Chapter 21 Green's function: Spherical Bessel function

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Free particle wave function
Spherical Bessel functions
Spherical Neumann function
Spherical Hankel function
Rayleigh formulas
Plane wave expression
Rayleigh's expansion
Bessel-Fourier transform
Green's function for the spherical Bessel functions

21.1 Free Particle Wave function

Free particle wave function ψ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E_k\psi ,$$

where *m* is the mass of particle, E_k (= $\frac{\hbar^2}{2m}k^2$) is the energy of the particle, and *k* is the wave number. This equation can be rewritten as

$$(\nabla^2 + k^2)\psi = 0 .$$

This equation is solved in a formal way as

$$\psi = \varphi_{k\ell m}(r,\theta,\phi) = \langle r\theta\phi | k\ell m\rangle$$

$$\frac{1}{2m}(p_r^{\ 2} + \frac{\mathbf{L}^2}{r^2})\varphi_{k\ell m}(r,\theta,\phi) = E_k\varphi_{k\ell m}(r,\theta,\phi)$$

(separation variables), where ***L*** is the angular momentum:

$$\varphi_{k\ell m}(r,\theta,\phi) = R_{k\ell}(r)Y_{\ell m}(\theta,\phi)$$

with

$$\mathbf{L}^2 Y_{\ell m}(\theta,\phi) = \hbar^2\ell(\ell+1)Y_{\ell m}(\theta,\phi)$$

Since $p_r = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}r$ (see Chapter 1), we have

$$p_r^{\,2} R_{k\ell}(r) = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}r(\frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}r)R_{k\ell}(r) = -\hbar^2\frac{1}{r}\frac{\partial^2}{\partial r^2}[rR_{k\ell}(r)]$$

or

$$-\frac{1}{r}\frac{\partial^2}{\partial r^2}[rR_{k\ell}(r)] + \frac{1}{r^2}\ell(\ell+1)R_{k\ell}(r) = k^2 R_{k\ell}(r)$$

or

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}[rR_{k\ell}(r)] + [k^2 - \frac{1}{r^2}\ell(\ell+1)]R_{k\ell}(r) = 0.$$

with

$$E_k = \frac{\hbar^2 k^2}{2m}.$$

((**Note**))

In the limit of $r \to \infty$, we have

$$\frac{\partial^2}{\partial r^2}[rR_{k\ell}(r)] + k^2[rR_{k\ell}(r)] = 0$$

Then we get

$$R_{kl} = \frac{e^{\pm ikr}}{r} \qquad \text{(outgoing and incoming spherical waves)}$$

We put $x = kr$ (dimensionless)

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} = k\frac{\partial}{\partial x}, \qquad \frac{\partial^2}{\partial r^2} = k\frac{\partial}{\partial x}(k\frac{\partial}{\partial x}) = k^2\frac{\partial^2}{\partial x^2}$$

$$[-\frac{k}{x}k^2\frac{\partial^2}{\partial r^2}(\frac{x}{k}) + \frac{k^2}{x^2}\ell(\ell+1)]R = k^2 R$$

or

$$\frac{1}{x}\frac{d^2}{dx^2}(xR)+[1-\frac{1}{x^2}\ell(\ell+1)]R=0$$ (Spherical Bessel equation).

or

$$\frac{1}{x}[xR''+2R']+[1-\frac{1}{x^2}\ell(\ell+1)]R=0$$

or

$$R''+\frac{2}{x}R'+[1-\frac{\ell(\ell+1)}{x^2}]R=0$$

or

$$\frac{d}{dx}(x^2R')+[x^2-\ell(\ell+1)]R=0.$$

This is a Sturm-Liouville-type differential equation.
Here we suppose that

$$R=\frac{J(x)}{\sqrt{x}},$$

$$\frac{d^2J}{dx^2}+\frac{1}{x}\frac{dJ}{dx}+[1-\frac{(\ell+\frac{1}{2})^2}{x^2}]J=0.$$

