Chapter 22
Legendre function
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: November 06, 2010)

Laplace's equation
Spherical harmonics
Legendre polynomials
Rodrigues' formula
Generating function
Addition theorem

22.1 Laplace's equation in the spherical coordinate

We consider the solution of Laplace's equation,

$$\nabla^2\Phi(\mathbf{r}) = 0 .$$

where $\Phi(\mathbf{r})$ is a scalar electric potential. The Laplacian in the spherical coordinate is given by

$$\begin{aligned}\nabla^2 &= -\frac{1}{\hbar^2 r^2}\mathbf{L}^2 + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) \\ &= -\frac{1}{\hbar^2 r^2}\mathbf{L}^2 + \frac{1}{r}\frac{\partial^2}{\partial r^2}(r)\end{aligned}$$

where ***L*** is the angular momentum. The differential equation of the potential $\Phi(\mathbf{r})$ is given by

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi(\mathbf{r})) - \frac{1}{\hbar^2 r^2}\mathbf{L}^2\Phi(\mathbf{r}) = 0 .$$

Here we assume that

$$\Phi(\mathbf{r}) = U(r)Y_l^m(\theta,\phi) .$$

(separation variable). Then we have

$$\frac{1}{r}Y_l^m(\theta,\phi)\frac{\partial^2}{\partial r^2}(rU(r)) - \frac{1}{\hbar^2 r^2}U(r)\mathbf{L}^2 Y_l^m(\theta,\phi) = 0 .$$

We use the relation

$$\mathbf{L}^2 Y_l^m(\theta,\phi) = \hbar^2 l(l+1) Y_l^m(\theta,\phi) .$$

Thus we get

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}[rU(r)] - \frac{l(l+1)}{r^2}U(r) = 0 .$$

The solution of $U(r)$ is given by

$$U(r) = Ar^l + Br^{-(l+1)} .$$

where A and B are constants. Then the general solution is expressed by

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[A_{lm}r^l + B_{lm}r^{-(l+1)}]Y_l^m(\theta,\phi) .$$

((**Mathematica**))

```
eq1 = 1/r D[r U[r], {r, 2}] - (l (l + 1))/r^2 U[r] == 0;

DSolve[eq1, U[r], r] // Simplify[#, l > 0] & //
 Expand

{{U[r] → r^l C[1] + r^(-1-l) C[2]}}
```

22.2 Spherical harmonics

$$\langle \mathbf{n}|l,m\rangle = \langle \theta,\phi|l,m\rangle = Y_l^m(\theta,\phi)$$

$$\hat{L}_z|l,m\rangle = m\hbar|l,m\rangle ,$$

or

$$\langle \mathbf{n}|\hat{L}_z|l,m\rangle = -i\hbar\frac{\partial}{\partial\phi}\langle \mathbf{n}|l,m\rangle = m\hbar\langle \mathbf{n}|l,m\rangle ,$$

$$|\mathbf{n}\rangle = |\theta,\phi\rangle .$$

The closure relation

$$\int |\theta,\phi\rangle d\Omega \langle \theta,\phi| = \hat{1} .$$

where

$$d\Omega = \sin\theta d\theta d\phi .$$

The θ and ϕ dependence of $\langle \mathbf{n}|l,m\rangle$ is given by

$$\begin{aligned}\langle \mathbf{n}|\hat{L}^2|lm\rangle &= -\hbar^2[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})]Y_l^m(\theta,\phi) \\ &= \hbar^2 l(l+1)Y_l^m(\theta,\phi)\end{aligned} \quad (1)$$

$$\langle \mathbf{n}|\hat{L}_z|lm\rangle = \frac{\hbar}{i}\frac{\partial}{\partial\phi}Y_l^m(\theta,\phi) = \hbar m Y_l^m(\theta,\phi) . \quad (2)$$

Equation (2) shows that

$$Y_l^m(\theta,\phi) = \Theta_l^m(\theta,\phi)e^{im\phi} .$$

We must require that the eigenfunctions be single valued

$$e^{im\phi} = e^{im(\phi+2\pi)},$$

which means that $m = 0, \pm 1, \pm 2,$ (integers). Equation (1) can be rewritten as

$$[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d}{d\theta}) - \frac{m^2}{\sin^2\theta} + l(l+1)]\Theta_l^m(\theta,\phi) = 0 .$$

