Chapter 27 Supplement
Stern-Gerlach experiment

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: November 19, 2010)

In 1921, the Stern-Gerlach experiment demonstrated the quantization of angular momentum .

Otto Stern (17 February 1888 – 17 August 1969) was a German physicist and Nobel laureate in physics. As an experimental physicist Stern contributed to the discovery of spin quantization in the Stern-Gerlach experiment with Walther Gerlach in 1922, demonstration of the wave nature of atoms and molecules; measurement of atomic magnetic moments; discovery of the proton's magnetic moment; and development of the molecular ray method which is utilized for the technique of molecular beam epitaxy. He was awarded the 1943 Nobel Prize in Physics, the first to be awarded since 1939.

http://en.wikipedia.org/wiki/Otto_Stern

Walt(h)er Gerlach (1 August 1889 - 10 August 1979) was a German physicist who co-discovered spin quantization in a magnetic field, the Stern-Gerlach effect. In 1920, he became a teaching assistant and lecturer at the Johann Wolfgang Goethe University of Frankfurt am Main. The next year, he took a position as extraordinarius professor at Frankfurt. It was in November 1921 that he and Otto Stern discovered space quantization in a magnetic field, known as the Stern-Gerlach effect.

27S.1 Stern-Gerlach (SG) experiment

We consider the Stern-Gerlach experiment, which provides a direct evidence of the quantization of magnetic moment and angular momentum. One way of measuring the angular momentum is by means of a Stern-Gerlach experiment. Suppose that we want to measure the angular momentum of the electrons in a given type of atom. A beam of these atoms is prepared by evaporation from the solid, and passing the evaporated atoms through a set of collimating slits. This beam then enters a region in which there is an inhomogeneous magnetic field that is normal to the direction of motion of atoms. The apparatus is shown schematically in Fig. The angular magnetic moment is related to the orbital angular momentum as

$$\boldsymbol{\mu}_L = -\frac{e}{2mc}\mathbf{L},$$

where e>0. In an inhomogeneous magnetic field, we have an interaction energy called the Zeeman energy,

$$V = -\boldsymbol{\mu}_L \cdot \mathbf{B}.$$

The atoms experience a force given by

$$\mathbf{F} = -\nabla V = -\nabla(-\boldsymbol{\mu}_L \cdot \mathbf{B}) = \nabla(\boldsymbol{\mu}_L \cdot \mathbf{B}) .$$

We consider the case when the magnetic field $\mathbf{B} = B_z\mathbf{e}_z$ is applied along the *z* axis. Then we have the force along the *z* axis,

$$F_z = \mu_{Lz} \frac{\partial B_z}{\partial z} = -\frac{e\hbar}{2mc}\frac{L_z}{\hbar}\frac{\partial B_z}{\partial z} = -\mu_B \frac{L_z}{\hbar}\frac{\partial B_z}{\partial z}$$

where μ_B $(=e\hbar/2mc)$ is the Bohr magneton. Thus, each atom experiences a force which is proportional to the *z* component of the orbital angular momentum. The beam is collected some distance from the magnet at a point that is far enough away so that atoms of different L_z is separated. By measuring the deflection one can calculate L_z.

$$\hat{L}_z|l,m\rangle = m\hbar|l,m\rangle$$

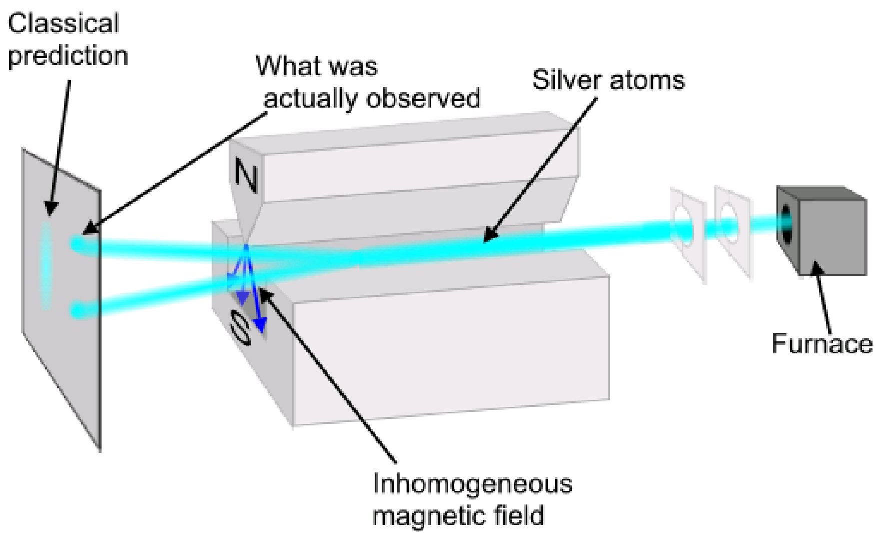
where $m = -l, -l+1, ..., l$.

