Chapter 24 Bessel function

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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Friedrich Wilhelm Bessel (22 July 1784 – 17 March 1846) was a German mathematician, astronomer, and systematizer of the Bessel functions (which were discovered by Daniel Bernoulli). He was a contemporary of Carl Gauss, also a mathematician and astronomer. The asteroid 1552 Bessel was named in his honour.

http://en.wikipedia.org/wiki/Friedrich_Bessel

24.1 Bessel functions and Neuman functions

((**Mathematica**))

The four Bessel functions $J_\nu(z)$, $I_\nu(z)$, $K_\nu(z)$, and $Y_\nu(z)$ are the best known and most frequently used special functions. That is why we will devote a slightly longer section to them and present a couple of applications. Following *Mathematica*'s naming convention, they are written as follows.

Bessel functions

BesselJ[n,z]	for $J_n(z)$
BesselY[n,z]	for $N_n(z)$ (or $Y_n(z)$)

Modified Bessel functions

BesselI[n,z]	for $I_n(z)$
BesselK[n,z]	for $K_n(z)$

Hankel functions

HankelH1[n,z]	for $H_n^{(1)}(z)$
HankelH2[n,z]	for $H_n^{(2)}(z)$

Spherical Bessel functions

SphericalBesselJ[n,z]	for $j_n(z)$
SphericalBesselI[n,z]	for $i_n(z)$
SphericalBesselK[n.z]	for $k_n(z)$
SphericalBesselY[n,z]	for $n_n(z)$

Spherical Hankel function

SphericalHankelH1[n,z]	for $h_n^{(1)}(z)$
SphericalKankelH2[n,z]	for $h_n^{(2)}(z)$

<u>Hankel function</u>

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$$
$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$$

<u>Modified Bessel function</u>

$$I_\nu(x) = e^{-\frac{i\nu\pi}{2}} J_\nu(ix)$$
$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) = \frac{\pi}{2} i^{\nu+1} [J_\nu(ix) + iN_\nu(ix)]$$

<u>Spherical Bessel function</u>

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x) \quad (l \text{ is integer})$$
$$n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x)$$
$$h_l^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H_{l+\frac{1}{2}}{}^{(1)}(x) = j_l(x) + in_l(x)$$

$$h_l^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H_{l+\frac{1}{2}}^{(2)}(x) = j_l(x) - in_l(x)$$

$$k_l(x) = -i^l h_l^{(1)}(ix)$$

$$i_l(x) = i^{-l} j_l(ix)$$

24.2 Bessel functions of the second kind (Neuman function)

$$N_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}.$$

For $\nu = n$ (integer),

L'Hospital's rule:

$$N_n(x) = \lim_{\nu\to n} N_\nu(x) = \lim_{\nu\to n} \frac{\frac{d}{d\nu}[\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)]}{\frac{d\sin\nu\pi}{d\nu}}$$

$$= \lim_{\nu\to n} \frac{[-\pi\sin(\nu\pi)J_\nu(x) + \cos(\nu\pi)\frac{dJ_\nu(x)}{d\nu} - \frac{dJ_{-\nu}(x)}{d\nu}]}{\pi\cos\nu\pi}$$

$$= \frac{1}{\pi}[\frac{dJ_\nu}{d\nu} - (-1)^n \frac{dJ_{-\nu}}{d\nu}]_{\nu=n}$$

In order to verify that $N_\nu(x)$ satisfies the Bessel's equation for integral ν, we may proceed

$$x^2 \frac{d^2}{dx^2} J_{\pm\nu}(x) + x\frac{d}{dx} J_{\pm\nu}(x) + (x^2 - \nu^2)J_{\pm\nu}(x) = 0$$

Differentiating this with respect to ν,

$$x^2 \frac{d^2}{dx^2}\frac{dJ_{\pm\nu}(x)}{d\nu} + x\frac{d}{dx}\frac{dJ_{\pm\nu}(x)}{d\nu} + (x^2 - \nu^2)\frac{dJ_{\pm\nu}(x)}{d\nu} = 2\nu J_{\pm\nu}(x)$$

Then

$$x^2 \frac{d^2}{dx^2} N_n(x) + x\frac{d}{dx} N_n(x) + (x^2 - n^2)N_n(x) = \frac{2n}{\pi}[J_n(x) - (-1)^n J_{-n}(x)] = 0$$

Thus $N_n(x)$ is seen to be a solution of Bessel's equation.

(b) For $\nu \neq n$,

$J_\nu(x)$ and $J_{-\nu}(x)$ are independent solutions.

or

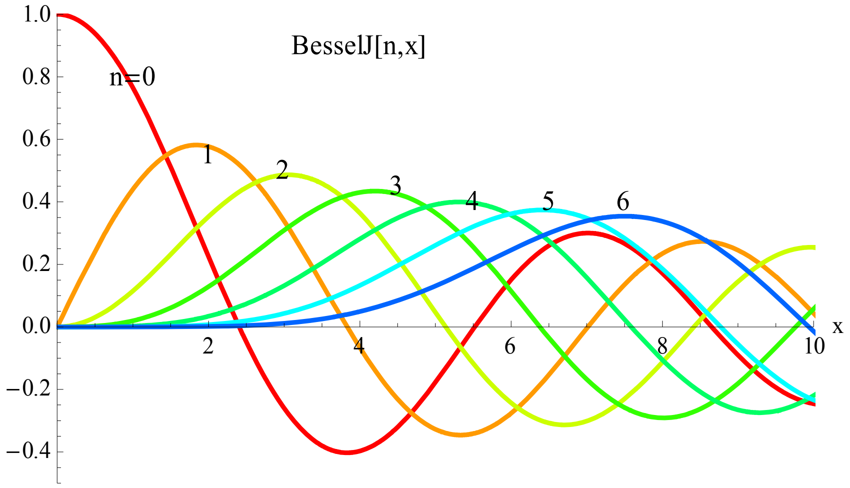
$J_\nu(x)$ and $N_\nu(x)$ are independent solutions.