The orhogonality relation $\langle l',m'|l,m\rangle = \delta_{l,l'}\delta_{m,m'}$ leads to

$$\delta_{l,l'}\delta_{m,m'} = \int d\Omega \langle l',m'|\mathbf{n}\rangle\langle \mathbf{n}|l,m\rangle = \iint \sin\theta d\theta d\phi Y_{l'}^{m'^*}(\theta,\phi)Y_l^m(\theta,\phi) .$$

To obtain the form of $Y_l^m(\theta,\phi)$, we may start with $m = l$.

$$\hat{L}_+|l,m=l\rangle = 0 ,$$

or

$$\langle \mathbf{n}|\hat{L}_+|l,m=l\rangle = -i\hbar e^{i\phi}(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi})\langle \mathbf{n}|l,m=l\rangle = 0 ,$$

Since $\langle \mathbf{n} | l, m=l \rangle = Y_l^l(\theta,\phi) = \Theta_l^l(\theta) e^{il\phi}$,

$$(\frac{d}{d\theta} - l\cot\theta)\Theta_l^l(\theta) = 0,$$

or

$$Y_l^l(\theta,\phi) = C_l e^{il\phi} \sin^l\theta,$$

where C_l is a normalization constant.

$$C_l = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)(2l)!}{4\pi}},$$

The result for $m \geq 0$ is

$$Y_l^m(\theta,\phi) = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)}{4\pi}\frac{(l+m)!}{(l-m)!}} e^{im\phi} \frac{1}{\sin^m\theta} \frac{d^{l-m}}{d(\cos\theta)^{l-m}} (\sin\theta)^{2l},$$

and we define $Y_l^{-m}(\theta,\phi)$ by

$$Y_l^{-m}(\theta,\phi) = (-1)^m [Y_l^m(\theta,\phi)]^*,$$

or

$$[Y_l^m(\theta,\phi)]^* = (-1)^m Y_l^{-m}(\theta,\phi).$$

22.3 Legendre polynomial

We consider the special case when $m = 0$ (the system has axis symmetry). This means that potential $\Phi(\mathbf{r})$ is independent of ϕ,

$$[\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d}{d\theta}) + l(l+1)]\Theta_l(\theta) = 0,$$

$$Y_l^0(\theta,\phi) = \Theta_l(\theta),$$

and

$\Theta_l(\theta)$ can be also expressed by

$$\Theta_l(\theta) = \sqrt{\frac{2l+1}{4\pi}} P_l^0(\cos\theta) = \sqrt{\frac{2l+1}{4\pi}} P_l(\mu)$$

where and $\mu = \cos\theta$ and $P_l(\mu)$ is the Legendre function. The solution for Φ of the system with axis symmetry is given by

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} (A_l r^l + \frac{B_l}{r^{l+1}}) P_l(\mu).$$

((Note)) Rodrigues' formula

The Legendre polynominals $P_l(x)$ are defined by the formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

where l is an integer (Rodrigues' formula).

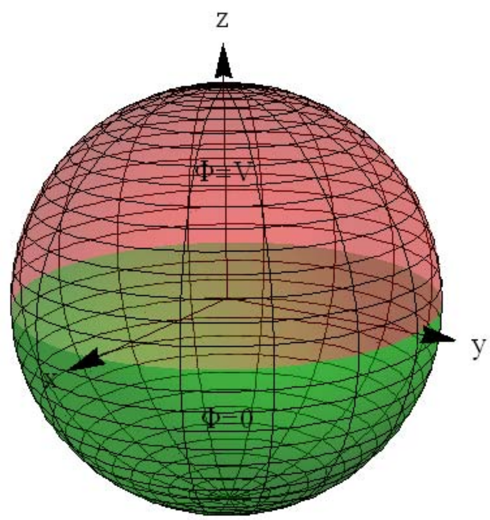
$P_l(x)$ satisfy the Legendre's differential equation.

$$(1 - x^2) P_l''(x) - 2x P_l'(x) + l(l+1) P_l(x) = 0$$

$$(|x| \leq 1)$$

22.4 Example

A hollow copper sphere of radius *a* is divided into two halves at the equator by a thin insulating strip. The top half of the sphere is held at potential *V*, and the bottom is grounded, what is the potential inside?