The experiment can also be used to reveal the existence of electron spin. For example, if we send a beam of hydrogen atoms in their ground state, the beam split into two parts. Note that the spin magnetic moment is related to the spin angular momentum as

$$\boldsymbol{\mu}_S = -2\mu_B \frac{\mathbf{S}}{\hbar}$$

where $\boldsymbol{\mu}_S$ is the spin magnetic moment and $\mathbf{S}$ $(= \frac{\hbar}{2}\boldsymbol{\sigma})$ is the spin angular momentum, and

$$\hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle , \qquad \hat{S}_z|-\rangle = -\frac{\hbar}{2}|-\rangle .$$



http://en.wikipedia.org/wiki/Stern%E2%80%93Gerlach_experiment

Fig. Stern-Gerlach (SG) apparatus. A beam of particles with magnetic moment enters the inhomogeneous magnetic field. Classically, the beam is expected to fan out and a produce a continuous trace. In fact, the atomic beam is split into two beams, indicating that the magnetic moments of the atoms are quantized to two orientation in space.

27S.2 Stern-Gerlach for S = 1/2 with the magnetic field along the z axis

Spin angular momentum is related to the Paili matrices as

$$\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x, \qquad \hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y \qquad \hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z$$

The eigenkets of $\hat{S}_z$ are given by

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The Pauli matrices are defined as

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The commutation relations

$$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z \qquad [\hat{\sigma}_y, \hat{\sigma}_z] = 2i\hat{\sigma}_x, \qquad [\hat{\sigma}_z, \hat{\sigma}_x] = 2i\hat{\sigma}_y$$

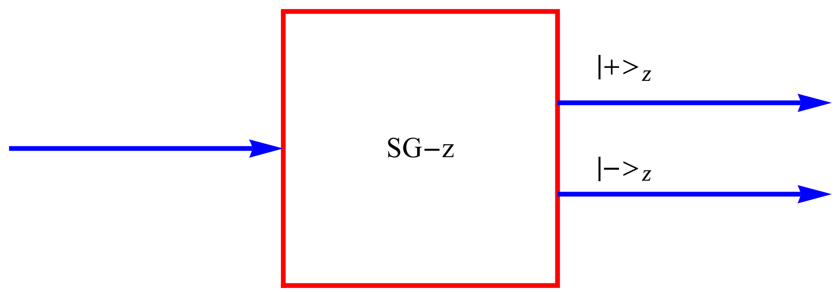
SG_Z stands for an apparatus with the inhomogeneous magnetic field along the *z* direction. We assume that $\frac{\partial B_z}{\partial z}>0$. The atom with $\mu_Z > 0$ ($S_Z < 0$) experiences a downward force, while the atom with $\mu_Z < 0$ ($S_Z > 0$) experiences a upward force, where the force F_z along the *z* axis is given by

$$F_z = 2\mu_B \frac{S_z}{\hbar}\frac{\partial B_z}{\partial z},$$

where

$$\boldsymbol{\mu} = -2\mu_B \frac{\mathbf{S}}{\hbar}$$

where μ_B is the Bohr magneton, and the magnetic moment $\boldsymbol{\mu}$ is antiparallel to the spin angular momentum $\boldsymbol{S}$.



The beam is then expected to get slit according to the values of μ (or S_Z). In other words, the SG apparatus measures the *z*-component of $\boldsymbol{\mu}$, or equivalently, the *z*-componenet of $\boldsymbol{S}$.

$$\hat{S}_z|+\rangle = \frac{\hbar}{2}\hat{\sigma}_z|+\rangle = \frac{\hbar}{2}|+\rangle$$

$$\hat{S}_z|-\rangle = \frac{\hbar}{2}\hat{\sigma}_z|-\rangle = -\frac{\hbar}{2}|-\rangle$$

where

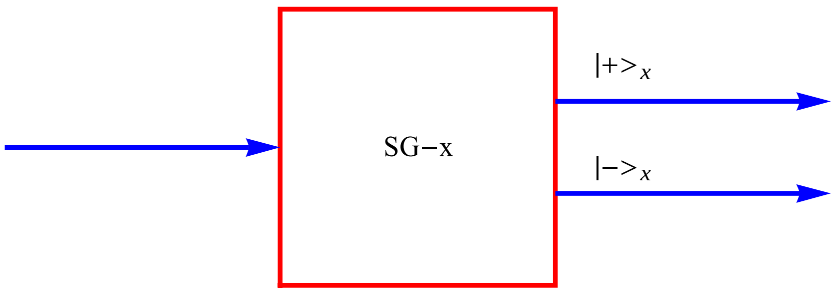
$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We have a closure relation.

$$|+\rangle\langle+|+|-\rangle\langle-|=\hat{1}$$

$$\hat{S}_z=\hat{S}_z(|+\rangle\langle+|+|-\rangle\langle-|)=\frac{\hbar}{2}|+\rangle\langle+|-\frac{\hbar}{2}|-\rangle\langle-|$$

27S.4 Stern-Gerlach for S = 1/2 with the magnetic field along the *x* axis



Here we discuss the expression for $|\pm\rangle_x$

$$\hat{\sigma}_x|\pm\rangle_x=\pm|\pm\rangle_x$$

$$\hat{\sigma}_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$|+\rangle_x=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}=\frac{1}{\sqrt{2}}(|+\rangle+|-\rangle)$$

$$|-\rangle_x=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}=\frac{1}{\sqrt{2}}(|+\rangle-|-\rangle)$$

((Mathematica))

```
Clear["Global`*"];

SuperStar /: expr_* := expr /. {Complex[a_, b_] :> Complex[a, -b]};

σx = {{0, 1}, {1, 0}};

eq1 = Eigensystem[σx]
```

{{-1, 1}, {{-1, 1}, {1, 1}}}

```
ψ2 = -Normalize[eq1[[2, 1]]]
```

$\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$

```
ψ1 = Normalize[eq1[[2, 2]]]
```