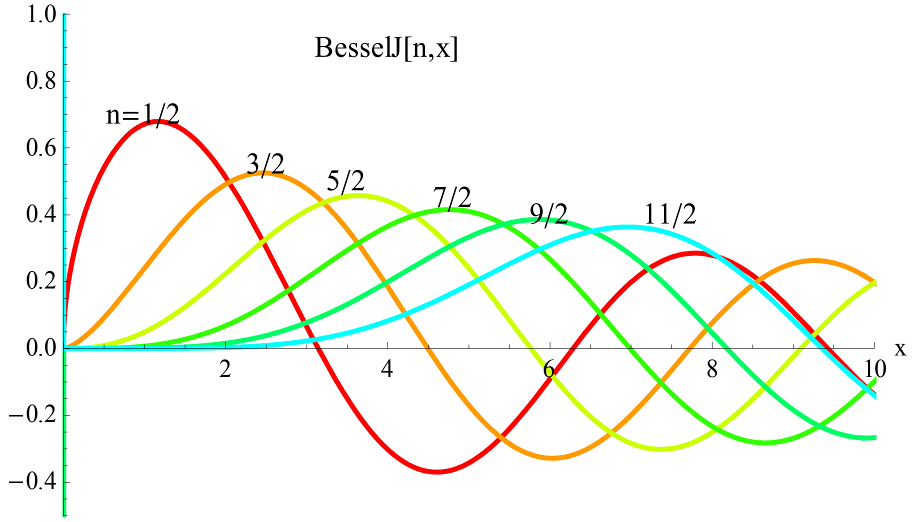
For $\nu = n$,

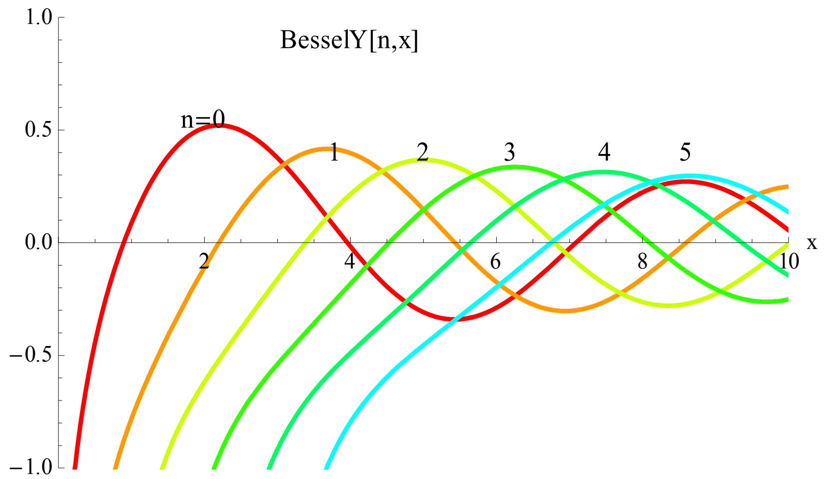
$J_n(x)$ and $J_{-n}(x)$ are dependent.

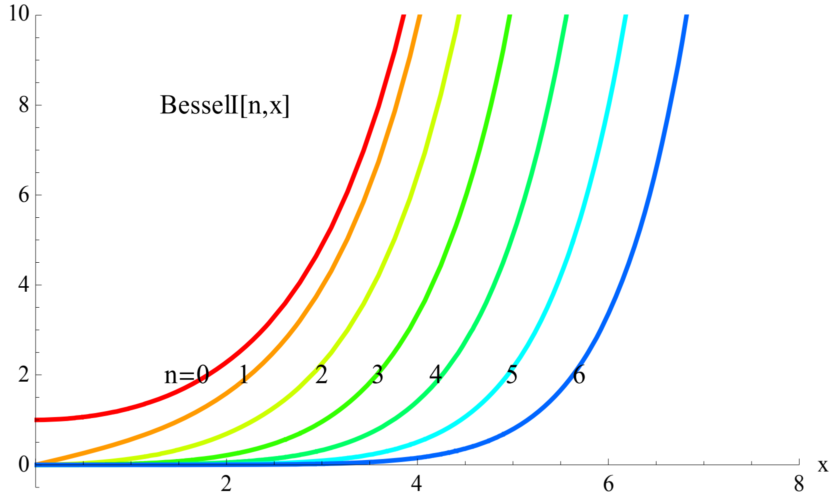
$J_n(x)$ and $N_n(x)$ are independent solutions.