The solution of this differential equation is

$$J(x)=J_{\ell+1/2}(x), \qquad \text{or} \qquad J(x)=N_{\ell+1/2}(x).$$

Then the solutions of R are obtained as the spherical Bessel functions defined by

$$j_\ell(x)=\sqrt{\frac{\pi}{2x}}J_{\ell+1/2}(x),$$

and spherical Neumann function defined by

$$n_\ell(x)=\sqrt{\frac{\pi}{2x}}N_{\ell+1/2}(x).$$

Since the spherical Neumann function diverges at x =0, it cannot be chosen as a solution. Finally we have

$$\varphi_{k\ell m}(r,\theta,\phi) = \langle r,\theta,\phi | k,l,m \rangle = \sqrt{\frac{2k^2}{\pi}} j_\ell(kr) Y_{\ell m}(\theta,\phi),$$

with

$$E_k = \frac{\hbar^2 k^2}{2m},$$

and

$$\langle k' l' m' | klm \rangle = \delta(k-k')\delta_{l,l'}\delta_{m,m'}.$$

21.2 Expression of the spherical Bessel function

For $\ell = 0$ (s-state): $m = 0$

$$\varphi = \varphi_{k00}(r,\theta,\phi) = \sqrt{\frac{2k^2}{\pi}} j_0(kr) Y_{00}(\theta,\phi)$$

where

$$Y_{00}(\theta,\phi) = \sqrt{\frac{1}{4\pi}}$$

$$j_0(kr) = \frac{\sin(kr)}{kr} = \frac{e^{ikr} - e^{-ikr}}{2ikr} = \frac{1}{2i}\left(\frac{e^{ikr}}{kr} - \frac{e^{-ikr}}{kr}\right)$$

which is the superposition of outgoing wave and incoming wave.

((**Note**))

L.D. Landau and E.M. Lifshitz, Quantum mechanics (Non-relativistic Theory) (Pergamon Press, Oxford, 1977).

$$\frac{1}{x}\frac{\partial^2}{\partial x^2}(xR) + [1 - \frac{1}{x^2}\ell(\ell+1)]R = 0$$

(a) When $\ell = 0$,

$$\frac{\partial^2}{\partial x^2}(xR) + xR = 0$$

$$R_0 = j_0(x) = \sqrt{\frac{\pi}{2x}} J_{1/2}(x) = \sqrt{\frac{\pi}{2x}} \frac{\sin x}{\sqrt{x}} = \sqrt{\frac{\pi}{2}} \frac{\sin x}{x}$$

(b) For $\ell \neq 0$

$$\frac{d}{dx}(x^2 R_\ell') + [x^2 - \ell(\ell+1)]R_\ell = 0$$

Substitution of $R_l(x) \approx x^\rho$ in the limit of $\rho \to 0$ gives an equation for ρ:

$$(\rho - l)(\rho + l + 1) = 0$$

Thus $R_l(x)$ has the following form.

$$R_l(x) \approx Ax^l + Bx^{-(l+1)}$$

In order for $R_l(x)$ to be finite, we must set $B = 0$. This gives

$R_l(x) \approx x^l$ in the limit of $x \to 0$

When we put

$$R_\ell = x^\ell \chi_\ell(x),$$

we have

$$x\chi_\ell''(x) + 2(1+\ell)\chi_\ell'(x) + x\chi_\ell(x) = 0.$$

or

$$\chi_\ell''(x) + \frac{2(1+\ell)}{x}\chi_\ell'(x) + \chi_\ell(x) = 0$$

If we differentiate this equation with respect to x, we obtain

$$\chi_\ell^{(3)}(x) + \frac{2(1+\ell)}{x}\chi_\ell''(x) + [1 - \frac{2(1+\ell)}{x^2}]\chi_\ell'(x) = 0.$$