$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$

```
UT = {ψ1, ψ2}
```

$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$

Unitary operator U

```
U = Transpose[UT]
```

$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$

```
U // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Hermite conjugate of U

```
UH = UT*
```

$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$

```
UH // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
UH.U
```

{{1, 0}, {0, 1}}

```
U.UH
```

{{1, 0}, {0, 1}}

((**Summary**))

Eigenvalues: ±1

$$|\pm\rangle_x = \hat{U}|\pm\rangle$$

Eigenkets:

$$|+\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \hat{U}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \hat{U}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The Unitary operator:

$$\hat{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

((**Another method**))

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

or

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\lambda = \pm 1.$$

For $\lambda = 1$,

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_{1+} \\ c_{2+} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$|+\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Similarly,

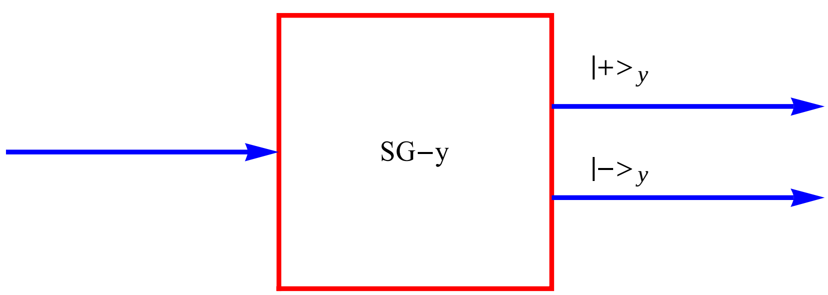
For $\lambda = -1$,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_{1-} \\ c_{2-} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$|-\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\hat{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

27S.5 Stern-Gerlach for S = 1/2 with the magnetic field along the *y* axis



Expression for $|\pm\rangle_y$

$$\hat{\sigma}_y|\pm\rangle_y = \pm|\pm\rangle_y$$

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\left|+\right\rangle_y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(\left|+\right\rangle + i\left|-\right\rangle),$$

$$\left|-\right\rangle_y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(\left|+\right\rangle - i\left|-\right\rangle)$$

((**Mathematica**))

```
Clear["Global`*"];

SuperStar /: expr_* := expr /. {Complex[a_, b_] :> Complex[a, -b]};

σy = {{0, -ⅈ}, {ⅈ, 0}};

eq1 = Eigensystem[σy]
```

{{-1, 1}, {{ⅈ, 1}, {-ⅈ, 1}}}

```
ψ2 = (-ⅈ) Normalize[eq1[[2, 1]]]
```

$\left\{\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}$

```
ψ1 = ⅈ Normalize[eq1[[2, 2]]]
```

$\left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}$

```
UT = {ψ1, ψ2}
```

$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}\right\}$

Unitary operator U

```
U = Transpose[UT]
```

$\left\{\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}\right\}$

```
U // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

Hermite conjugate of U

```
UH = UT*
```

$\left\{\left\{\frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}\right\}$

```
UH // MatrixForm
```

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

```
UH.U
```

{{1, 0}, {0, 1}}

```
U.UH
```

((**Summary**))

The eigenkets:

$$|+\rangle_y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \hat{U}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$|-\rangle_y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \hat{U}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

The Unitary operator:

$$\hat{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

((**Another method**))

Expression for $|\pm\rangle_y$

$$\hat{\sigma}_y|\pm\rangle_y = \pm|\pm\rangle_y$$

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

We assume that

$$|\pm\rangle_x = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

We solve the eigenvalue problem

$$\hat{\sigma}_y|\pm\rangle_y = \lambda|\pm\rangle_y$$

where λ is an eigenvalue.

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \lambda\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

or

$$\begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$M = \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix}$$

$$\det M = \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

or

$$\lambda = \pm 1$$

For $\lambda = 1$

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

or

$$C_1 = -iC_2$$

The normalization condition: $|C_1|^2 + |C_2|^2 = 1$. We choose $C_1 = -iC_2 = \frac{1}{\sqrt{2}}$. Then we have

$$|+\rangle_y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$$

Similarly, for λ = -1

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

or

$$C_1 = iC_2$$

The normalization condition: $|C_1|^2 + |C_2|^2 = 1$. We choose $C_1 = iC_2 = \frac{1}{\sqrt{2}}$. Then we have

$$|-\rangle_y = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -i\frac{i}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle)$$

Unitary operator $\hat{U}$

$$\hat{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

$$\hat{U}^+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

((**Note**))

$$\hat{U}^+\hat{\sigma}_y\hat{U} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{U}^+\hat{\sigma}_y{}^2\hat{U} = \hat{U}^+\hat{\sigma}_y\hat{U}\hat{U}^+\hat{\sigma}_y\hat{U} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{U}^+\hat{\sigma}_y{}^3\hat{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general

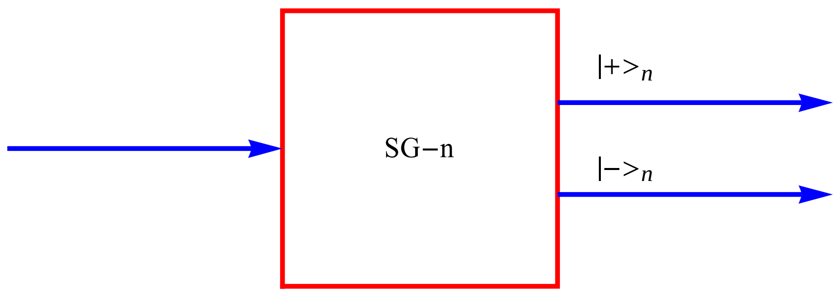
$$\hat{U}^+\hat{\sigma}_y{}^n\hat{U} = \begin{pmatrix} 1^n & 0 \\ 0 & (-1)^n \end{pmatrix}$$

$$\hat{U}^+ \exp(-\frac{i\theta}{2}\hat{\sigma}_y)\hat{U} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

From this we can calculate the matrix of $\hat{\sigma}_y$.