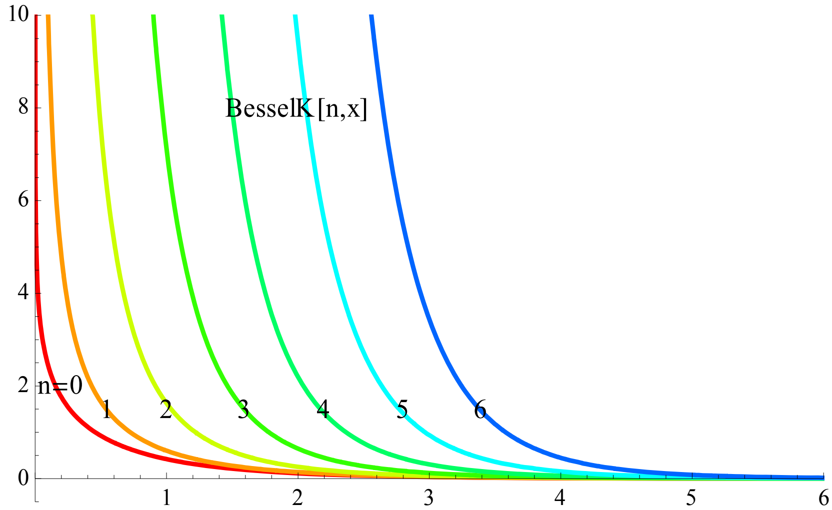
((**Mathematica**))











24.3 Hankel function

Definitions

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$$

where ν>0, ν = integral and nonintegral values.

In the limit of $x \approx 0$,

$$H_0^{(1)}(x) = i\frac{2}{\pi}\ln x + 1 + i\frac{2}{\pi}(\gamma - \ln 2) + ...$$

$$H_\nu^{(1)}(x) = -i\frac{(\nu-1)!}{\pi}(\frac{2}{x})^\nu + ...$$

$$H_0^{(2)}(x) = -i\frac{2}{\pi}\ln x + 1 - i\frac{2}{\pi}(\gamma - \ln 2) + ...$$

$$H_\nu^{(2)}(x) = i\frac{(\nu-1)!}{\pi}(\frac{2}{x})^\nu + ...$$

Since the Hankel functions are linear combinations of J_ν and N_ν, they satisfy the same recurrence relations.

$$H_{\nu-1}(x) + H_{\nu+1}(x) = \frac{2\nu}{x}H_\nu(x)$$

$$H_{\nu-1}(x) - H_{\nu+1}(x) = 2H_\nu'(x)$$

24.4 Integral representation of Bessel functions

The generating function is defined as

$$g(x,t) = \exp[\frac{x}{2}(t - \frac{1}{t})] = \sum_{n=-\infty}^{\infty} J_n(x)t^n$$

Here we substitute $t = e^{i\theta}$

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta} = J_0(x) + \sum_{n=1}^{\infty} J_n(x)e^{in\theta} + \sum_{n=-\infty}^{-1} J_n(x)e^{in\theta}$$

Using the property of $J_{-n}(x) = (-1)^n J_n(x)$, we have

$$e^{ix\sin\theta} = J_0(x) + \sum_{n=1}^{\infty} J_n(x)e^{in\theta} + \sum_{n=1}^{\infty} J_{-n}(x)e^{-in\theta} ,$$

or

$$\begin{aligned} e^{ix\sin\theta} &= \cos(x\sin\theta) + i\sin(x\sin\theta) \\ &= J_0(x) + \sum_{n=1}^{\infty} J_n(x)e^{in\theta} + \sum_{n=1}^{\infty} (-1)^n J_n(x)e^{-in\theta} . \end{aligned}$$

Then we have the Fourier series,

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta)$$

$$\sin(x\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin[(2n-1)\theta]$$

$$\frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos(n\theta)d\theta = J_n(x) ,$$

for n = even, and zero for n = odd.

$$\frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin(n\theta)d\theta = J_n(x) ,$$

for n = odd and zero for n = even. Thus, if these two equations are added together,

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \{\cos(x\sin\theta)\cos(n\theta) + \sin(x\sin\theta)\sin(n\theta)\}d\theta$$

or

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta$$

for n = 0, 1, 2, 3,….. When n = 0,

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos(x\sin\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}d\theta$$

24.5 Contour integral representation

We derive the Schalaefli formula given by

$$J_\nu(x) = \frac{1}{2\pi i}\int e^{(x/2)(t-1/t)}\frac{1}{t^{\nu+1}}dt \quad \text{(Schlaefli integral)}$$

using the generating function of the Bessel function.

For $\nu = n$,

$$\exp[\frac{x}{2}(t-\frac{1}{t})] = \sum_{n=-\infty}^{\infty} J_n(x)t^n$$

For a complex variable *z*, we have the same result,

$$\exp[\frac{x}{2}(z-\frac{1}{z})] = \sum_{n=-\infty}^{\infty} J_n(x)z^n = ... + J_0(x) + ... + J_n(x)z^n + ...$$

$$\frac{1}{z^{n+1}}\exp[\frac{x}{2}(z-\frac{1}{z})] = ... + \frac{J_0(x)}{z^{n+1}} + ... + J_n(x)\frac{1}{z} + ...$$

