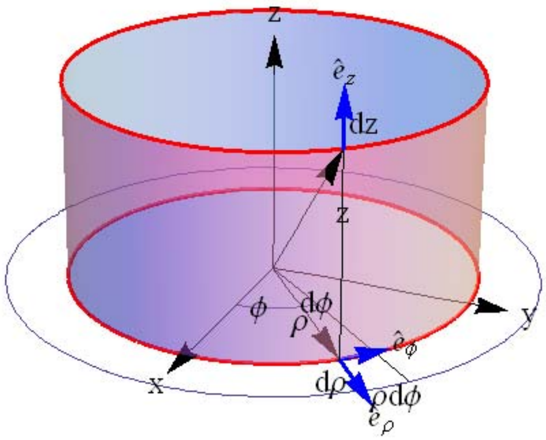
Chapter 20 Green's function in cylindrical coordinate

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: November 6, 2010)

Dirac delta function in the cylindrical coordinate
Green's function in the cylindrical coordinate
Modified Bessel functions



20.1 Dirac delta function

Arfken: 14.3.12

We verify that

$$\delta(\phi_1 - \phi_2) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp[im(\phi_1 - \phi_2)].$$

is a Dirac delta function by showing that it satisfies the definition of a Dirac delta function:

$$\int_{-\pi}^{\pi} f(\phi_1) \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp[im(\phi_1 - \phi_2)] d\phi_1 = f(\phi_2) .$$

Hint. Represent $f(\phi_1)$ by an exponential Fourier series. Note. The most important application of this expression is in the determination of Green's functions.

((**Proof**))
Suppose that

$$f(\phi) = f(\phi + 2\pi)\,.$$

Then the function $f(\phi)$ can be described as

$$f(\phi) = \sum_{n=-\infty}^{\infty} N_n \exp(in\phi)\,.$$

where

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)e^{-im\phi}d\phi = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} N_n \int_{-\pi}^{\pi} d\phi e^{i(n-m)\phi} = \sum_{n=-\infty}^{\infty} N_n \pi\delta_{n,m} = N_m\,.$$

Then we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(\phi_1)\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp[im(\phi_1-\phi_2)]d\phi_1 &= \sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} N_n \exp(in\phi_1)\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp[im(\phi_1-\phi_2)]d\phi_1 \\ &= \sum_{n=-\infty}^{\infty} N_n \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp[-im\phi_2]\int_{-\pi}^{\pi}\exp[i(m+n)\phi_1]d\phi_1\end{aligned}$$

Here we note that

$$\begin{aligned}\int_{-\pi}^{\pi}\exp[i(m+n)\phi_1]d\phi_1 &= \exp[i(m+n)\pi]\int_{0}^{2\pi}\exp[i(m+n)\phi]d\phi \\ &= 2\pi\exp[i(m+n)\pi]\delta_{m+n,0} \\ &= 2\pi\delta_{m+n,0}\end{aligned}$$

Then

$$\begin{aligned}\int_{-\pi}^{\pi} f(\phi_1)\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp[im(\phi_1-\phi_2)]d\phi_1 &= \sum_{n=-\infty}^{\infty} N_n \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp[-im\phi_2]2\pi\delta_{m+n,0} \\ &= \sum_{n=-\infty}^{\infty} N_n \exp[in\phi_2] = f(\phi_2)\end{aligned}$$

20.2 Dirac delta function for the *z* component

$$\delta(z-z')=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\exp[ik(z-z')]=\frac{1}{\pi}\int_{0}^{\infty}dk\cos[k(z-z')]$$

20.3 Dirac delta function in the cylindrical co-ordinate

We define the Dirac delta function as

$$\int\delta(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}'=1$$

Suppose that

$$\delta(\mathbf{r}-\mathbf{r}')=A(\rho')\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z'),$$

Then we have

$$\begin{aligned}\int\delta(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}'&=\int_0^{\infty}\rho'd\rho'\int_0^{2\pi}d\phi'\int_{-\infty}^{\infty}dz'A(\rho')\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')\\&=\int_0^{\infty}A(\rho')\delta(\rho-\rho')\rho'd\rho'=A(\rho)\rho=1\end{aligned}$$

or

$$A(\rho)=\frac{1}{\rho}.$$

In summary,

$$\delta(\mathbf{r}-\mathbf{r}')=\frac{1}{\rho}\delta(r-r')\delta(\phi-\phi')\delta(z-z').$$