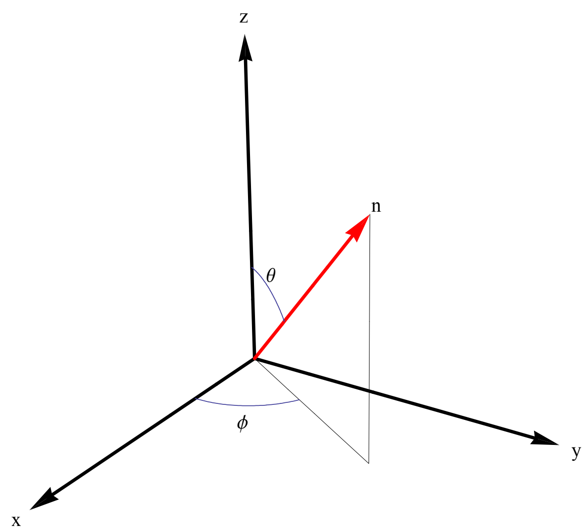
$$\exp(-\frac{i\theta}{2}\hat{\sigma}_y) = \hat{U}\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}\hat{U}^+ = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

27S.6 Stern-Gerlach for S = 1/2 with the magnetic field along the *n* direction



$$\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\hat{\sigma}_n = \hat{\sigma}\cdot\mathbf{n} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$



((**Eigenvalue problem**))

Here we use the rotation operator for j = 1/2.

$$\hat{\sigma}_n|+\rangle_n = |+\rangle_n \qquad \hat{\sigma}_n|-\rangle_n = -|-\rangle_n$$

Unitary operator $\hat{U}$ is given by the rotation operator for j = 1/2.

$$\hat{U} = D^{(1/2)}(\theta,\phi) = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos(\frac{\theta}{2}) & -e^{-i\frac{\phi}{2}}\sin(\frac{\theta}{2}) \\ e^{i\frac{\phi}{2}}\sin(\frac{\theta}{2}) & e^{i\frac{\phi}{2}}\cos(\frac{\theta}{2}) \end{pmatrix}$$

The eigenkets $|+\rangle_n$ and $|-\rangle_n$ are obtained as

$$|+\rangle_n = \hat{U}|+\rangle = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos(\frac{\theta}{2}) \\ e^{i\frac{\phi}{2}}\sin(\frac{\theta}{2}) \end{pmatrix},$$

and

$$\left|-\right\rangle_n = \hat{U}\left|-\right\rangle = \begin{pmatrix} -e^{-i\frac{\phi}{2}}\sin(\dfrac{\theta}{2}) \\ e^{i\frac{\phi}{2}}\cos(\dfrac{\theta}{2}) \end{pmatrix}.$$

We note that

$$\hat{\sigma}_n\hat{U}\left|+\right\rangle = \hat{U}\left|+\right\rangle \qquad \hat{U}^+\hat{\sigma}_n\hat{U}\left|+\right\rangle = \left|+\right\rangle$$

$$\hat{\sigma}_n\left|+\right\rangle_n = -\left|+\right\rangle_n$$

$$\hat{\sigma}_n\hat{U}\left|-\right\rangle = \hat{U}\left|-\right\rangle \qquad \hat{U}^+\hat{\sigma}_n\hat{U}\left|-\right\rangle = -\left|-\right\rangle$$

$$\hat{U}^+\hat{\sigma}_n\hat{U} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

((**Another method**))
The above formula can be derived without Mathematica.

$$(\hat{\sigma}\cdot\mathbf{n})\left|\psi\right\rangle = \lambda\left|\psi\right\rangle$$

Since $(\hat{\sigma}\cdot\mathbf{n})^2 = \hat{1}$,

$$(\hat{\sigma}\cdot\mathbf{n})^2\left|\psi\right\rangle = \lambda(\hat{\sigma}\cdot\mathbf{n})\left|\psi\right\rangle = \lambda^2\left|\psi\right\rangle \text{ (eigenvalue problem)}$$

We get $\lambda^2 = 1$, or $\lambda = \pm 1$.