By the substitution

$$\chi'_\ell(x) = x\chi_{\ell+1}(x)$$

we have

$$\chi_{\ell+1}''(x)+\frac{2(2+\ell)}{x}\chi_{\ell+1}'(x)+\chi_{\ell+1}(x)=0$$

which is in fact the equation satisfied by $\chi_{\ell+1}(x)$. Thus the successive function $\chi_{\ell+1}(x)$ is related by

$$\chi_{\ell+1}(x)=\frac{\chi'_{\ell}(x)}{x}$$

or

$$\frac{R_{\ell+1}}{x^{\ell+1}}=\frac{1}{x}\frac{d}{dx}\left(\frac{R_{\ell}}{x^{\ell}}\right)$$

Since $R_{\ell}\approx j_{\ell}(x)$

$$j_{\ell}(x)=(-x)^{\ell}\left(\frac{1}{x}\frac{d}{dx}\right)^{\ell}\frac{\sin x}{x}$$

((**Mathematica**))

```
Clear["Global`*"]; OP := (1/x D[#, x] &);

f[x_, ℓ_] := (-x)^ℓ Nest[OP, Sin[x]/x, ℓ];

Table[{n, f[x, n]}, {n, 0, 5}] // Simplify // TableForm
```

0	$\frac{\text{Sin}[x]}{x}$
1	$\frac{-x\,\text{Cos}[x]+\text{Sin}[x]}{x^2}$
2	$-\frac{3\,x\,\text{Cos}[x]+(-3+x^2)\,\text{Sin}[x]}{x^3}$
3	$\frac{x\,(-15+x^2)\,\text{Cos}[x]+3\,(5-2\,x^2)\,\text{Sin}[x]}{x^4}$
4	$\frac{5\,x\,(-21+2\,x^2)\,\text{Cos}[x]+(105-45\,x^2+x^4)\,\text{Sin}[x]}{x^5}$
5	$\frac{-x\,(945-105\,x^2+x^4)\,\text{Cos}[x]+15\,(63-28\,x^2+x^4)\,\text{Sin}[x]}{x^6}$

21.3 Spherical Hankel functions

We define the spherical Hankel functions as

$$h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H^{(1)}_{n+\frac{1}{2}}(x) = j_n(x) + i n_n(x)$$

$$h_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H^{(2)}_{n+\frac{1}{2}}(x) = j_n(x) - i n_n(x)$$

where the spherical Bessel function and spherical Neumann function are given by

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x)$$

$$n_n(x) = \sqrt{\frac{\pi}{2x}} N_{n+\frac{1}{2}}(x) = (-1)^n \sqrt{\frac{\pi}{2x}} J_{-n-\frac{1}{2}}(x)$$

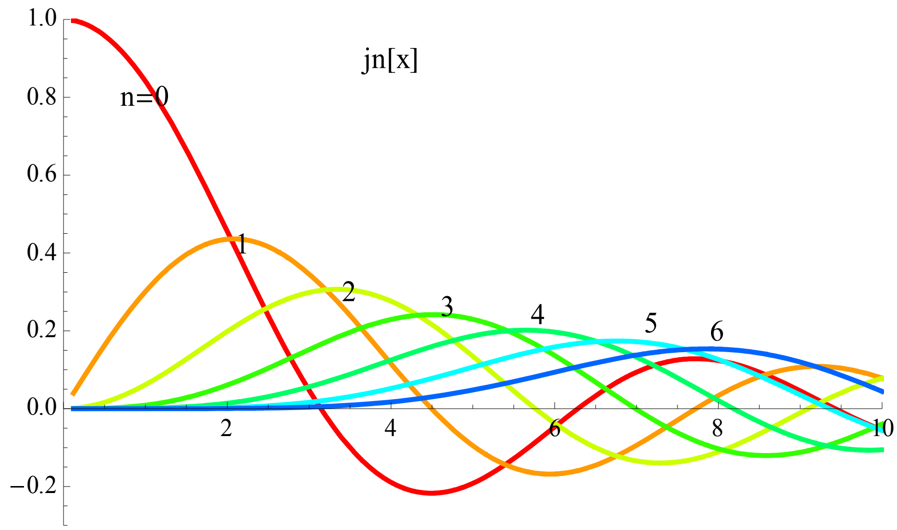

Fig. $j_n(x)$ with n = 0, 1, 2, 3, 4, 5, and 6.

