $\boldsymbol{r}\,\nabla^2 = -\,\nabla\,(1 + \mathrm{r}\,\frac{\partial}{\partial r}) = i\,\nabla\times\boldsymbol{L}$

```
Clear["Gobal`"]

<< "VectorAnalysis`"

SetCoordinates[Spherical[r, θ, ϕ]]

Spherical [r, θ, ϕ]

Clear[ψ]

L = (-ⅈ ℏ Cross[(ur r), Grad[#]]) &

-ⅈ ℏ (ur r) × Grad [#1] &

Lx := (ux.(-ⅈ ℏ Cross[(ur r), Grad[#]]) &) // Simplify

Ly := (uy.(-ⅈ ℏ Cross[(ur r), Grad[#]]) &) // Simplify

Lz := (uz.(-ⅈ ℏ Cross[(ur r), Grad[#]]) &) // Simplify

Lx[ψ[r, θ, ϕ]] // Simplify
```

$i\hbar\left(\text{Cos}[\phi]\,\text{Cot}[\theta]\,\psi^{(0,0,1)}[r,\theta,\phi]+\text{Sin}[\phi]\,\psi^{(0,1,0)}[r,\theta,\phi]\right)$

```
Ly[ψ[r, θ, ϕ]] // Simplify
```

$i\hbar\left(\text{Cot}[\theta]\,\text{Sin}[\phi]\,\psi^{(0,0,1)}[r,\theta,\phi]-\text{Cos}[\phi]\,\psi^{(0,1,0)}[r,\theta,\phi]\right)$

Arfken Problem 2-5-13

```
Lz[ψ[r, θ, ϕ]] // Simplify
```

$-i\hbar\,\psi^{(0,0,1)}[r, \theta, \phi]$

Arfken Problem 2-5-14

```
Lx [ψ[r, θ, ϕ]] + i Ly [ψ[r, θ, ϕ]] // FullSimplify
```

$e^{i\phi}\,\hbar\,\left(i\,\mathrm{Cot}[\theta]\,\psi^{(0,0,1)}[r, \theta, \phi] + \psi^{(0,1,0)}[r, \theta, \phi]\right)$

```
Lx [ψ[r, θ, ϕ]] - i Ly [ψ[r, θ, ϕ]] // FullSimplify
```

$\hbar\,(i\,\mathrm{Cos}[\phi] + \mathrm{Sin}[\phi])\,\left(\mathrm{Cot}[\theta]\,\psi^{(0,0,1)}[r, \theta, \phi] + i\,\psi^{(0,1,0)}[r, \theta, \phi]\right)$

Arfken Problem 2-5-15

```
Lx [Ly[ψ[r, θ, ϕ]]] - Ly [Lx[ψ[r, θ, ϕ]]] - i ħ Lz[ψ[r, θ, ϕ]] //
  Expand // FullSimplify
```

0

```
Ly [Lz[ψ[r, θ, ϕ]]] - Lz [Ly[ψ[r, θ, ϕ]]] - i ħ Lx[ψ[r, θ, ϕ]] //
  Expand // FullSimplify
```

0

```
Lz [Lx[ψ[r, θ, ϕ]]] - Lx [Lz[ψ[r, θ, ϕ]]] - i ħ Ly[ψ[r, θ, ϕ]] //
  Expand // FullSimplify
```

0

Arfken Problem 2-5-16 (a)

```
L[ψ[r, θ, ϕ]] // Simplify
```

$\{0,\ i\hbar\,\mathrm{Csc}[\theta]\,\psi^{(0,0,1)}[r,\theta,\phi],\ -i\hbar\,\psi^{(0,1,0)}[r,\theta,\phi]\}$

Arfken Problem 2-5-16 (b)

```
Lx[ψ[r, θ, ϕ]] // Simplify
```

$i\hbar\left(\mathrm{Cos}[\phi]\,\mathrm{Cot}[\theta]\,\psi^{(0,0,1)}[r,\theta,\phi]+\mathrm{Sin}[\phi]\,\psi^{(0,1,0)}[r,\theta,\phi]\right)$

```
Ly[ψ[r, θ, ϕ]] // Simplify
```

$i\hbar\left(\mathrm{Cot}[\theta]\,\mathrm{Sin}[\phi]\,\psi^{(0,0,1)}[r,\theta,\phi]-\mathrm{Cos}[\phi]\,\psi^{(0,1,0)}[r,\theta,\phi]\right)$

```
Lz[ψ[r, θ, ϕ]] // Simplify
```