Thus

$$(\hat{\sigma}\cdot\mathbf{n})\left|+\right\rangle_n = \left|+\right\rangle_n$$

$$(\hat{\sigma}\cdot\mathbf{n})\left|-\right\rangle_n = -\left|-\right\rangle_n$$

$$\left|+\right\rangle_n = \hat{U}\left|+\right\rangle = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix}$$

$$\left|-\right\rangle_n = \hat{U}\left|-\right\rangle = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix}$$

where

$$\begin{pmatrix} U_{11}{}^* & U_{21}{}^* \\ U_{12}{}^* & U_{22}{}^* \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

Derivation of the eigenket $|+\rangle_n$

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix} = \begin{pmatrix} U_{11} \\ U_{21} \end{pmatrix}$$

or

$$\cos\theta U_{11} + \sin\theta e^{-i\phi} U_{21} = U_{11}$$

$$\sin\theta e^{i\phi} U_{11} - \cos\theta U_{21} = U_{21}$$

or

$$U_{21} = \frac{\sin\theta e^{i\phi}}{(1+\cos\theta)} U_{11} = \tan\frac{\theta}{2} e^{i\phi} U_{11}$$

Since

$$|U_{11}|^2 + |U_{21}|^2 = 1 \text{ (normalization),}$$

we get

$$|U_{11}|^2 = \cos^2\frac{\theta}{2}.$$

When we choose

$$U_{11} = e^{-i\frac{\phi}{2}} \cos\frac{\theta}{2}$$

we have

$$U_{21} = e^{i\frac{\phi}{2}} \sin\frac{\theta}{2}$$

Derivation of the eigenket $|+\rangle_n$

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix} = -\begin{pmatrix} U_{12} \\ U_{22} \end{pmatrix}$$

$$\cos\theta U_{12} + \sin\theta e^{-i\phi} U_{22} = -U_{12}$$

$$\sin\theta e^{i\phi} U_{12} - \cos\theta U_{22} = -U_{22}$$

$$U_{22} = -\frac{(1+\cos\theta)e^{i\phi}}{\sin\theta} U_{12} = -\cot(\frac{\theta}{2}) e^{i\phi} U_{12}$$

Since

$$|U_{12}|^2 + |U_{22}|^2 = 1 \text{ (normalization)},$$

we get

$$|U_{22}|^2 = \cos^2(\frac{\theta}{2}).$$

When we choose

$$U_{22} = e^{i\frac{\phi}{2}} \cos\frac{\theta}{2}$$

we have

$$U_{12} = -e^{-i\frac{\phi}{2}} \sin\frac{\theta}{2}$$

27S.7 Derivation of the eigenkets in each SG configuration from the above formula

(i) SG_x experiment:

$\theta = \pi/2$ and $\phi = 0$.

$$|+\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad |-\rangle_x = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

The eigenket of $|-\rangle_x$ thus obtained is different from the conventional eigenket

$$|-\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

except for the phase factor exp(iπ).

(ii) SG_y experiment:

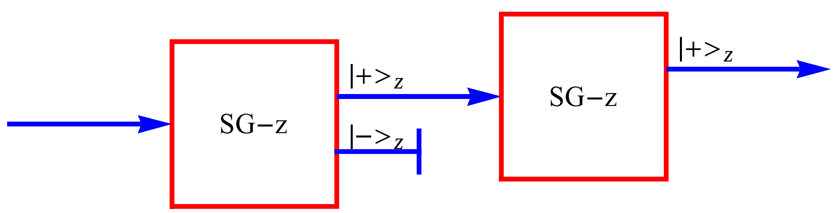
$\theta = \pi/2$ and $\phi = \pi/2$.

$$|+\rangle_y = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{-i\pi/4} \\ \frac{1}{\sqrt{2}} e^{i\pi/4} \end{pmatrix} = e^{-i\pi/4} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \qquad |-\rangle_y = \begin{pmatrix} -\frac{1}{\sqrt{2}} e^{-i\pi/4} \\ \frac{1}{\sqrt{2}} e^{i\pi/4} \end{pmatrix} = -e^{-i\pi/4} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{pmatrix}$$

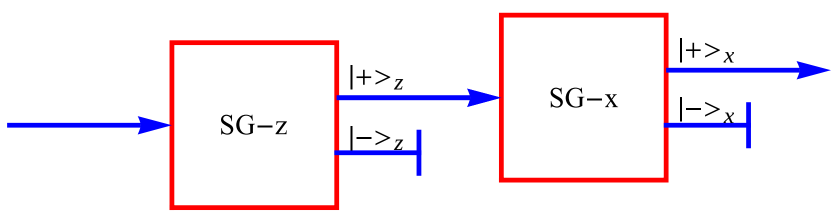
The eigenket of $|+\rangle_y$ is different from the conventional $|+\rangle_y$ except for the phase factor $e^{-i\pi/4}$. The eigenket of $|-\rangle_y$ is different from the conventional $|-\rangle_y$ except for the phase factor (-$e^{-i\pi/4}$).

27S.8 SG Thinking experiment

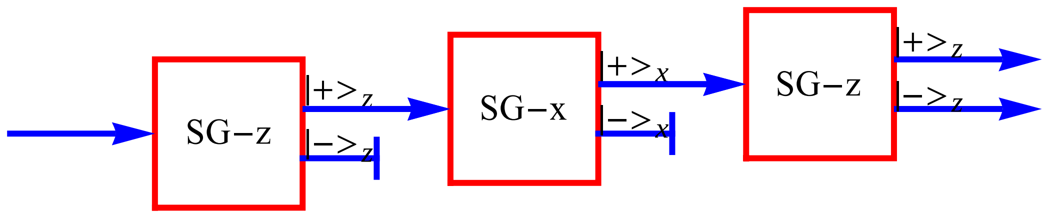
1. Experiment



2. Experiment



3. Experiment



Analysis of experiment-3

$$|+\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad |-\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P_1 = \left|{}_x\langle + | + \rangle\right|^2 = \left|\langle + | + \rangle_x\right|^2 = \frac{1}{2}$$