We need to set all the coefficients B_l to zero. Otherwise, the potential becomes divergent at $r \rightarrow 0$.

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\mu),$$

When $r = a$,

$$\Phi(a,\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\mu) = V \text{ for } 0<\mu\leq 1, \text{ and } 0 \text{ for } -1\leq\mu<0.$$

We note that

$$\int_{-1}^{1} P_l(\mu) P_{l'}(\mu) d\mu = \frac{2}{2l+1}\delta_{l,l'}. \quad \text{(orthogonality)}$$

Then we have

$$\int_{-1}^{1}\Phi(a,\theta)P_l(\mu)d\mu = \sum_{l'=0}^{\infty}A_{l'}a^{l'}\int_{-1}^{1}P_{l'}(\mu)P_l(\mu)d\mu$$

$$= \sum_{l'=0}^{\infty}A_{l'}a^{l'}\frac{2}{2l+1}\delta_{l',l}$$

$$= A_l a^l \frac{2}{2l+1}$$

or

$$A_l = \frac{2l+1}{2a^l}\int_{-1}^{1}\Phi(a,\theta)P_l(\mu')d\mu' = \frac{2l+1}{2a^l}V\int_{0}^{1}P_l(\mu')d\mu',$$

Then we have

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(\frac{2l+1}{2}\right)\frac{r^l}{a^l}VP_l(\mu)\int_{0}^{1}P_l(\mu')d\mu',$$

((Mathematica)) The form of $\Phi(r,\theta)$

$$\left(\frac{V}{2} + \frac{3\,r\,V\,\text{Cos}[\theta]}{4\,a} - \frac{7\,r^3\,V\,\left(-3\,\text{Cos}[\theta] + 5\,\text{Cos}[\theta]^3\right)}{32\,a^3} + \frac{11\,r^5\,V\,\left(15\,\text{Cos}[\theta] - 70\,\text{Cos}[\theta]^3 + 63\,\text{Cos}[\theta]^5\right)}{256\,a^5} - \frac{75\,r^7\,V\,\left(-35\,\text{Cos}[\theta] + 315\,\text{Cos}[\theta]^3 - 693\,\text{Cos}[\theta]^5 + 429\,\text{Cos}[\theta]^7\right)}{4096\,a^7} + \frac{1}{65536\,a^9}133\,r^9\,V\,\left(315\,\text{Cos}[\theta] - 4620\,\text{Cos}[\theta]^3 + 18018\,\text{Cos}[\theta]^5 - 25740\,\text{Cos}[\theta]^7 + 12155\,\text{Cos}[\theta]^9\right)\right)$$