$-i\hbar\,\psi^{(0,0,1)}[r,\theta,\phi]$

Arfken Problem 2-5-16 (c)

```
seq1 = Lx[Lx[ψ[r, θ, ϕ]]] // FullSimplify
```

$$\frac{1}{4}\hbar^2\Big((3+\mathrm{Cos}[2\theta])\,\mathrm{Csc}[\theta]^2\,\mathrm{Sin}[2\phi]\,\psi^{(0,0,1)}[r,\theta,\phi]-4\,\mathrm{Cot}[\theta]\left(\mathrm{Cos}[\phi]^2\left(\mathrm{Cot}[\theta]\,\psi^{(0,0,2)}[r,\theta,\phi]+\psi^{(0,1,0)}[r,\theta,\phi]\right)+\mathrm{Sin}[2\phi]\,\psi^{(1,1,0)}[r,\theta,\phi]\right)-4\,\mathrm{Sin}[\phi]^2\,\psi^{(0,2,0)}[r,\theta,\phi]\Big)$$

```
seq2 = Ly[Ly[ψ[r, θ, ϕ]]] // FullSimplify
```

$$-\frac{1}{4}\hbar^2\Big((3+\mathrm{Cos}[2\theta])\,\mathrm{Csc}[\theta]^2\,\mathrm{Sin}[2\phi]\,\psi^{(0,0,1)}[r,\theta,\phi]+4\,\mathrm{Cot}[\theta]\,\mathrm{Sin}[\phi]\left(\mathrm{Sin}[\phi]\left(\mathrm{Cot}[\theta]\,\psi^{(0,0,2)}[r,\theta,\phi]+\psi^{(0,1,0)}[r,\theta,\phi]\right)-2\,\mathrm{Cos}[\phi]\,\psi^{(0,1,1)}[r,\theta,\phi]\right)+4\,\mathrm{Cos}[\phi]^2\,\psi^{(0,2,0)}[r,\theta,\phi]\Big)$$

```
seq3 = Lz[Lz[ψ[r, θ, ϕ]]] // Simplify
```

$-\hbar^2\,\psi^{(0,0,2)}[r,\theta,\phi]$

```
seq123 = seq1 + seq2 + seq3 // Expand // FullSimplify
```

$$-\hbar^2\left(\mathrm{Csc}[\theta]^2\,\psi^{(0,0,2)}[r,\theta,\phi]+\mathrm{Cot}[\theta]\,\psi^{(0,1,0)}[r,\theta,\phi]+\psi^{(0,2,0)}[r,\theta,\phi]\right)$$

```
seq4 = -ℏ^2 r^2 Laplacian[ψ[r, θ, ϕ]] + ℏ^2 D[r^2 D[ψ[r, θ, ϕ], r], r] //
  Simplify
```

$$-\hbar^2\left(\mathrm{Csc}[\theta]^2\,\psi^{(0,0,2)}[r,\theta,\phi]+\mathrm{Cot}[\theta]\,\psi^{(0,1,0)}[r,\theta,\phi]+\psi^{(0,2,0)}[r,\theta,\phi]\right)$$

```
seq123 - seq4 // Simplify
```

0

Arfken Problem 2-5-17(a)

```
Grad[ψ[r, θ, ϕ]] + (ⅈ/ℏ) (1/r^2) Cross[{r, 0, 0}, L[ψ[r, θ, ϕ]]] //
 Simplify
```

```
{ψ^(1,0,0)[r, θ, ϕ], 0, 0}
```

Arfken Problem 2-5-17(b)

```
(ⅈ/ℏ) Curl[L[ψ[r, θ, ϕ]]] - {r, 0, 0} Laplacian[ψ[r, θ, ϕ]] +
   Grad[ψ[r, θ, ϕ] + r ∂r ψ[r, θ, ϕ]] // Expand // FullSimplify
```

```
{0, 0, 0}
```

1.3.6 Radial momentum operator in the quantum mechanics

(a) In classical mechanics, the radial momentum of the radius r is defined by

$$p_{rc} = \frac{1}{r}(\mathbf{r} \cdot \mathbf{p})$$

(b) In quantum mechanics, this definition becomes ambiguous since the component of p and r do not commute. Since pr should be Hermitian operator, we need to define as the radial momentum of the radius r is defined by

$$p_{rq} = \frac{1}{2}(\frac{\mathbf{r}}{r} \cdot \mathbf{p} + \mathbf{p} \cdot \frac{\mathbf{r}}{r})$$

This symmetric expression is indeed the canonical conjugate of r.

$$p_{rq} r - r p_{rq} = \frac{\hbar}{i}$$

Note that

$$p_{rq} = (-i\hbar)(\frac{\partial}{\partial r} + \frac{1}{r}) = (-i\hbar)\frac{1}{r}\frac{\partial}{\partial r} r$$

((**Mathematica**))

```
Clear["Gobal`"]

<< "VectorAnalysis`"

SetCoordinates[Spherical[r, θ, ϕ]]
```

Spherical [r, θ, ϕ]

```
Clear[ψ]

prc := (-ⅈ ℏ {1, 0, 0}.Grad[#] &)

prc[ψ[r, θ, ϕ]]
```