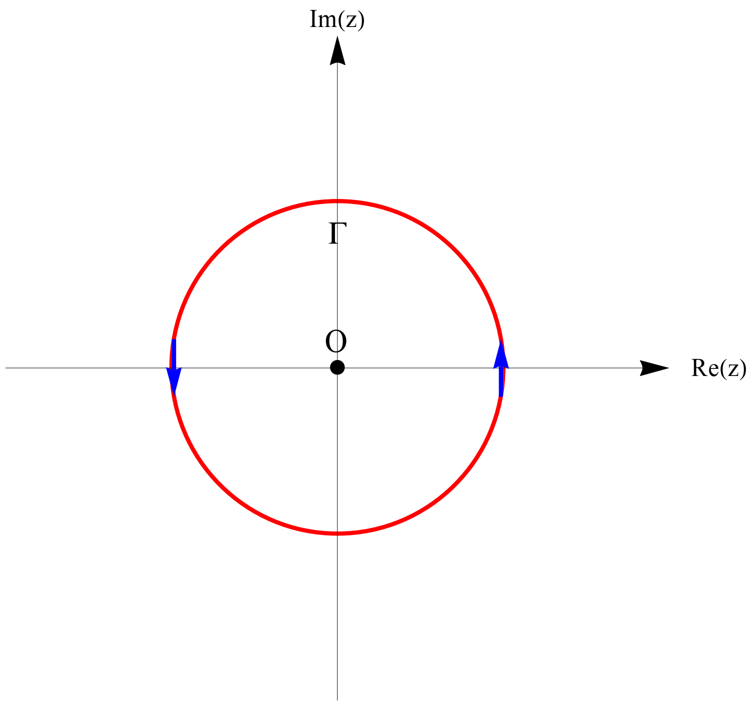
Then we apply the Cauchy theorem.

$$\oint \frac{1}{z^{n+1}}\exp[\frac{x}{2}(z-\frac{1}{z})]dz = 2\pi i \operatorname{Res}[z=0] = 2\pi i J_n(x)$$

or

$$J_n(x) = \frac{1}{2\pi i}\oint_\Gamma \frac{1}{z^{n+1}}\exp[\frac{x}{2}(z-\frac{1}{z})]dz$$

where Γ is a circle (counter-clock wise) around *z* = 0.



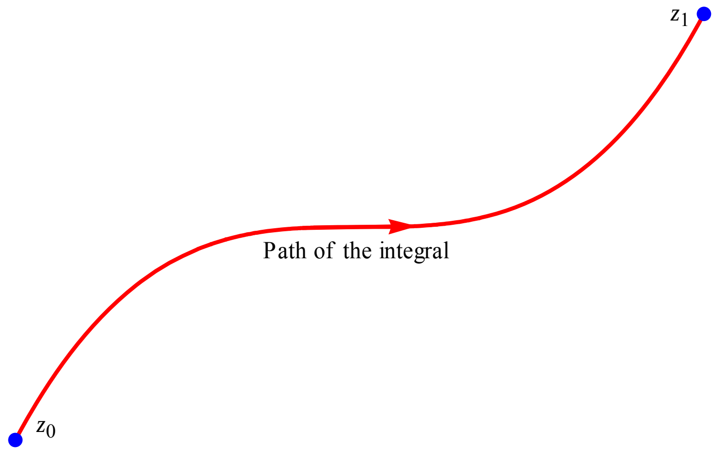
For $z = e^{i\theta}$,

$$J_n(x) = \frac{1}{2\pi i}\int_0^{2\pi} e^{-i(n+1)\theta} i e^{i\theta} \exp[\frac{x}{2}(e^{i\theta} - e^{-i\theta})]d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \exp[i(x\sin\theta - n\theta)]d\theta$$
$$= \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta$$

Suppose that ν is not an integer. We can show that

$$f_\nu(x) = \frac{1}{2\pi i}\int \frac{1}{z^{\nu+1}} \exp[\frac{x}{2}(z - \frac{1}{z})]dz$$

satisfies the Bessel's differential equation, where, except for avoiding the origin, we take *an arbitrary integration path for the moment.*



$$x^2 f_\nu''(x) + x f_\nu'(x) + (x^2 - \nu^2) f_\nu(x)$$
$$= \frac{1}{2\pi i}\int_{z_0}^{z_1} \frac{1}{z^{\nu+1}} \exp[\frac{x}{2}(z - \frac{1}{z})][\frac{x^2}{4}(z - \frac{1}{z})^2 + \frac{x}{2}(z - \frac{1}{z}) + x^2 - \nu^2]dz$$

the resulting integrand is a perfect differential, such that

$$x^2 f_\nu''(x) + x f_\nu'(x) + (x^2 - \nu^2) f_\nu(x)$$
$$= \frac{1}{2\pi i}\Delta F_\nu[z,x] = \frac{1}{2\pi i} F_\nu[z,x]|_{z_0}^{z_1}$$

where

$$F_\nu[z,x] = \frac{1}{z^\nu}\exp[\frac{x}{2}(z - \frac{1}{z})][\frac{x}{2}(z + \frac{1}{z}) + \nu)]$$

$$\frac{d}{dz}F_\nu[z,x] = \frac{1}{z^{\nu+1}}\exp[\frac{x}{2}(z - \frac{1}{z})][\frac{x^2}{4}(z - \frac{1}{z})^2 + \frac{x}{2}(z - \frac{1}{z}) + x^2 - \nu^2]$$

where Δ indicates the difference between values at the endpoints (z_0, z_1) of integration.

(a) When $\nu \to n$ is an integer, $F_n[z,x]$ is a single-valued and the right-hand side vanishes for any closed path (including the contour Γ (a circle around the olrigin). Paths which do not enclose the origin, though , degenerate to the trivial solution $f_\nu(x) \to 0$). Hence $f_\nu(x)$ is a solution to Bessel's equation under those conditions.