20.4 Green's function

We consider the Green's function

$$\begin{aligned}\nabla^2G(r,r')&=-\delta(\mathbf{r}-\mathbf{r}')\\&=-\frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')\end{aligned}$$

In the cylindrical coordinates, the Laplacian is expressed by

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

We expand the Green's function as

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\phi-\phi')}\exp[ik(z-z')]G_m(k,\rho,\rho')$$

The reason for expressing the delta function as an integral over positive *k* is that it will result in Bessel functions of positive argument in the final form. Then we have

$$\begin{aligned}\nabla^2 G(\mathbf{r},\mathbf{r}') &= \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}]e^{im(\phi-\phi')}e^{ik(z-z')}G_m(k,\rho,\rho') \\ &= \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) - \frac{1}{\rho^2}m^2 - k^2]e^{im(\phi-\phi')}e^{ik(z-z')}G_m(k,\rho,\rho') \\ &= \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\phi-\phi')}e^{ik(z-z')}[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) - (\frac{m^2}{\rho^2} + k^2)]G_m(k,\rho,\rho')\end{aligned}$$

and

$$\begin{aligned}\delta(\mathbf{r}-\mathbf{r}') &= \frac{1}{\rho}\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z') \\ &= \frac{1}{4\pi^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dk e^{im(\phi-\phi')}e^{ik(z-z')}\frac{1}{\rho}\delta(\rho-\rho')\end{aligned}$$

Thus we get

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) - (k^2 + \frac{1}{\rho^2}m^2)]G_m(k,\rho,\rho') = -\frac{1}{\rho}\delta(\rho-\rho')$$

or

$$\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) - (k^2\rho + \frac{1}{\rho}m^2)]G_m(k,\rho,\rho') = -\delta(\rho-\rho')$$

(1) The discontinuity of $\partial G_m(k,\rho,\rho')/\partial\rho$ of at $\rho = \rho'$:

$$\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} d\rho[\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho})G_m(k,\rho,\rho') - (k^2\rho + \frac{1}{\rho}m^2)G_m(k,\rho,\rho')] = -\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} d\rho\delta(\rho-\rho') = -1,$$

or

$$\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} d\rho[\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho})G_m(k,\rho,\rho') - (k^2\rho + \frac{1}{\rho}m^2)G_m(k,\rho,\rho')] = -\int_{\rho'-\varepsilon}^{\rho'+\varepsilon} d\rho\delta(\rho-\rho') = -1,$$

or

$$[\rho\frac{\partial}{\partial\rho}G_m(k,\rho,\rho')]_{\rho'-\varepsilon}^{\rho'+\varepsilon} - \int_{\rho'-\varepsilon}^{\rho'+\varepsilon}(k^2\rho + \frac{1}{\rho}m^2)G_m(k,\rho,\rho') = -1,$$

or

$$\frac{\partial G_m(k,\rho,\rho')}{\partial\rho}|_{\rho'+\varepsilon} - \frac{\partial G_m(k,\rho,\rho')}{\partial\rho}|_{\rho'-\varepsilon} = -\frac{1}{\rho'}.$$

(2) Solution of the differential equation

The solution of the differential equation

$$\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) - (k^2\rho + \frac{1}{\rho}m^2)]G_m(k,\rho,\rho') = 0,$$

is the modified Bessel functions;

$$I_m(k\rho), \qquad K_m(k\rho)$$

((Mathematica))

$I_n(x)$	BesselI[*n*, x]
$K_n(x)$	BesselK[*n*, x]