$-\mathrm{i}\hbar\,\psi^{(1,0,0)}[r, \theta, \phi]$

$$\text{prq} := \left(\frac{-\mathrm{i}\hbar}{2}\{1, 0, 0\}.\text{Grad}[\#] + \frac{-\mathrm{i}\hbar}{2}\text{Div}[\#\{1, 0, 0\}]\ \&\right)$$

```
prq[ψ[r, θ, ϕ]] // Simplify
```

$$-\frac{\mathrm{i}\hbar\,(\psi[r, \theta, \phi] + r\,\psi^{(1,0,0)}[r, \theta, \phi])}{r}$$

((Commutation relation))

```
prq[r ψ[r, θ, ϕ]] - r prq[ψ[r, θ, ϕ]] // Simplify
```

$-\mathrm{i}\hbar\,\psi[r, \theta, \phi]$

■

Arfken 2-5-18

Show that the following three forms (spherical coordinates) of $\nabla^2\psi(r)$ are equivalent:

(a) $\frac{1}{r^2}\frac{d}{dr}[r^2\frac{d\psi(r)}{dr}]$; (b) $\frac{1}{r}\frac{d^2}{dr^2}[r\,\psi(r)]$; (c) $\frac{d^2\psi(r)}{dr^2}+\frac{2}{r}\frac{d\psi(r)}{dr}$

```
Clear["Gobal`"]
```

```
<< "VectorAnalysis`"
```

```
SetCoordinates[Spherical[r, θ, ϕ]]
```

```
Spherical [r, θ, ϕ]
```

```
Clear[ψ]
```

$$kr := -i\frac{1}{r}D[r\,\#, r]\ \&$$

```
-kr[kr[ψ[r]]] // Expand
```

$$\frac{2\psi'[r]}{r}+\psi''[r]$$

$$\frac{1}{r}D[r\,\psi[r], \{r, 2\}]\ //\ Simplify$$

$$\frac{2\psi'[r]}{r}+\psi''[r]$$

$$\frac{1}{r^2}D\left[r^2 D[\psi[r], r], r\right]\ //\ Simplify$$

$$\frac{2\psi'[r]}{r}+\psi''[r]$$

1.3.7 Cylindrical coordinates

The position of a point in space P having Cartesian coordinates x, y, and z may be expressed in terms of cylindrical co-ordinates

$$x=\rho\cos\phi,\qquad y=\rho\sin\phi,\qquad z=z.$$

The position vector r is written as

$$\mathbf{r} = \rho\cos\phi\mathbf{e}_x + \rho\sin\phi\mathbf{e}_y + z\mathbf{e}_z$$

$$d\mathbf{r} = \sum_{j=1}^{3}\mathbf{e}_j h_j dq_j = \mathbf{e}_\rho d\rho + \mathbf{e}_\phi \rho d\phi + \mathbf{e}_z dz$$

where

$$h_1 = h_\rho = 1$$
$$h_2 = h_\phi = \rho$$
$$h_3 = h_z = 1$$

The unit vectors are written as

$$\mathbf{e}_\rho = \frac{1}{h_\rho}\frac{\partial \mathbf{r}}{\partial \rho} = \frac{\partial \mathbf{r}}{\partial \rho} = \cos\phi\mathbf{e}_x + \sin\phi\mathbf{e}_y$$
$$\mathbf{e}_\phi = \frac{1}{h_\phi}\frac{\partial \mathbf{r}}{\partial \phi} = \frac{1}{\rho}\frac{\partial \mathbf{r}}{\partial \phi} = -\sin\phi\mathbf{e}_x + \cos\phi\mathbf{e}_y$$
$$\mathbf{e}_z = \frac{1}{h_z}\frac{\partial \mathbf{r}}{\partial z} = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{e}_z$$

We note that

$$\frac{\partial \mathbf{e}_\rho}{\partial \phi} = \mathbf{e}_\phi, \qquad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_\rho,$$

$$\frac{\partial \mathbf{e}_\rho}{\partial \rho} = 0, \qquad \frac{\partial \mathbf{e}_\phi}{\partial \rho} = 0.$$