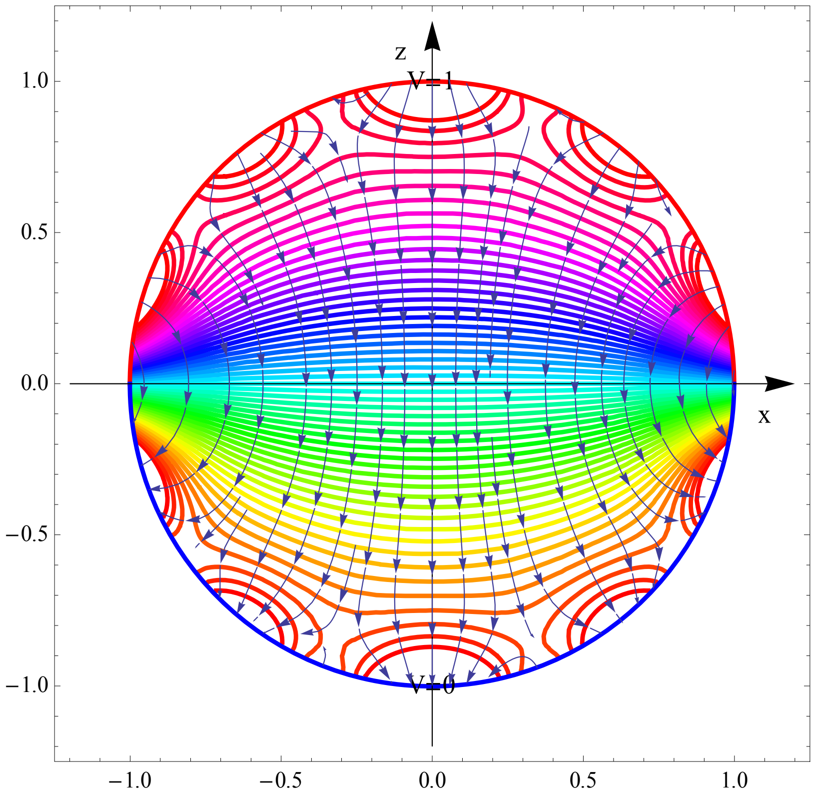


Fig. V = 1. a = 1. The distribution of potential $\Phi(r,\theta)$ in the z-x plane. The electric field ($E = -\nabla\Phi$) line is perpendicular to the $\Phi(r,\theta)$ = constant line.

22.5 Mathematica

LegendreP[n,x]: gives the Legendre polynomial $P_n(x)$.
LegendreQ[n,x] gives the Legendre function of the second kind $Q_n(z)$. gives the Legendre function of the second kind $Q_n(z)$.

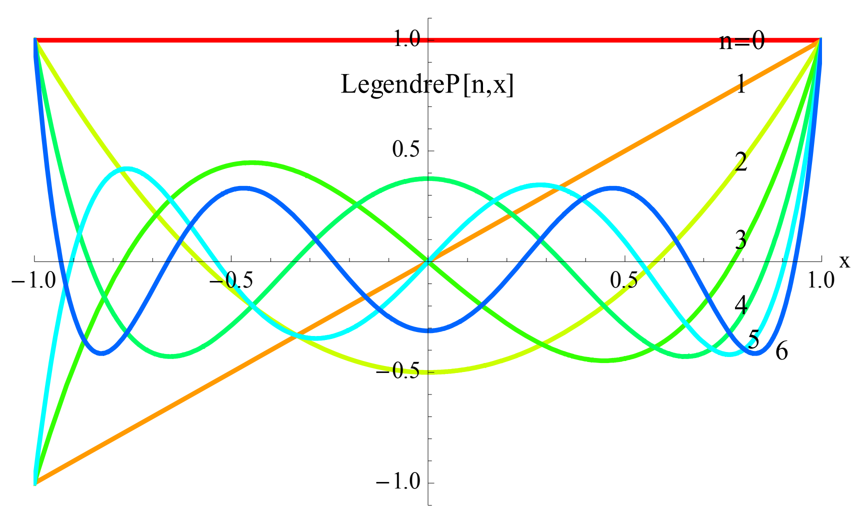


Fig. Legendre polynomial $P_n(x)$ with n = 0, 1, 2, 3, 4, 5, and 6.

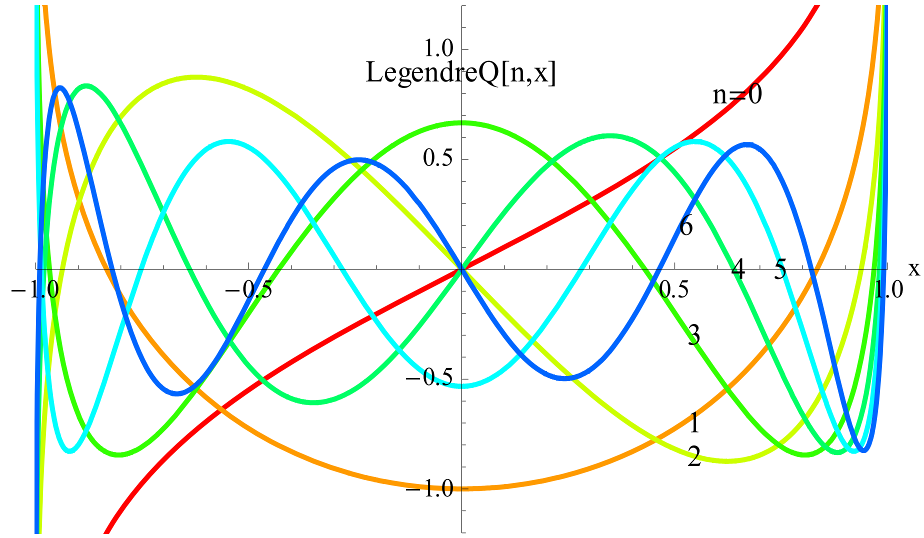


Fig. Legendre function of the second kind $Q_n(x)$ with n = 0, 1, 2, 3, 4, 5, and 6.