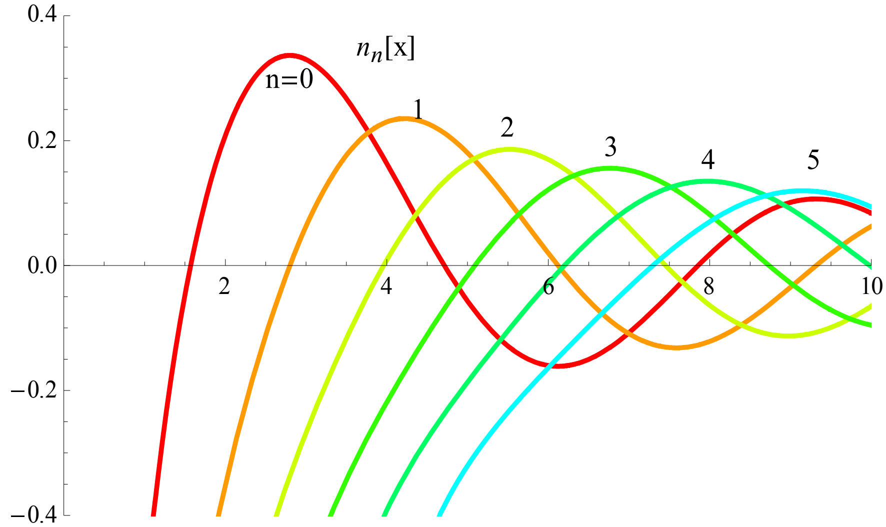

Fig. $n_n(x)$ with n = 0, 1, 2, 3, 4, and 5.

21.4 Rayleigh formulas

$$j_\ell(x) = (-1)^\ell x^l (\frac{1}{x}\frac{d}{dx})^\ell \frac{\sin x}{x},$$

$$n_\ell(x) = -(-1)^\ell x^l (\frac{1}{x}\frac{d}{dx})^\ell \frac{\cos x}{x},$$

$$h_\ell^{(1)}(x) = -i(-1)^\ell x^l (\frac{1}{x}\frac{d}{dx})^\ell \frac{e^{ix}}{x}$$

$$h_\ell^{(2)}(x) = i(-1)^\ell x^l (\frac{1}{x}\frac{d}{dx})^\ell \frac{e^{-ix}}{x}$$

21.5 Asymptotic forms

The asymptotic values of the spherical Bessel functions and spherical Hankel functions may be obtained from the Bessel asymptotic form.

$$j_\ell(x) \approx \frac{1}{x}\sin(x - \frac{l\pi}{2}),$$

$$n_\ell(x) \approx -\frac{1}{x}\cos(x - \frac{l\pi}{2}),$$

$$h_\ell^{(1)}(x) \approx -i\frac{e^{i(x-l\pi/2)}}{x}$$ (outgoing spherical wave)

$$h_\ell^{(2)}(x) \approx i\frac{e^{-i(x-l\pi/2)}}{x}$$ (incoming spherical wave)

21.6 Plane wave expression

The wave function ψ can be described by

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_l^m(\theta,\phi) j_l(kr)$$

We consider the plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$, which is one of the solution of the Schrödinger equation.

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_l^m(\theta,\phi) j_l(kr)$$

We choose the direction of ***k*** along the *z* direction.

$$\mathbf{k} = (0,0,k), \qquad \mathbf{k}\cdot\mathbf{r} = kr\cos\theta$$

We note that $e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\theta}$ is independent of ϕ. $Y_l^m(\theta,\phi)$ is independent of ϕ only for $m = 0$.