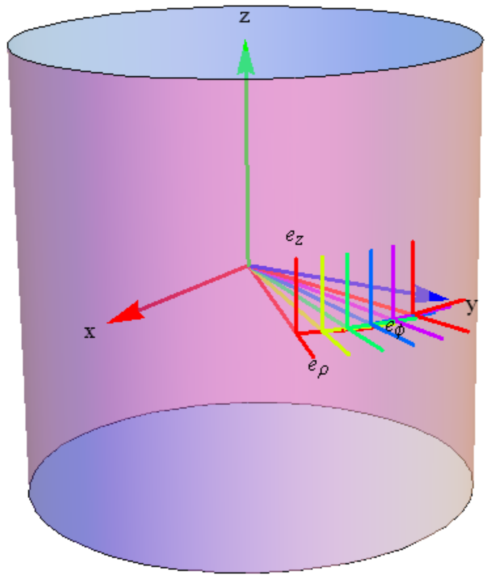
$$\dot{\mathbf{e}}_\rho = \dot{\phi}\mathbf{e}_\phi, \qquad \dot{\mathbf{e}}_\phi = -\dot{\phi}\mathbf{e}_\rho, \qquad \dot{\mathbf{e}}_z = 0 \qquad \text{(time derivative)}$$

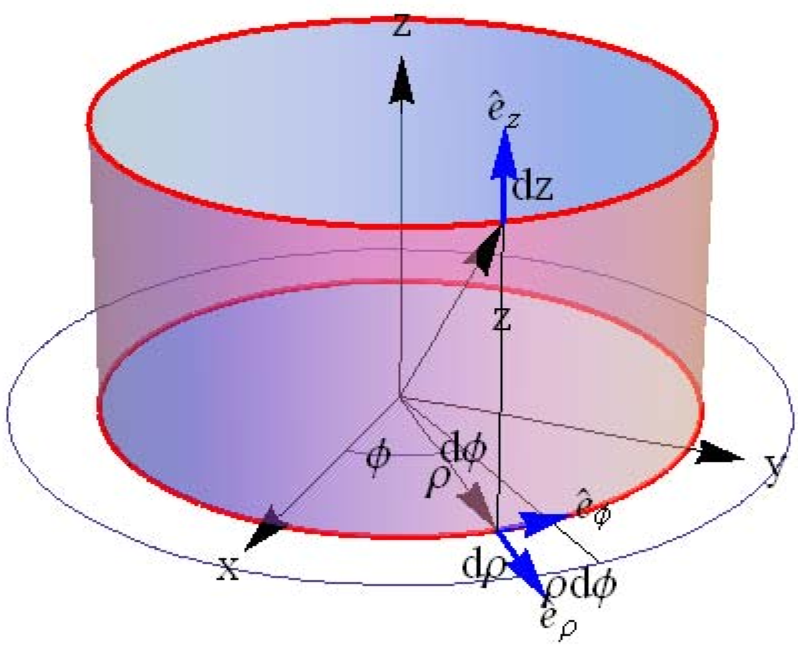
The above expression can be described using a matrix A as

$$\begin{pmatrix}\mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z\end{pmatrix} = \mathbf{A}\begin{pmatrix}\mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z\end{pmatrix}.$$

or by using the inverse matrix A^{-1} as

$$\begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix} = \mathbf{A}^T \begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix}$$





1.3.8 Differential operations in the cylindrical coordinate

The differential operations involving ∇ are as follows.

$$\nabla\psi = \mathbf{e}_\rho \frac{\partial\psi}{\partial\rho} + \mathbf{e}_\phi \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}V_\phi + \frac{\partial}{\partial z}V_z$$

$$\nabla\times\mathbf{V} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\mathbf{e}_\phi & \mathbf{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\phi & V_z \end{vmatrix}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\psi}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

where $\boldsymbol{V}$ is a vector and ψ is a scalar.

1.3.9 Mathematica

We use the cylindrical co-ordinate.
We need a Vector Analysis Package. We also need SetCordinatinates.In this system the vector is expressed in terms of (Aρ, Aϕ, Az)

```
Clear["Gobal`"]

Needs["VectorAnalysis`"]

SetCoordinates[Cylindrical[ρ, ϕ, z]]

Cylindrical[ρ, ϕ, z]
```

Vector analysis
Grad, Curl, Laplacian which are expressed in terms of the cylindrical coordinates

```
eq1 = Laplacian[ψ[ρ, ϕ, z]] // Simplify
```

$$\psi^{(0,0,2)}[\rho, \phi, z] + \frac{\psi^{(0,2,0)}[\rho, \phi, z]}{\rho^2} + \frac{\psi^{(1,0,0)}[\rho, \phi, z]}{\rho} + \psi^{(2,0,0)}[\rho, \phi, z]$$

```
eq2 = Grad[ψ[ρ, ϕ, z]]
```

$$\left\{\psi^{(1,0,0)}[\rho, \phi, z], \frac{\psi^{(0,1,0)}[\rho, \phi, z]}{\rho}, \psi^{(0,0,1)}[\rho, \phi, z]\right\}$$

```
B = {Bρ[ρ, ϕ, z], Bϕ[ρ, ϕ, z], Bz[ρ, ϕ, z]}
```