22.6 Generating function

We introduce a generating function

$$g(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n ,$$

for $|t|<1$.

(i) Recurrence relations

If the generating function is differentiated with respect to t, we obtain

$$\frac{\partial g(t,x)}{\partial t} = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1} ,$$

or

$$(1-2xt+t^2)\sum_{n=0}^{\infty} n P_n(x) t^{n-1} + (t-x)\sum_{n=0}^{\infty} P_n(x) t^n = 0 .$$

From this equation we find

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \qquad (1)$$

If we take $n = 1$,

$$3xP_1(x) = 2P_2(x) + P_0(x)$$

or

$$P_2(x) = \frac{3}{2}xP_1(x) - \frac{1}{2}P_0(x)$$

Since $P_0(x) = 1$ and $P_1(x) = x$, we have $P_2(x) = \frac{1}{2}(3x^2 - 1)$

(ii) Differential equation

$$\frac{\partial g(t,x)}{\partial x} = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)t^n$$

or

$$(1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)t^n = t\sum_{n=0}^{\infty} P_n(x)t^n$$

The co-efficient of each power of t is set equal to zero and we obtain

$$P_{n+1}'(x) + P_{n-1}'(x) = 2xP_n'(x) + P_n(x) \qquad (2)$$

By differentiating Eq.(1) with respect to *x*, and using Eq.(2),

$$\begin{aligned}(2n+1)[P_n(x) + xP_n'(x)] &= (n+1)P_{n+1}'(x) + nP_{n-1}'(x) \\ &= P_{n+1}'(x) + n[P_{n+1}'(x) + P_{n-1}'(x)] \\ &= P_{n+1}'(x) + n[2xP_n'(x) + P_n(x)]\end{aligned}$$

or

$$P_{n+1}'(x) = (n+1)P_n(x) + xP_n'(x) \qquad (3)$$

Similarly, we have

$$P_{n+1}'(x) = -nP_n(x) + xP_n'(x) \qquad (3')$$

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x) = 0$$

$$(1-x^2)P_n'(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x) = 0$$

From these relations, we have the differential equation.

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

22.7 Addition theorem

$$g(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \qquad \text{for } |t|<1$$

In physics, this equation often appears in the vector form

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^{\,l}}{r_>^{\,l+1}} P_l(\cos\theta) = \frac{1}{r>} \sum_{l=0}^{\infty} \frac{r_<^{\,l}}{r_>^{\,l}} P_l(\cos\theta)$$

Note

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 - 2rr'\cos\theta + r'^2}$$

(i)

$$r_> = r, \quad r_< = r', \qquad \text{for } r>r'$$

$$|\mathbf{r}-\mathbf{r}'| = r\left(1 - 2\frac{r'}{r}\cos\theta + \frac{r'^2}{r^2}\right)^{1/2}$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\frac{1}{\sqrt{1-2\frac{r'}{r}\cos\theta + \frac{r'^2}{r^2}}} = \frac{1}{r}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\frac{r'}{r}\right)^l$$

(ii)

$$r_> = r', \quad r_< = r, \qquad \text{for } r<r'$$

$$|\mathbf{r}-\mathbf{r}'| = r'\left(1 - 2\frac{r}{r'}\cos\theta + \frac{r^2}{r'^2}\right)^{1/2}$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{r'}\frac{1}{\sqrt{1-2\frac{r}{r'}\cos\theta+\frac{r^2}{r'^2}}}=\frac{1}{r'}\sum_{l=0}^{\infty}P_l(\cos\theta)\left(\frac{r}{r'}\right)^l$$

APPENDIX

1. $$\int_{-1}^{1} d\mu P_l(\mu)P_{l'}(\mu)=\delta_{l,l'}\frac{2}{2l+1}$$

2.

$$\int_{-1}^{1}\mu d\mu P_l(\mu)P_{l'}(\mu)=\frac{2(l+1)}{(2l+1)(2l+3)}\delta_{l;,l+1}+\frac{2l}{(2l-1)(2l+1)}\delta_{l;,l-1}$$

(Jackson, p.100)