$$P_2 = \left|{}_x\langle - | + \rangle\right|^2 = \left|\langle + | - \rangle_x\right|^2 = \frac{1}{2}$$

$$P_3 = \left|\langle + | + \rangle_x\right|^2 = \frac{1}{2}$$

$$P_4 = \left|\langle - | + \rangle_x\right|^2 = \frac{1}{2}$$

$$\langle S_z \rangle = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0$$

or

$$\langle S_z \rangle = \frac{\hbar}{2}P_3 + (-\frac{\hbar}{2})P_4 = 0$$

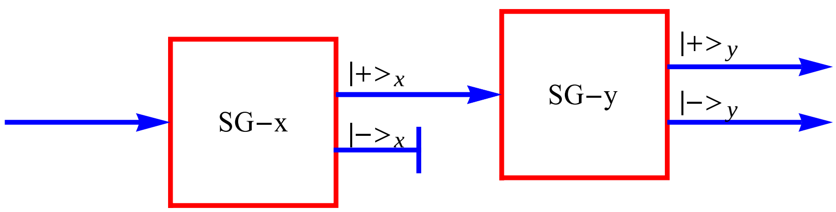
$$\langle S_z{}^2 \rangle = \frac{\hbar^2}{4}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\hbar^2}{4}$$

or

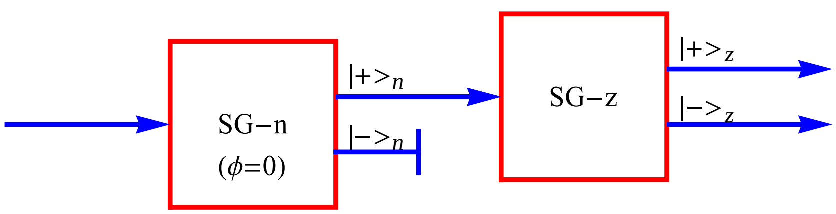
$$\langle S_z^2 \rangle = \frac{\hbar^2}{4} P_3 + \frac{\hbar^2}{4} P_4 = \frac{\hbar^2}{4}$$

$$\Delta S_z^2 = \langle S_z^2 \rangle - \langle S_z \rangle^2 = \frac{\hbar^2}{4}$$

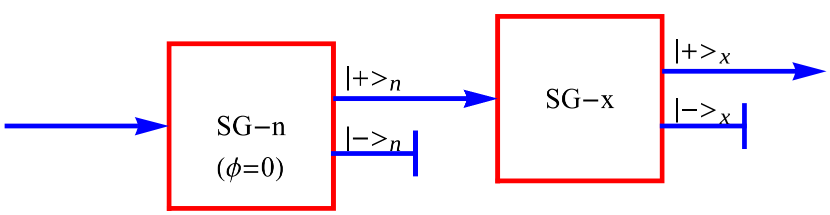
4. Experiment



5. Experiment



6. Experiment



Analysis of experiment-6

$$|+\rangle_n = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad |+\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad |+\rangle_x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$ {}_x\langle +|+\rangle_n = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \frac{1}{\sqrt{2}}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}) = \frac{1}{\sqrt{2}}[\cos(\frac{\theta}{2} - \frac{\pi}{4})] $$

$$ {}_x\langle -|+\rangle_n = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \frac{1}{\sqrt{2}}(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}) = \frac{1}{\sqrt{2}}[\cos(\frac{\theta}{2} + \frac{\pi}{4})] $$

$$ P_{+x} = \left|{}_x\langle +|+\rangle_n\right|^2 = \frac{1}{4}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})^2 = \frac{1}{4}(1 + \sin\theta)\,, $$

$$ P_{-x} = \left|{}_x\langle -|+\rangle_n\right|^2 = \frac{1}{4}(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})^2 = \frac{1}{4}(1 - \sin\theta), $$

$$ \langle S_x\rangle = {}_n\langle +|\hat{S}_x|+\rangle_n = \frac{\hbar}{2}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} $$

$$ = \hbar\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\hbar}{2}\sin\theta $$

or

$$ \langle S_x\rangle = \frac{\hbar}{2}P_{+x} + (-\frac{\hbar}{2})P_{-x} = \frac{\hbar}{2}\sin\theta $$

$$ \langle S_z\rangle = {}_n\langle +|\hat{S}_z|+\rangle_n = \frac{\hbar}{2}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} $$

$$ = \frac{\hbar}{2}(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}) = \frac{\hbar}{2}\cos\theta $$

27S.9 Stern-Gerlach experiment with $J = 1$

$$\hat{R}_y(\theta) = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & e^{i\phi}\frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}$$

Using the Mathematica, one can get the matrix representation

$$\hat{J}_n = \hat{J}\cdot\mathbf{n} = \hat{J}_x\cdot\mathbf{n}_x + \hat{J}_y\cdot\mathbf{n}_y + \hat{J}_z\cdot\mathbf{n}_z = \hbar\begin{pmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{2}}e^{-i\phi i} & 0 \\ \frac{\sin\theta}{\sqrt{2}}e^{i\phi i} & 0 & \frac{\sin\theta}{\sqrt{2}}e^{-i\phi i} \\ 0 & \frac{\sin\theta}{\sqrt{2}}e^{i\phi i} & -\cos\theta \end{pmatrix}$$