$$Y_l^{m=0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

Then we get

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} c_l P_l(\theta) j_l(kr)$$

where

$$c_l = i^l(2l+1)$$

((Proof))

$$\int_0^\pi e^{ikr\cos\theta} P_l(\cos\theta)\sin\theta d\theta = \sum_{s=0}^{\infty} c_s j_s(kr)\int_0^\pi P_s(\cos\theta)P_l(\cos\theta)\sin\theta d\theta$$

or

$$\int_0^{\pi} e^{ikr\cos\theta} P_l(\cos\theta)\sin\theta d\theta = \sum_{s=0}^{\infty} c_s j_s(kr)\frac{1}{2s+1}\delta_{l,s} = \frac{2}{2l+1}c_l j_l(kr)$$

Differentiate l times with respect to $x = kr$.

$$\frac{d^l}{dx^l}\int_0^{\pi} e^{ix\cos\theta} P_l(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}c_l \frac{d^l}{dx^l} j_l(x)$$

or

$$\int_0^{\pi} (i\cos\theta)^l e^{ix\cos\theta} P_l(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}c_l \frac{d^l}{dx^l} j_l(x)$$

Note that

$$j_l(x) \approx \frac{2^l l!}{(2l+1)!}x^l \qquad \text{for } x \ll 1.$$

$$\frac{d^l j_l(x)}{dx^l} = \frac{2^l (l!)^2}{(2l+1)!}$$

When $x = 0$,

$$\frac{2}{2l+1}c_l \frac{2^l (l!)^2}{(2l+1)!} = i^l \int_0^{\pi} \cos^l \theta P_l(\cos\theta)\sin\theta d\theta = i^l \int_0^{\pi} \varsigma^l P_l(\varsigma) d\varsigma = i^l \frac{2^{l+1}(l!)^2}{(2l+1)!}$$

or

$$c_l = i^l (2l+1)$$

or

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1) P_l(\cos\theta) j_l(kr) \qquad \text{(Rayleigh's expansion)}$$

This formula is especially useful in scattering theory. For $kr \gg 1$, we get

$$
\begin{aligned}
e^{i\mathbf{k}\cdot\mathbf{r}} &\approx \sum_{l=0}^{\infty} i^l (2l+1) P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}) j_l(kr) \\
&\approx \sum_{l=0}^{\infty} i^l (2l+1) P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}) \frac{\sin(kr - \frac{l\pi}{2})}{kr} \\
&= \sum_{l=0}^{\infty} i^l (2l+1) P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}) \frac{\cos[kr - \frac{(l+1)\pi}{2}]}{kr} \\
&= \frac{1}{2ikr} \sum_{l=0}^{\infty} \{ e^{\frac{il\pi}{2}} (2l+1) P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}) [e^{i(kr - \frac{(l+1)\pi}{2})} + e^{-i(kr - \frac{(l+1)\pi}{2})}] \\
&= \frac{e^{ikr}}{2ikr} \sum_{l=0}^{\infty} (2l+1) P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}) - \frac{e^{-ikr}}{2ikr} \sum_{l=0}^{\infty} (2l+1)(-1)^l P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}) \\
&= \frac{2\pi e^{ikr}}{ikr} \delta(\hat{\mathbf{k}},\hat{\mathbf{r}}) - \frac{2\pi e^{-ikr}}{ikr} \delta(\hat{\mathbf{k}},-\hat{\mathbf{r}})
\end{aligned}
$$

where

$$
\hat{\mathbf{k}}\cdot\hat{\mathbf{r}} = \frac{\mathbf{k}\cdot\mathbf{r}}{kr} = \cos\theta
$$

From Chapter 23, we have

$$
\langle \mathbf{n} | \mathbf{n}' \rangle = \delta(\mathbf{n},\mathbf{n}') = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\mathbf{n}\cdot\mathbf{n}') .
$$

21.7 Bessel-Fourier transform

$$
e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l (2l+1) P_l(\theta) j_l(kr)
$$

$$
\int_0^{\pi} e^{ikr\cos\theta} P_l(\cos\theta) \sin\theta d\theta = \sum_{s=0}^{\infty} i^s (2s+1) j_s(kr) \int_0^{\pi} P_s(\theta) P_l(\cos\theta) \sin\theta d\theta
$$

$$
\int_0^{\pi} e^{ikr\cos\theta} P_l(\cos\theta) \sin\theta d\theta = \sum_{s=0}^{\infty} i^s (2s+1) j_s(kr) \frac{2}{2s+1} \delta_{s,l} = 2i^l j_l(kr)
$$

or

$$j_l(kr) = \frac{1}{2i^l}\int_0^{\pi} e^{ikr\cos\theta} P_l(\cos\theta)\sin\theta d\theta = \frac{1}{2i^l}\int_{-1}^{1} e^{ikrx} P_l(x)dx$$