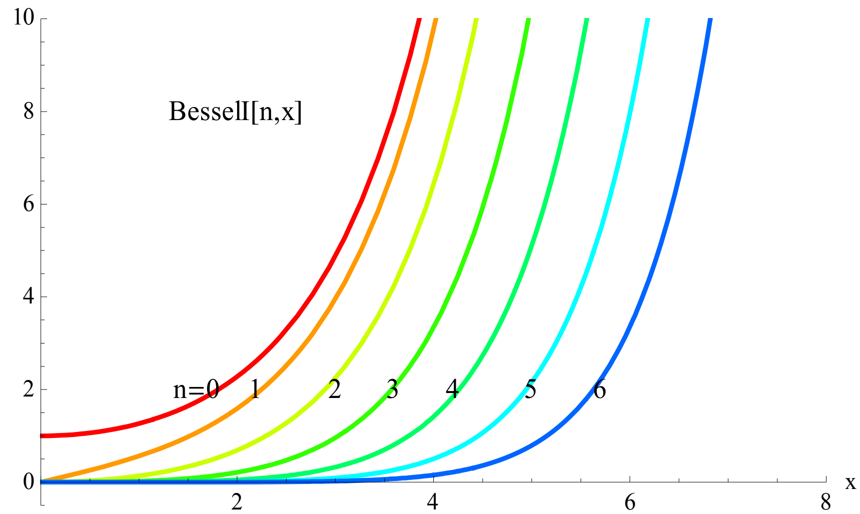


Fig. $I_n(x)$ vs x with n = 0, 1, 2, 3, 4, 5, and 6.

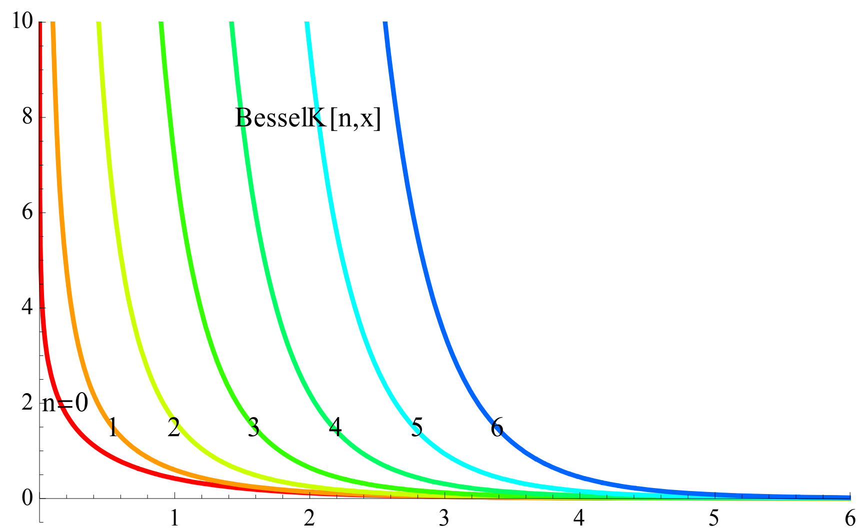


Fig. $K_n(x)$ vs x with n = 0, 1, 2, 3, 4, 5, and 6.

Here we demand that $G_m(k,\rho,\rho')$ be finite at $\rho\rightarrow 0$ and vanishes as $\rho\rightarrow\infty$,

For $\rho<\rho'$,

$$G_m(k,\rho,\rho') = AI_m(k\rho), \qquad \frac{d}{d\rho}G_m(k,\rho,\rho') = AkI_m'(k\rho).$$

For $a>\rho>\rho'$,

$$G_m(k,\rho,\rho') = BK_m(k\rho), \qquad \frac{d}{d\rho}G_m(k,\rho,\rho') = BkK_m'(k\rho).$$

From the condition of continuity of $G_m(k,\rho,\rho')$ at $\rho = \rho'$:

$$AI_m(k\rho') = BK_m(k\rho'), \qquad \frac{A}{K_m(k\rho')} = \frac{B}{I_m(k\rho')} = C$$

or

$$A = CK_m(k\rho')$$

$$B = CI_m(k\rho')$$

where C is constant.

Then the Green's function can be rewritten as

$$G_m^{I}(k,\rho,\rho') = CI_m(k\rho)K_m(k\rho'), \qquad \text{for } \rho<\rho' \text{ (region I)}$$

and

$$G_m^{II}(k,\rho,\rho') = CI_m(k\rho')K_m(k\rho), \qquad \text{for } \rho>\rho' \text{ (region II)}$$

which is symmetric with respect to the exchange between ρ and ρ'.

(iii) The discontinuity of $\partial G_m(k,\rho,\rho')/\partial\rho$ *at* $\rho = \rho'$;

$$\frac{dG_m^{II}(k,\rho,\rho')}{d\rho}\bigg|_{\rho'+\varepsilon} - \frac{dG_m^{I}(k,\rho,\rho')}{d\rho}\bigg|_{\rho'-\varepsilon} = -\frac{1}{\rho'}$$

or

$$kC[I_m(k\rho')K_m'(k\rho') - I_m'(k\rho')K_m(k\rho')] = -\frac{1}{\rho'}$$

Here the Wronskian determinant is given by

$$W = \begin{vmatrix} I_m(k\rho') & K_m(k\rho') \\ I_m'(k\rho') & K_m'(k\rho') \end{vmatrix} = -\frac{1}{k\rho'}$$