The above result can be obtained by solving the eigenvalue problem:

$$(\hat{J}\cdot\mathbf{n})\left|1\right\rangle_n = \hbar\left|1\right\rangle_n, \qquad (\hat{J}\cdot\mathbf{n})\left|0\right\rangle_n = 0, \qquad (\hat{J}\cdot\mathbf{n})\left|-1\right\rangle_n = -\hbar\left|-1\right\rangle_n$$

Use the Mathematica to obtain the eigenkets and the eigenvalues:
Eigensystem[J_{n}]

$$\left|1\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \left|0\right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \left|-1\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For $\theta = \pi/2$ and $\phi = 0$ (corresponding to the x axis)

$$\left|1\right\rangle_x = \hat{R}_y(\frac{\pi}{2})\left|1\right\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} = \frac{1}{2}\left|1\right\rangle + \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{2}\left|-1\right\rangle,$$

$$\left|0\right\rangle_x = \hat{R}_y(\frac{\pi}{2})\left|0\right\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = -\frac{1}{\sqrt{2}}\left|1\right\rangle + \frac{1}{\sqrt{2}}\left|-1\right\rangle,$$

$$\left|-1\right\rangle_x = \hat{R}_y(\frac{\pi}{2})\left|-1\right\rangle = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} = \frac{1}{2}\left|1\right\rangle - \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{2}\left|-1\right\rangle$$

More generally for the unit vector ***n*** in the *x*-*z* plane,

$$\left|1\right\rangle_{\mathbf{n}} = \frac{1+\cos\theta}{2}\left|1\right\rangle + \frac{\sin\theta}{\sqrt{2}}\left|0\right\rangle + \frac{1-\cos\theta}{2}\left|-1\right\rangle,$$

$$\left|0\right\rangle_{\mathbf{n}} = -\frac{\sin\theta}{\sqrt{2}}\left|1\right\rangle + \cos\theta\left|0\right\rangle + \frac{\sin\theta}{\sqrt{2}}\left|-1\right\rangle,$$

$$\left|-1\right\rangle_{\mathbf{n}} = \frac{1-\cos\theta}{2}\left|1\right\rangle - \frac{\sin\theta}{\sqrt{2}}\left|0\right\rangle + \frac{1+\cos\theta}{2}\left|-1\right\rangle,$$

or, inversely

$$\begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1+\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ -\frac{\sin\theta}{\sqrt{2}} & \cos\theta & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}$$

When $\theta = \pi/2$, this matrix is expressed by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

or

$$\left|1\right\rangle = \frac{1}{2}\left|1\right\rangle_x - \frac{1}{\sqrt{2}}\left|0\right\rangle_x + \frac{1}{2}\left|-1\right\rangle_x$$

$$\left|0\right\rangle = \frac{1}{\sqrt{2}}\left|1\right\rangle_x - \frac{1}{2}\left|-1\right\rangle_x$$

$$\left|-1\right\rangle = \frac{1}{2}\left|1\right\rangle_x + \frac{1}{\sqrt{2}}\left|0\right\rangle_x + \frac{1}{2}\left|-1\right\rangle_x$$

more generally

$$\left|1\right\rangle = \frac{1+\cos\theta}{2}\left|1\right\rangle_{\mathbf{n}} - \frac{\sin\theta}{\sqrt{2}}\left|0\right\rangle_{\mathbf{n}} + \frac{1-\cos\theta}{2}\left|-1\right\rangle_{\mathbf{n}}$$

$$\left|0\right\rangle = \frac{\sin\theta}{\sqrt{2}}\left|1\right\rangle_{\mathbf{n}} + \cos\theta\left|0\right\rangle_{\mathbf{n}} - \frac{\sin\theta}{\sqrt{2}}\left|-1\right\rangle_{\mathbf{n}}$$

$$\left|-1\right\rangle = \frac{1-\cos\theta}{2}\left|1\right\rangle_{\mathbf{n}} + \frac{\sin\theta}{\sqrt{2}}\left|0\right\rangle_{\mathbf{n}} + \frac{1+\cos\theta}{2}\left|-1\right\rangle_{\mathbf{n}}$$

For $\theta = \pi/2$ and $\phi = \pi/2$ (corresponding to the y axis)

$$\hat{R}\left|1\right\langle = \begin{pmatrix} -i/2 \\ 1/\sqrt{2} \\ i/2 \end{pmatrix} = -i\begin{pmatrix} 1/2 \\ i/\sqrt{2} \\ -1/2 \end{pmatrix},$$

$$\hat{R}\left|0\right\rangle = \begin{pmatrix} i/\sqrt{2} \\ 0 \\ i/\sqrt{2} \end{pmatrix} = i\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix},$$

$$\hat{R}\left|-1\right\rangle = \begin{pmatrix} -i/2 \\ -1/\sqrt{2} \\ i/2 \end{pmatrix} = -i\begin{pmatrix} 1/2 \\ -i/\sqrt{2} \\ -1/2 \end{pmatrix}.$$

Conventionally we use

$$\left|1\right\rangle_y = \begin{pmatrix} 1/2 \\ i/\sqrt{2} \\ -1/2 \end{pmatrix},$$

$$\left|0\right\rangle_y = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix},$$

$$|-1\rangle_y = \begin{pmatrix} 1/2 \\ -i/\sqrt{2} \\ -1/2 \end{pmatrix}$$

or inversely,

$$|1\rangle = \frac{1}{2}|1\rangle_y + \frac{1}{\sqrt{2}}|0\rangle_y + \frac{1}{2}|-1\rangle_y$$

$$|0\rangle = -\frac{i}{\sqrt{2}}|1\rangle_y + \frac{i}{\sqrt{2}}|-1\rangle_y$$

$$|-1\rangle = -\frac{1}{2}|1\rangle_y + \frac{1}{\sqrt{2}}|0\rangle_y - \frac{1}{2}|-1\rangle_y$$