(b) When ν is not an integral, we require a branch cut to interpret the integrated term. It is customary to cut the z-plane below the negative real axis. We also need to choose a contour for which $F_\nu[z,x]$ has the same value at both ends for any x within a usefully large domain.

The integrand is not a single-valued function. There is a cut-line between z = 0 and $z = \infty e^{\pm i\pi}$.

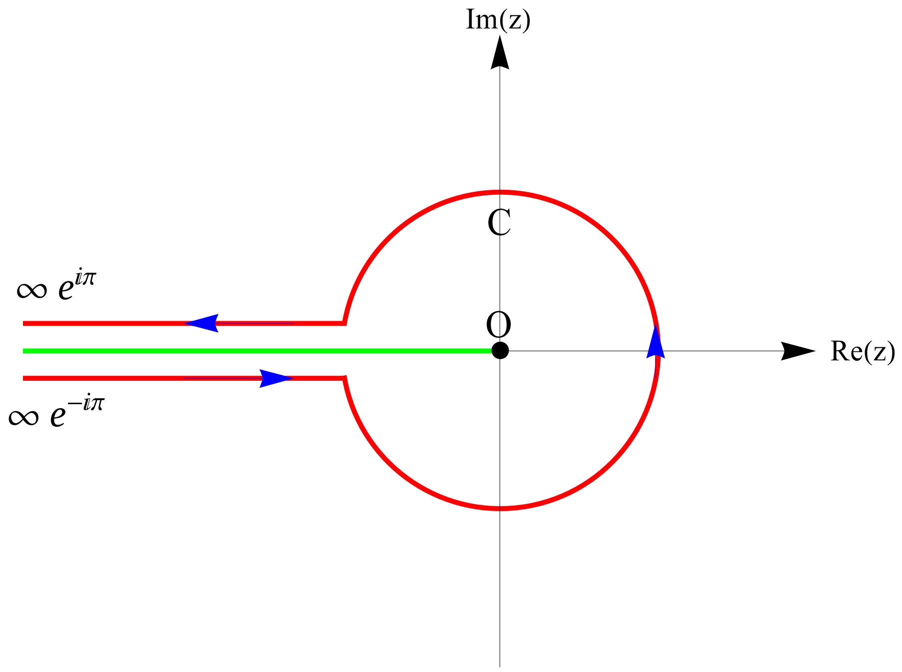


Fig. Contour for $J_n(x)$.

For $z \to \infty e^{\pm i\pi}$ $\qquad F_\nu[z,x] = \frac{1}{z^\nu}\exp[\frac{xz}{2}](\frac{xz}{2}+\nu) \to 0$ $\qquad$ for x>0

We now deform the above contour so that it approaches the origin along the positive real axis.

For $z \to 0^+$, $\qquad F_\nu[z,x] = \frac{1}{z^\nu}\exp(-\frac{x}{2z})](\frac{x}{2z}+\nu) \to 0$, $\qquad$ for x>0

Thus the two contours shown in the Fig. provide integral representation of the two independent solutions to the Bessel's differential equation; Hankel functions, $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$

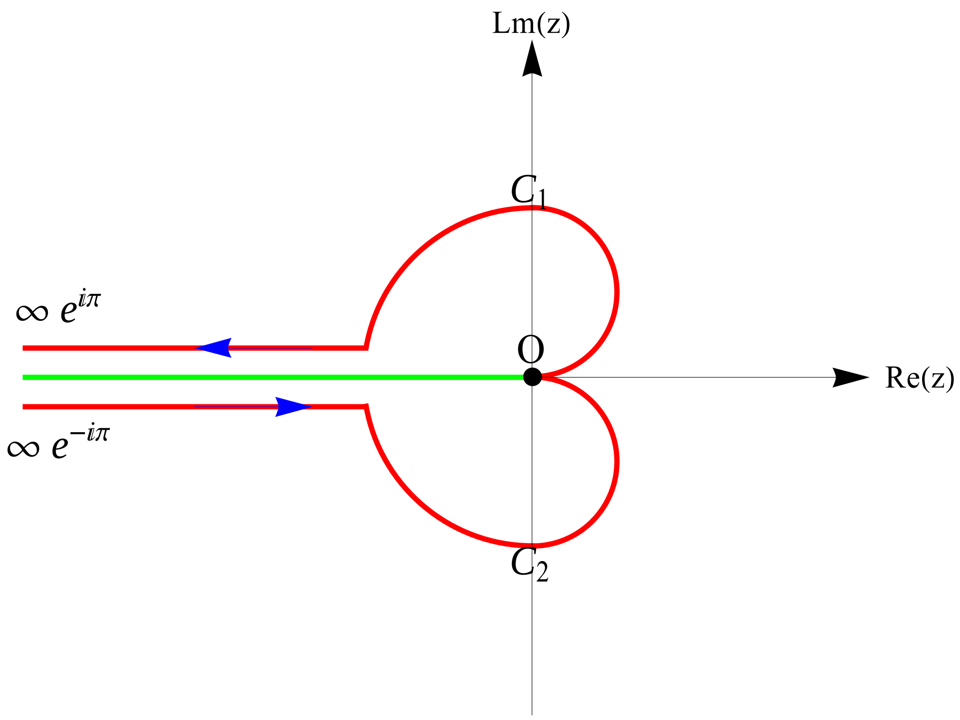


Fig. Contours C_1 for $H_\nu^{(1)}(x)$ and C_2 for $H_\nu^{(2)}(x)$

$$H_\nu^{(1)}(x) = \frac{1}{\pi i}\int_{C_1} \frac{1}{z^{\nu+1}} \exp[\frac{x}{2}(z - \frac{1}{z})]dz$$