((Note)) We can calculate W by using Mathematica.

```
Wronskian[{BesselI[m, ρ], BesselK[m, ρ]}, ρ] // Simplify
```

$$-\frac{1}{\rho}$$

Thus we get

$$C = 1.$$

More generally,

$$G_m(k,\rho,\rho') = I_m(k\rho_<)K_m(k\rho_>)$$

For $\rho<\rho'$,

$$\rho_< = \rho$$
$$\rho_> = \rho'$$

For $\rho>\rho'$,

$$\rho_< = \rho'$$
$$\rho_> = \rho$$

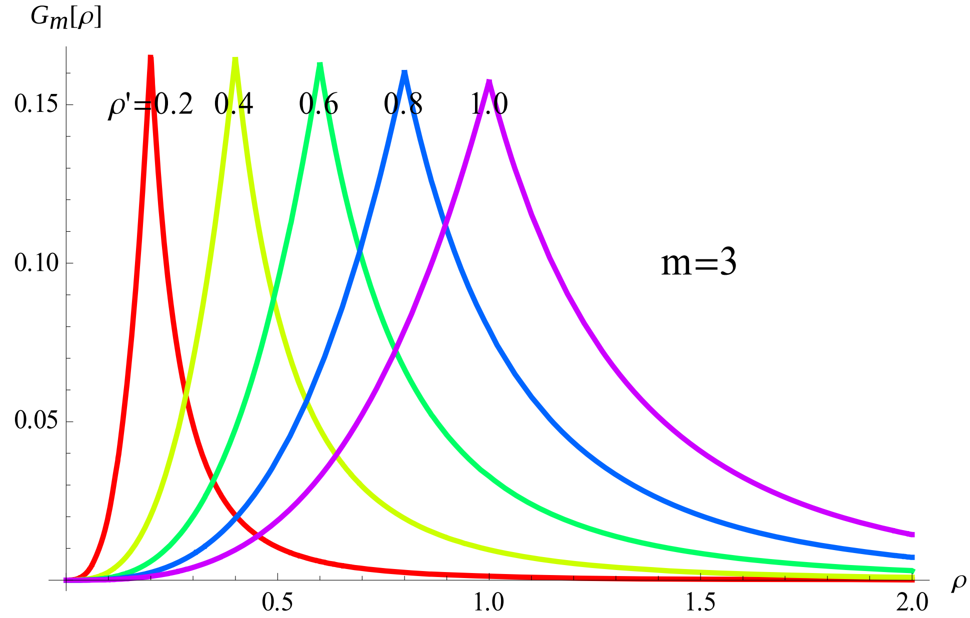


Fig. $G_m(k, \rho, \rho')$ vs ρ with $m = 3$. $k = 1$. ρ' is changed as a parameter. $\rho' = 0.2$, 0.4, 0.6, 0.8, and 1.0.

20.5 Application-1

The Green's function in the cylindrical coordinate is obtained as

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$
$$= \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\phi-\phi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>)$$

or

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk e^{im(\phi-\phi')}\cos[k(z-z')]I_m(k\rho_<)K_m(k\rho_>)$$
$$= \frac{4}{\pi}\int_0^{\infty} dk\cos[k(z-z')]\{\frac{1}{2}I_0(k\rho_<)K_0(k\rho_>) + \sum_{m=1}^{\infty}\cos[m(\phi-\phi')]I_m(k\rho_<)K_m(k\rho_>)\}$$

When $\mathbf{r}'\rightarrow 0$, $\rho_> = \rho$ and $\rho_< = \rho' = 0$.

$$\frac{1}{|\mathbf{r}|} = \frac{1}{\sqrt{\rho^2 + z^2}} = \frac{2}{\pi}\int_0^\infty dk\cos(kz)K_0(k\rho)$$

where

$I_0(0) = 1, \qquad I_m(0) = 0.$

REFERENCES

G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists*, Sixth edition (Elsevier Academic Press, New York, 2005).

J.D. Jackson, *Classical Electrodynamics*, third edition (John Wiley & Sons, Inc., New York, 1999).

J. Schwinger et al, *Classical Elecrodynamics* (Oerseus Bookis, Reading, MA, 1998).

Susan M. Lea *Mathematics for Physicists* (Thomson Brooks/Cole, 2004).