This means that (apart from constant factor) the spherical Bessel function $j_l(kr)$ is the Fourier transform of the Legendre polynomial $P_l(x)$.

21.8 Green's function for the spherical Bessel function

We consider the Green's function given by

$$(\nabla^2 + k^2)G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'),$$

The solution of the Green's function is given by

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

with the boundary condition

$$G(\mathbf{r},\mathbf{r}') \to 0 \quad \text{for } r \to 0 \text{ and for } r \to \infty.$$

where ***r*** is the variable and ***r***' is fixed.

Within each region (region I (0<*r*<*r*') and region II (*r*'<*r*), we have the simpler equation

$$(\nabla^2 + k^2)G(\mathbf{r},\mathbf{r}') = 0\ .$$

The solution of the Green's function is given by the form

$$G(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}(r,r',\theta',\phi')Y_l^m(\theta,\phi).$$

Then the differential equation of the Green's function is given by

$$\sum_{l',m'}[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{l'm'}) + (k^2 - \frac{l'(l'+1)}{r^2})A_{l'm'}]Y_{l'}^{m'}(\theta,\phi) = -\frac{\delta(r-r')}{r^2}\delta(\phi-\phi')\delta(\mu-\mu').$$

Note that

$$\delta_{l,l'}\delta_{m,m'} = \int d\Omega\langle l',m'|\mathbf{n}\rangle\langle\mathbf{n}|l,m\rangle = \iint \sin\theta d\theta d\phi Y_{l'}^{m'^*}(\theta,\phi)Y_l^m(\theta,\phi)$$

where

$d\Omega = \sin\theta d\theta d\phi$.

Then

$$\sum_{l',m'} \int d\Omega Y_l^{m^*}(\theta,\phi) Y_{l'}^{m'}(\theta,\phi)[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{l'm'}) + (k^2 - \frac{l'(l'+1)}{r^2})A_{l'm'}]$$
$$= -\int d\Omega Y_l^{m^*}(\theta,\phi)\frac{\delta(r-r')}{r^2}\delta(\phi-\phi')\delta(\mu-\mu')$$

or

$$\sum_{l',m'}[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{l'm'}) + (k^2 - \frac{l'(l'+1)}{r^2})A_{l'm'}]\delta_{l,l'}\delta_{m,m'}$$
$$= -\int d\Omega Y_l^{m^*}(\theta,\phi)\frac{\delta(r-r')}{r^2}\delta(\phi-\phi')\delta(\mu-\mu')$$

or

$$\begin{aligned}\frac{1}{r}\frac{\partial^2}{\partial r^2}(rA_{lm}) + [k^2 - \frac{l(l+1)}{r^2}]A_{lm} &= -\frac{\delta(r-r')}{r^2}\int d\Omega Y_l^{m^*}(\theta,\phi)\delta(\phi-\phi')\delta(\mu-\mu') \\ &= -\frac{\delta(r-r')}{r^2}Y_l^{m^*}(\theta',\phi')\int d\mu d\phi \delta(\phi-\phi')\delta(\mu-\mu') \\ &= -\frac{\delta(r-r')}{r^2}Y_l^{m^*}(\theta',\phi')\end{aligned}$$

Since $Y_l^{m^*}(\theta',\phi')$ is constant, we put

$$G_l(r,r') = \frac{A_{lm}(r,r',\theta',\phi')}{Y_l^{m^*}(\theta',\phi')}.$$

Then we get

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rG_l) + [k^2 - \frac{l(l+1)}{r^2}]G_l = -\frac{\delta(r-r')}{r^2},$$

The possible solutions of G_l are $j_l(kr)$, $n_l(kr)$, $h_l^{(1)}(kr)$, $h_l^{(1)}(kr)$, or a linear combination of these functions.