$$H_\nu^{(2)}(x) = \frac{1}{\pi i}\int_{C_2} \frac{1}{z^{\nu+1}} \exp[\frac{x}{2}(z - \frac{1}{z})]dz$$

where the contour C_1 and C_2 are not closed.
or

$$H_\nu^{(1)}(x) = \frac{1}{\pi i}\int_{0^+}^{\infty e^{i\pi}} \frac{1}{z^{\nu+1}} \exp[\frac{x}{2}(z - \frac{1}{z})]dz$$

$$H_\nu^{(2)}(x) = \frac{1}{\pi i}\int_{\infty e^{-i\pi}}^{0^+} \frac{1}{z^{\nu+1}} \exp[\frac{x}{2}(z - \frac{1}{z})]dz$$

Noting that

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x),$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x).$$

we have

$$J_\nu(x) = \frac{1}{2}[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)],$$

and

$$N_\nu(x) = \frac{1}{2i}[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)].$$

24.6 Hankel's definite integral for $J_0(x)$

$$J_0(x) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(x\sin\theta)d\theta$$

$$t = \sin\theta, \qquad dt = \cos\theta d\theta = \sqrt{1-t^2}d\theta$$

or

$$d\theta = \frac{dt}{\sqrt{1-t^2}}$$

$$|\theta| \le \frac{\pi}{2} \to |t| \le 1$$

Then we have

$$J_0(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}dt = \frac{2}{\pi}\int_{0}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}dt = \frac{1}{\pi}\int_{-1}^{1}\frac{e^{ixt}}{\sqrt{1-t^2}}dt$$

24.7 Hankel's contour integral

We now consider the function

$$y = \int_a^b \frac{e^{ixt}}{\sqrt{1-t^2}}dt$$

where t is complex variable and $x>0$

$$xy''+y'+xy = [-i\sqrt{1-t^2}e^{ixt}]_a^b$$

If we put $a = \pm 1$, $b = i\eta$ and $\eta \to \infty$, y satisfies

$$xy''+y'+xy=0$$

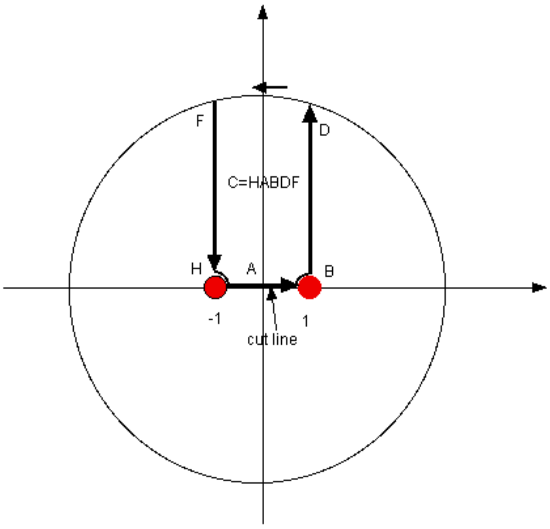
More generally, $a=\pm 1$, $b=\mathrm{R}e^{i\beta}$ ($R\to\infty$),

$$e^{ixR(\cos\beta+i\sin\beta)}=e^{ixR\cos\beta-xR\sin\beta)}$$

When $\sin\beta>0$, or , $0<\beta<\pi$, it follows that

$$xy''+y'+xy=0,$$

y is the solution of Bessel differential equation.



Path ($t=\pm 1$ to infinity in the upper half of the t-plane)

$$0=\oint_C \frac{e^{ixt}}{\sqrt{1-t^2}}dt=\int_{-1}^{1}\frac{e^{ixt}}{\sqrt{1-t^2}}dt+\int_{1}^{1+i\infty}\frac{e^{ixt}}{\sqrt{1-t^2}}dt+\int_{-1+i\infty}^{-1}\frac{e^{ixt}}{\sqrt{1-t^2}}dt$$

or

We define

$$\pi J_0(x)=\int_{-1}^{1}\frac{e^{ixt}}{\sqrt{1-t^2}}dt$$

$$-\frac{\pi}{2}H_0^{(1)}(x)=\int_{1}^{1+i\infty}\frac{e^{ixt}}{\sqrt{1-t^2}}dt$$

$$-\frac{\pi}{2}H_0^{(2)}(x) = \int_{-1+i\infty}^{-1} \frac{e^{ixt}}{\sqrt{1-t^2}}dt$$

or

$$J_0(x) = \frac{1}{2}[H_0^{(1)}(x) + H_0^{(2)}(x)]$$

From the path BD

$$t = 1 + i\eta, \qquad dt = id\eta \ (\eta \text{ is real})$$

$$\sqrt{1-t^2} = \sqrt{1-(1+i\eta)^2} = \sqrt{1-(1+2i\eta-\eta^2)} = \sqrt{2}e^{-i\pi/4}\sqrt{\eta + \frac{1}{2}i\eta^2}$$

$$-\frac{\pi}{2}H_0^{(1)}(x) = \int_0^{\infty} \frac{e^{ix}e^{-x\eta}}{\sqrt{2}e^{-i\pi/4}\sqrt{\eta + \frac{1}{2}i\eta^2}}id\eta$$

or

$$H_0^{(1)}(x) = \frac{\sqrt{2}}{\pi}e^{i(x-\frac{1}{4}\pi)}\int_0^{\infty}\frac{e^{-x\eta}}{\sqrt{\eta+\frac{1}{2}i\eta^2}}d\eta$$