{Bρ[ρ, ϕ, z], Bϕ[ρ, ϕ, z], Bz[ρ, ϕ, z]}

```
eq3 = Curl[B]
```

$$\left\{\frac{-\rho\, B\phi^{(0,0,1)}[\rho, \phi, z] + Bz^{(0,1,0)}[\rho, \phi, z]}{\rho}, B\rho^{(0,0,1)}[\rho, \phi, z] - Bz^{(1,0,0)}[\rho, \phi, z], \frac{B\phi[\rho, \phi, z] - B\rho^{(0,1,0)}[\rho, \phi, z] + \rho\, B\phi^{(1,0,0)}[\rho, \phi, z]}{\rho}\right\}$$

```
eq3 = Div[B]
```

$$\frac{1}{\rho}\left(B\rho[\rho, \phi, z] + \rho\, Bz^{(0,0,1)}[\rho, \phi, z] + B\phi^{(0,1,0)}[\rho, \phi, z] + \rho\, B\rho^{(1,0,0)}[\rho, \phi, z]\right)$$

1.3.10 Velocity and acceleration in the cylindrical coordinates

The velocity (***v***) and acceleration (***a***) in the cylindrical co-ordinates are given by

$$v_\rho = \dot{\rho} \qquad a_\rho = \ddot{\rho} - \rho\dot{\phi}^2$$

$$v_\phi = \rho\dot{\phi} \qquad a_\phi = \rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}$$

$$v_z = \dot{z} \qquad a_z = \ddot{z}$$

((**Mathematica**))

We drive the above formula using the Mathematica.

Velocity and acceleration in the cylindrical coordinates

```
Clear["Gobal`"]

<< "VectorAnalysis`"

SetCoordinates[Cartesian[x, y, z]]

Cartesian[x, y, z]

RR[t_] := {ρ[t] Cos[ϕ[t]], ρ[t] Sin[ϕ[t]], z[t]}

D[RR[t], t] // FullSimplify

{Cos[ϕ[t]] ρ′[t] - Sin[ϕ[t]] ρ[t] ϕ′[t],
 Sin[ϕ[t]] ρ′[t] + Cos[ϕ[t]] ρ[t] ϕ′[t], z′[t]}

D[RR[t], {t, 2}] // FullSimplify

{Cos[ϕ[t]] (-ρ[t] ϕ′[t]^2 + ρ″[t]) - Sin[ϕ[t]] (2 ρ′[t] ϕ′[t] + ρ[t] ϕ″[t]),
 2 Cos[ϕ[t]] ρ′[t] ϕ′[t] +
  Sin[ϕ[t]] (-ρ[t] ϕ′[t]^2 + ρ″[t]) + Cos[ϕ[t]] ρ[t] ϕ″[t], z″[t]}

D[RR[t], {t, 3}] // FullSimplify

{Cos[ϕ[t]] (-3 ϕ′[t] (ρ′[t] ϕ′[t] + ρ[t] ϕ″[t]) + ρ^(3)[t]) +
  Sin[ϕ[t]] (-3 (ϕ′[t] ρ″[t] + ρ′[t] ϕ″[t]) + ρ[t] (ϕ′[t]^3 - ϕ^(3)[t])),
 Sin[ϕ[t]] (-3 ϕ′[t] (ρ′[t] ϕ′[t] + ρ[t] ϕ″[t]) + ρ^(3)[t]) +
  Cos[ϕ[t]] (3 ϕ′[t] ρ″[t] + 3 ρ′[t] ϕ″[t] + ρ[t] (-ϕ′[t]^3 + ϕ^(3)[t])), z^(3)[t]}
```

Unit vectors along the ρ, ϕ, and z directions

```
uρ = ∂_ρ[t] RR[t] // Simplify

{Cos[ϕ[t]], Sin[ϕ[t]], 0}

uϕ = ∂_ϕ[t] RR[t] / ρ[t] // Simplify

{-Sin[ϕ[t]], Cos[ϕ[t]], 0}

uz = ∂_z[t] RR[t] // Simplify

{0, 0, 1}
```

■ Velocity and kinetic energy in the cylindrical coordinates

```
Vρ = D[RR[t], t].uρ // Simplify
```
$\rho'[t]$

```
Vϕ = D[RR[t], t].uϕ // Simplify
```
$\rho[t]\,\phi'[t]$

```
Vz = D[RR[t], t].uz // Simplify
```
$z'[t]$

$$K1 = \frac{m}{2}\left(V\rho^2 + V\phi^2 + Vz^2\right) \;//\; \text{Simplify}$$

$$\frac{1}{2}\,m\left(z'[t]^2 + \rho'[t]^2 + \rho[t]^2\,\phi'[t]^2\right)$$

■ Acceleration in the spherical coordinate

```
Aρ = D[RR[t], {t, 2}].uρ // Simplify
```
$-\rho[t]\,\phi'[t]^2 + \rho''[t]$

```
Aϕ = D[RR[t], {t, 2}].uϕ // Simplify
```
$2\,\rho'[t]\,\phi'[t] + \rho[t]\,\phi''[t]$

```
Az = D[RR[t], {t, 2}].uz // Simplify
```
$z''[t]$