$G_{lI} = Aj_l(kr)$ for $r<r'$ (region I)

$G_{lII} = Bh_l^{(1)}(kr)$ for $r>r'$ (region II) where A and B are constant. Note that If we use the positive sign for $G(\boldsymbol{r}, \boldsymbol{r}')$, we need to choose $h_l^{(1)}(kr)$;

$$h_\ell^{(1)}(kr) \approx -i\frac{e^{i(kr-l\pi/2)}}{kr} \approx \frac{e^{ikr}}{r} \qquad \text{(outgoing spherical wave)}$$

(i) The continuity of G_l at $r = r'$

$$Aj_l(kr') = Bh_l^{(1)}(kr')$$

or

$$\frac{A}{h_l^{(1)}(kr')} = \frac{B}{j_l(kr')} = C$$

(ii) The discontinuity of dG_l/dr at $r = r'$.

$$\int_{r'-\varepsilon}^{r'+\varepsilon} \{\frac{d^2}{dr^2}(rG_l) + [k^2 r - \frac{l(l+1)}{r}]G_l\}dr = -\int_{r'-\varepsilon}^{r'+\varepsilon} \frac{\delta(r-r')}{r}dr$$

or

$$[\frac{d}{dr}(rG_l)]_{r'-\varepsilon}^{r'+\varepsilon} = -\frac{1}{r'}$$

or

$$(G_l + r\frac{dG_l}{dr})|_{r'-\varepsilon}^{r'+\varepsilon} = -\frac{1}{r'}$$

$$\frac{dG_l^{II}(k,r,r')}{dr}|_{r'+\varepsilon} - \frac{dG_l^{I}(k,r,r')}{dr}|_{r'-\varepsilon} = -\frac{1}{r'^2}$$

or

$$kC[j_l(kr')h_l^{(1)}{}'(kr') - j_l{}'(kr')h_l^{(1)}(kr')] = -\frac{1}{r'^2}$$

We need to calculate the Wronskian

$$W=\begin{vmatrix} j_l(kr') & n_l(kr') \\ j_l'(kr') & n_l'(kr') \end{vmatrix}=\frac{i}{k^2 r'^2}$$

((Note)) We can calculate W by using Mathematica.

```
Wronskian[{SphericalBesselJ[l, x],
  SphericalHankelH1[l, x]}, x]
```

$$\frac{i}{x^2}$$

Thus we get

$$C = ik$$

In general, we have

$$G(\mathbf{r},\mathbf{r}')=ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr_<)h_l^{(1)}(kr_>)Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi').$$

This means that

$r_< = r$
$r_> = r'$ in the region I ($r<r'$)

$r_> = r$
$r_< = r'$ in the region II ($r'<r$)

We also get

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}=4\pi ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} j_l(kr_<)h_l^{(1)}(kr_>)Y_l^m(\theta,\phi)Y_l^{m^*}(\theta',\phi')$$

APPENDIX

Mathematica

Bessel functions

BesselJ[n,z] for $J_n(z)$
BesselI[n,z] for $I_n(z)$

BesselK[n,z]	for $K_n(z)$
BesselY[n,z]	for $N_n(z)$ (or $Y_n(z)$)

Hankel functions

HankelH1[n,z]	for $H_n^{(1)}(z)$
HankelH2[n,z]	for $H_n^{(2)}(z)$

Spherical Bessel functions

SphericalBesselJ[n,z]	for $j_n(z)$
SphericalBesselI[n,z]	for $i_n(z)$
SphericalBesselK[n.z]	for $k_n(z)$
SphericalBesselY[n,z]	for $n_n(z)$

Spherical Hankel functions

SphericalHankelH1 [n,z]	for $h_n^{(1)}(z)$
SphericalKankelH2[n,z]	for $h_n^{(2)}(z)$