Putting $u = x\eta$ (we assume x>0)

$$H_0^{(1)}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\frac{1}{\sqrt{\pi}}e^{i(x-\frac{1}{4}\pi)}\int_0^{\infty}\frac{e^{-u}}{\sqrt{u}}(1+\frac{iu}{2x})^{-1/2}du$$

In the same kind of way we find that

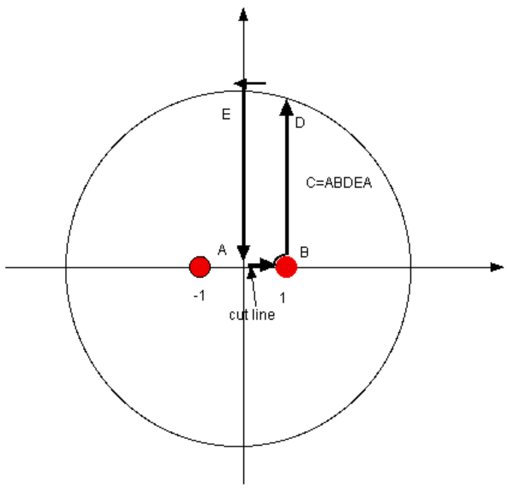
$$H_0^{(2)}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\frac{1}{\sqrt{\pi}}e^{-i(x-\frac{1}{4}\pi)}\int_0^{\infty}\frac{e^{-u}}{\sqrt{u}}(1-\frac{iu}{2x})^{-1/2}du$$

When x>0, $H_0^{(1)}(x)$ and $H_0^{(2)}(x)$ are a conjugate pair of complex numbers. $J_0(x)$ is the real part of either.

$$H_0^{(1)}(x) = J_0(x) + iN_0(x)$$

$$H_0^{(2)}(x) = J_0(x) - iN_0(x)$$

24.8 Use of the contour C=ABDEA



$$0 = \oint_C \frac{e^{ixt}}{\sqrt{1-t^2}} dt = \int_0^1 \frac{e^{ixt}}{\sqrt{1-t^2}} dt + \int_1^{1+i\infty} \frac{e^{ixt}}{\sqrt{1-t^2}} dt + \int_{i\infty}^0 \frac{e^{ixt}}{\sqrt{1-t^2}} dt$$

Along the path AE, we put

$$t = i\eta, \qquad dt = id\eta$$

$$\sqrt{1-t^2} = \sqrt{1+\eta^2}$$

From the definition,

$$\frac{\pi}{2} J_0(x) + \int_0^1 \frac{i\sin(xt)}{\sqrt{1-t^2}} dt - \frac{\pi}{2} H_0^{(1)}(x) + \int_\infty^0 \frac{e^{-x\eta}}{\sqrt{1+\eta^2}} id\eta = 0$$

Putting

$$H_0^{(1)}(x) = J_0(x) + iN_0(x)$$

$$\frac{\pi}{2} N_0(x) = -\int_0^\infty \frac{e^{-x\eta}}{\sqrt{1+\eta^2}} d\eta + \int_0^1 \frac{\sin(xt)}{\sqrt{1-t^2}} dt$$

We may also write, with

$\eta = \sinh u$, $\quad t = \sin\theta$, $\quad v = e^u$

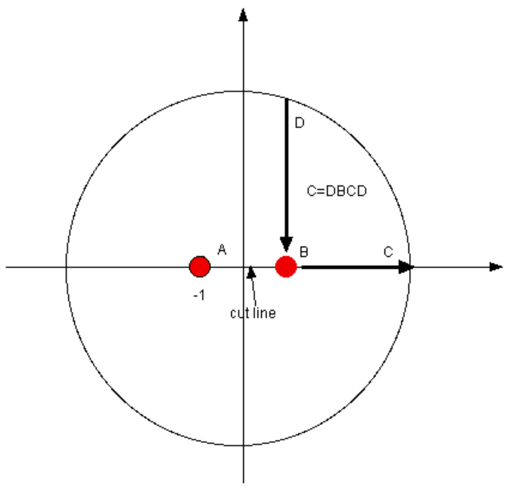
$$\frac{\pi}{2} N_0(x) = -\int_0^\infty e^{-x\sinh u} du + \int_0^{\pi/2} \sin(x\sin\theta) d\theta$$

or

$$\frac{\pi}{2} N_0(x) = -\int_t^\infty e^{-\frac{x}{2}(v-\frac{1}{v})} \frac{dv}{v} + \int_0^{\pi/2} \sin(x\sin\theta) d\theta$$

((Note)) $v - \frac{1}{v} = e^u - e^{-u} = 2\sinh u$

24.9 Use of the contour DBCD



$$0 = \oint_C \frac{e^{ixt}}{\sqrt{1-t^2}} dt = \int_{1+i\infty}^{1} \frac{e^{ixt}}{\sqrt{1-t^2}} dt + \int_1^\infty \frac{e^{ixt}}{\sqrt{1-t^2}} dt$$

or

$$\frac{\pi}{2} H_0^{(1)}(x) + \int_1^\infty \frac{e^{ixt}}{\sqrt{1-t^2}} dt = 0$$