■ Some application

```
Sρ = D[RR[t], {t, 3}].uρ // Simplify
```
$-3\,\rho'[t]\,\phi'[t]^2 - 3\,\rho[t]\,\phi'[t]\,\phi''[t] + \rho^{(3)}[t]$

```
Sϕ = D[RR[t], {t, 3}].uϕ // Simplify
```
$3\,(\phi'[t]\,\rho''[t] + \rho'[t]\,\phi''[t]) + \rho[t]\left(-\phi'[t]^3 + \phi^{(3)}[t]\right)$

```
Sz = D[RR[t], {t, 3}].uz // Simplify
```
$z^{(3)}[t]$

1.3.11 Jacobian

$$dV = dxdydz = \frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)}dq_1dq_2dq_3 = h_1h_2h_3dq_1dq_2dq_3$$

Jacobian determinant is defined as;

$$\frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)} = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{vmatrix}$$

(a) Spherical coordinate

$$h_1h_2h_3dq_1dq_2dq_3 = h_rh_\theta h_\phi drd\theta d\phi = r^2\sin\theta drd\theta d\phi$$

(b) Cylindrical co-ordinate

$$h_1h_2h_3dq_1dq_2dq_3 = h_\rho h_\phi h_z d\rho d\phi dz = \rho d\rho d\phi dz$$

((**Mathematica**))
This is the program to determine the Jacobian determinant.

JacobianDeterminant[pt, coordsys]:

to give the determinant of the Jacobian matrix of the transformation from the coordinate system coordinate system to the Cartesian coordinate system at the point pt.

```
Clear["Gobal`"]
```

```
<< "VectorAnalysis`"
```

Jacobian determinant for transformation from cylindrical to Cartesian coordinates:

```
jdet = JacobianDeterminant[{ρ, ϕ, z}, Cylindrical]
```

ρ

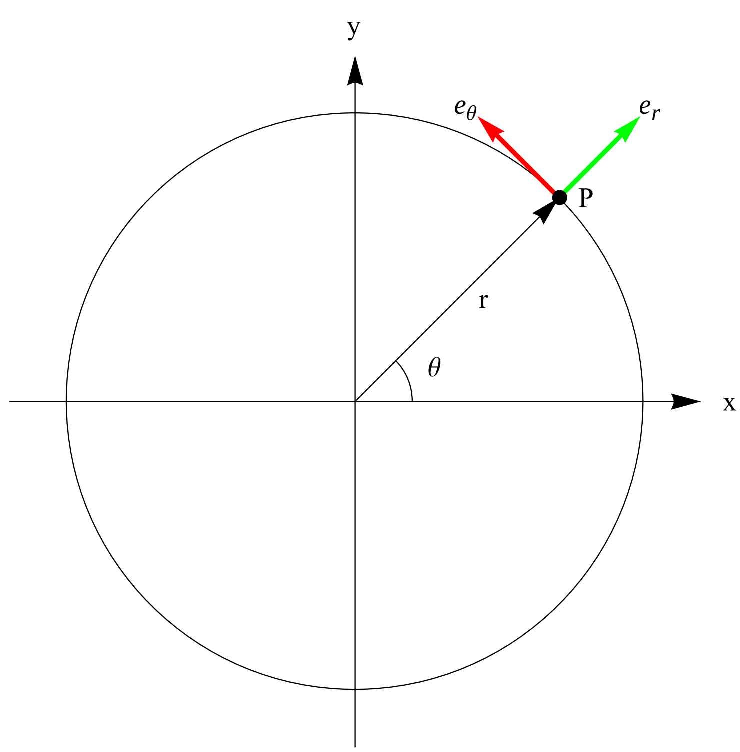
Jacobian determinant for transformation from cylindrical to Spherical coordinates:

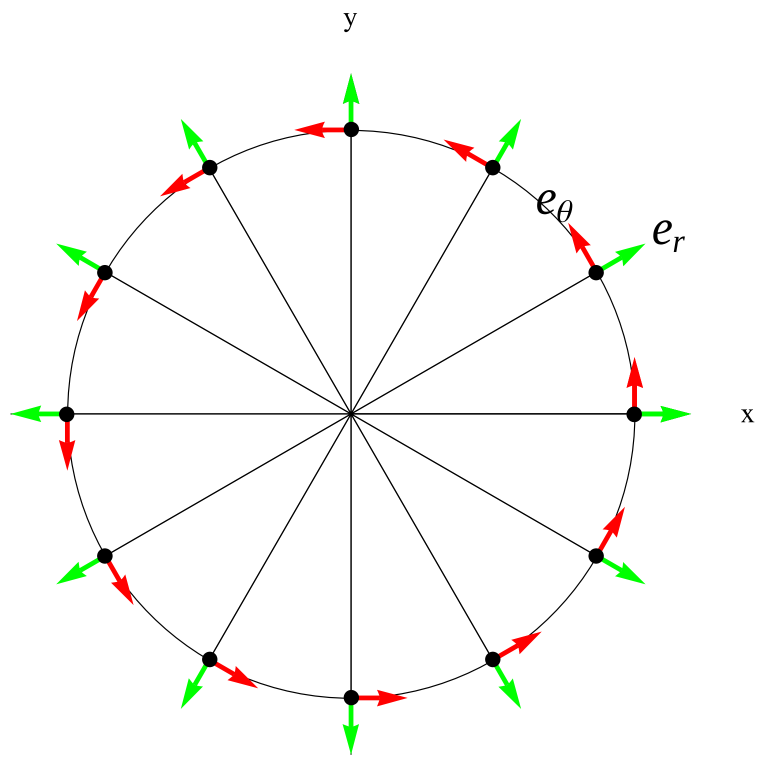
```
jdet = JacobianDeterminant[{r, θ, ϕ}, Spherical]
```

r^2 Sin[θ]

1.3.12 Plane polar coordinate for 2D system

The point P is located at (r, θ), where r is the distance from the origin and θ is the measured counterclockwise from the reference line (the x axis).





We introduce the unit vectors given by

$$\mathbf{e}_r = (\cos\theta, \sin\theta) = \cos\theta\,\mathbf{e}_x + \sin\theta\,\mathbf{e}_y$$

$$\begin{aligned}\mathbf{e}_\theta &= (\cos(\theta+\frac{\pi}{2}), \sin(\theta+\frac{\pi}{2}) = (-\sin\theta, \cos\theta)\,. \\ &= -\sin\theta\,\mathbf{e}_x + \cos\theta\,\mathbf{e}_y\end{aligned}$$

These expressions can be rewritten using a matrix A as

$$\begin{pmatrix}\mathbf{e}_r \\ \mathbf{e}_\theta\end{pmatrix} = A\begin{pmatrix}\mathbf{e}_x \\ \mathbf{e}_y\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathbf{e}_x \\ \mathbf{e}_y\end{pmatrix},$$

and using A^{-1} as

$$\begin{pmatrix}\mathbf{e}_x \\ \mathbf{e}_y\end{pmatrix} = A^{-1}\begin{pmatrix}\mathbf{e}_r \\ \mathbf{e}_\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathbf{e}_r \\ \mathbf{e}_\theta\end{pmatrix}.$$

Note that

$$\frac{\partial\mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta \qquad \frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r$$

The position vector (displacement vector) is given by

$$\mathbf{r} = (r\cos\theta, r\sin\theta) = r\cos\theta\mathbf{e}_x + r\sin\theta\mathbf{e}_y$$

The velocity and acceleration are

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta$$

or

$$v_r = \mathbf{v}\cdot\hat{r} = \dot{r} \qquad a_r = \mathbf{a}\cdot\hat{r} = \ddot{r} - r\dot{\theta}^2$$
$$v_\theta = \mathbf{v}\cdot\hat{\theta} = r\dot{\theta} \qquad a_\theta = \mathbf{a}\cdot\hat{\theta} = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

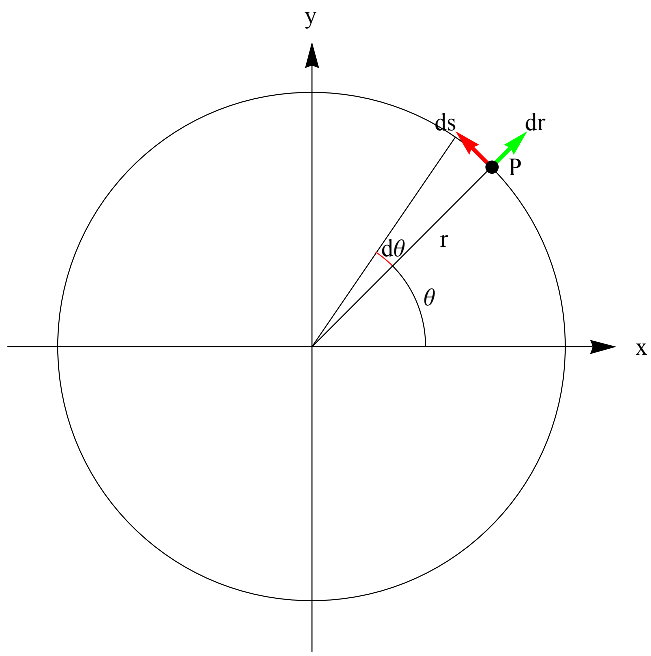
((Note))

Velocity along the $\mathbf{e}_\theta$ direction

$$ds = r d\theta$$
$$v_\theta = \frac{ds}{dt} = r\dot{\theta}$$

Velocity along the $\mathbf{e}_r$ direction

$$v_r = \frac{dr}{dt} = \dot{r}\ .$$



((Note))

$$\mathbf{r} = (r\cos\theta, r\sin\theta)$$

$$\mathbf{v} = \dot{\mathbf{r}} = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta, \dot{r}\sin\theta + r\dot{\theta}\cos\theta) = \dot{r}(\cos\theta, \sin\theta) + r\dot{\theta}(-\sin\theta, \cos\theta) = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = (\ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta, \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2 \mathrm{si}$$

or

$$\ddot{\mathbf{r}} = (\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta, \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)$$

or

$$\ddot{\mathbf{r}} = (\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta)\hat{i} + (\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)\hat{j}$$

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)(\cos\theta\hat{i} + \sin\theta\hat{j}) + (2\dot{r}\dot{\theta} + r\ddot{\theta})(-\sin\theta\hat{i} + \cos\theta\hat{j}) = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

((**Mathematica**))

```
R = {r[t] Cos[θ[t]], r[t] Sin[θ[t]]}
```

```
{Cos[θ[t]] r[t], r[t] Sin[θ[t]]}
```

```
V = D[R, t] // Simplify
```

```
{Cos[θ[t]] r'[t] - r[t] Sin[θ[t]] θ'[t],
 Sin[θ[t]] r'[t] + Cos[θ[t]] r[t] θ'[t]}
```

```
A = D[R, {t, 2}] // Simplify
```

```
{-2 Sin[θ[t]] r'[t] θ'[t] + Cos[θ[t]] r''[t] -
  r[t] (Cos[θ[t]] θ'[t]^2 + Sin[θ[t]] θ''[t]), 2 Cos[θ[t]] r'[t] θ'[t] +
  Sin[θ[t]] r''[t] + r[t] (-Sin[θ[t]] θ'[t]^2 + Cos[θ[t]] θ''[t])}
```

```
ru = {Cos[θ[t]], Sin[θ[t]]}
```

```
{Cos[θ[t]], Sin[θ[t]]}
```

```
θu = {-Sin[θ[t]], Cos[θ[t]]}
```

```
{-Sin[θ[t]], Cos[θ[t]]}
```

```
A.ru // Simplify
```

```
-r[t] θ'[t]^2 + r''[t]
```

```
A.θu // Simplify
```

```
2 r'[t] θ'[t] + r[t] θ''[t]
```

```
V.ru // Simplify
```

```
r'[t]
```

```
V.θu // Simplify
```

```
r[t] θ'[t]
```

1.13.13 Angular momentum

The angular momentum is defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m(\mathbf{r} \times \mathbf{v}) = m[r\mathbf{e}_r \times (v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta)] = mrv_\theta(\mathbf{e}_r \times \mathbf{e}_\theta) = mr^2\dot{\theta}(\mathbf{e}_r \times \mathbf{e}_\theta)$$

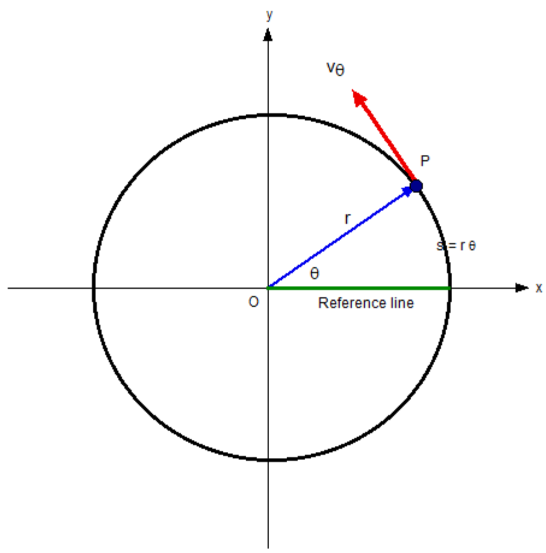
1.13.14 Circular motion (r = constant)

We consider a circular motion with r = constant. since $\dot{r} = 0$ and $\ddot{r} = 0$.

$$a_r = -r\dot{\theta}^2 \qquad v_r = 0$$
$$a_\theta = a_t = r\ddot{\theta} \qquad v_\theta = v_t = r\dot{\theta}$$

In summary, we have

$$v_\theta = v_t = r\dot{\theta} = v$$



$$a_r = -r\dot{\theta}^2 = -\frac{v^2}{r}$$
$$a_\theta = a_t = r\ddot{\theta} = \frac{dv}{dt}$$