Now along the path BC

$$\sqrt{1-t^2} = -i\sqrt{t^2-1}$$

$$H_0^{(1)}(x) = -\frac{2i}{\pi}\int_1^\infty \frac{e^{ixt}}{\sqrt{t^2-1}}dt = -\frac{2i}{\pi}\int_1^\infty \frac{\cos(xt)+i\sin(xt)}{\sqrt{t^2-1}}dt$$

or

$$H_0^{(1)}(x) = -\frac{2i}{\pi}\int_1^\infty \frac{e^{ixt}}{\sqrt{t^2-1}}dt = -\frac{2i}{\pi}\int_1^\infty \frac{\cos(xt)}{\sqrt{t^2-1}}dt + \frac{2}{\pi}\int_1^\infty \frac{\sin(xt)}{\sqrt{t^2-1}}dt = J_0(x) + iN_0(x)$$

In the definition given by

$$H_0^{(1)}(x) = -\frac{2i}{\pi}\int_1^\infty \frac{e^{ixt}}{\sqrt{t^2-1}}dt$$

we put

$$t = \cosh s = \frac{1}{2}(e^s + e^{-s}), \qquad dt = \frac{1}{2}(e^s - e^{-s})ds,$$

$$\sqrt{t^2-1} = \frac{1}{2}(e^s - e^{-s}),$$

$$\int_1^\infty \frac{e^{ixt}}{\sqrt{t^2-1}}dt = \int_0^\infty e^{ix\cosh s}ds.$$

Then we have

$$H_0^{(1)}(x) = -\frac{2i}{\pi}\int_0^\infty e^{ix\cosh s}ds.$$

24.10 Asymptotic expansion

We start our discussion from

$$H_0^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\frac{1}{\sqrt{\pi}}e^{i(x-\frac{1}{4}\pi)}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}(1+\frac{iu}{2x})^{-1/2}du,$$

$$f(x) = \int_0^\infty \frac{e^{-u}}{\sqrt{u}}(1+\frac{iu}{2x})^{-1/2}du.$$

When $x \to \infty$, we have

$$f(\infty) = \int_0^\infty \frac{e^{-u}}{\sqrt{u}} du = 2\int_0^\infty e^{-y^2} dy = \sqrt{\pi}$$

Therefore we get

$$H_0^{(1)}(x) \to \sqrt{\frac{2}{\pi x}} e^{i(x - \frac{1}{4}\pi)}$$

Similarly,

$$H_0^{(2)}(x) = H_0^{(1)}(x)^* \to \sqrt{\frac{2}{\pi x}} e^{-i(x - \frac{1}{4}\pi)}$$

Since

$$J_0(x) = \frac{1}{2}[H_0^{(1)}(x) + H_0^{(2)}(x)]$$

$$N_0(x) = \frac{1}{2}[H_0^{(1)}(x) - H_0^{(2)}(x)]$$

then we have the asymptotic forms

$$J_0(x) \to \sqrt{\frac{2}{\pi x}} \cos(x - \frac{1}{4}\pi) \qquad \text{for } x \to \infty .$$

$$N_0(x) \to \sqrt{\frac{2}{\pi x}} \sin(x - \frac{1}{4}\pi) \qquad \text{for } x \to \infty .$$

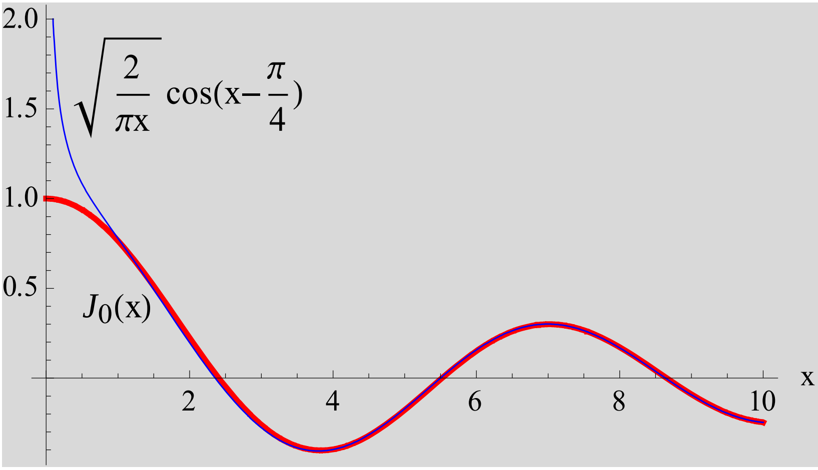


Fig. Plot of $J_0(x)$ and its asymptotic form.

APPENDIX
Mathematica

Bessel functions

BesselJ[n,z]	for $J_n(z)$
BesselI[n,z]	for $I_n(z)$
BesselK[n,z]	for $K_n(z)$
BesselY[n,z]	for $N_n(z)$ (or $Y_n(z)$)

Hankel functions

HankelH1[n,z]	for $H_n^{(1)}(z)$
HankelH2[n,z]	for $H_n^{(2)}(z)$

Spherical Bessel functions

SphericalBesselJ[n,z]	for $j_n(z)$
SphericalBesselI[n,z]	for $i_n(z)$
SphericalBesselK[n.z]	for $k_n(z)$
SphericalBesselY[n,z]	for $n_n(z)$

Spherical Hankel functions

SphericalHankelH1[n,z]	for $h_n^{(1)}(z)$
SphericalKankelH2[n,z]	for $h_n^{(2)}(z)$