Calculation of the rotation matrix with $J = 1$ without the use of Mathematica

Taylor expansion:

$$\exp(-\frac{i}{\hbar}\theta\hat{J}_y) = 1 + \frac{1}{1!}(-\frac{i}{\hbar}\theta\hat{J}_y) + \frac{1}{2!}(-\frac{i}{\hbar}\theta\hat{J}_y)^2 + \frac{1}{3!}(-\frac{i}{\hbar}\theta\hat{J}_y)^3 + \frac{1}{4!}(-\frac{i}{\hbar}\theta\hat{J}_y)^4 + ..$$

where

$$\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i}$$

Note that

$$\hat{J}_+|1\rangle = 0, \qquad \hat{J}_+|0\rangle = \sqrt{2}\hbar|1\rangle, \qquad \hat{J}_+|-1\rangle = \sqrt{2}\hbar|0\rangle$$

$$\hat{J}_-|1\rangle = \sqrt{2}\hbar|0\rangle, \qquad \hat{J}_-|0\rangle = \sqrt{2}\hbar|-1\rangle, \qquad \hat{J}_-|-1\rangle = 0$$

$$\hat{J}_y|1\rangle = \frac{i\hbar}{\sqrt{2}}|0\rangle, \qquad \hat{J}_y|0\rangle = \frac{-i\hbar}{\sqrt{2}}(|1\rangle - |-1\rangle), \qquad \hat{J}_y|-1\rangle = -\frac{i\hbar}{\sqrt{2}}|0\rangle$$

$$\hat{J}_y = \hbar\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \quad \hat{J}_y{}^2 = -\hbar^2\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\hat{J}_y{}^3 = \hbar^2 \hat{J}_y, \qquad \hat{J}_y{}^4 = \hat{J}_y{}^3 \hat{J}_y = \hbar^2 \hat{J}_y \hat{J}_y = \hbar^2 \hat{J}_y{}^2,$$

$$\hat{J}_y{}^5 = \hat{J}_y{}^4 \hat{J}_y = \hbar^2 \hat{J}_y{}^2 \hat{J}_y = \hbar^2 \hat{J}_y{}^3 = \hbar^4 \hat{J}_y$$

Therefore

$$\begin{aligned} \exp(-\frac{i}{\hbar}\theta\hat{J}_y) &= 1 + \frac{\hat{J}_y}{\hbar}[(-\theta) + \frac{1}{3!}(-i\theta)^3 + \frac{1}{5!}(-i\theta)^5 + ..] \\ &+ \frac{\hat{J}_y{}^2}{\hbar^2}[\frac{1}{2!}(-i\theta)^2 + \frac{1}{4!}(-i\theta)^4 + ..] \\ &= 1 - \frac{\hat{J}_y}{\hbar}(i\sin\theta) + \frac{\hat{J}_y{}^2}{\hbar^2}(\cos\theta - 1) \\ &= \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix} \end{aligned}$$

We also get

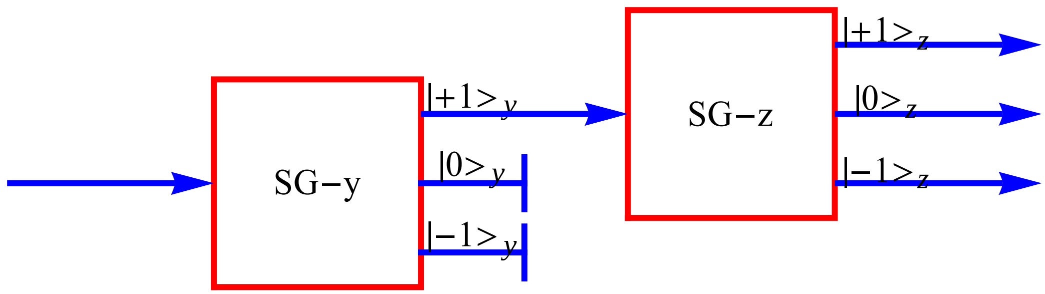
$$\exp(-\frac{i}{\hbar}\phi\hat{J}_z) = \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix}$$

which is a diagonal matrix.

27S.10 Examples of SG experiments

---

1. A spin-1 particle exists an SG$_y$ device in a state with $S_y = \hbar$. The beam then enters an SG$_z$ device. What is the probability that the measurement of S_z yields the value 0, $+\hbar$, and -$\hbar$?



$$\left|1\right\rangle_y = \begin{pmatrix} 1/2 \\ i/\sqrt{2} \\ -1/2 \end{pmatrix} = \frac{1}{2}\left|1\right\rangle + \frac{i}{\sqrt{2}}\left|0\right\rangle + \frac{-1}{2}\left|-1\right\rangle$$

The probability for finding the state $\left|1\right\rangle$ is

$$P_1 = \left|{}_y\left\langle 1 | 1 \right\rangle\right|^2 = \left|\left\langle 1 | 1 \right\rangle_y\right|^2 = \frac{1}{4}$$

The probability for finding the state $\left|0\right\rangle$ is

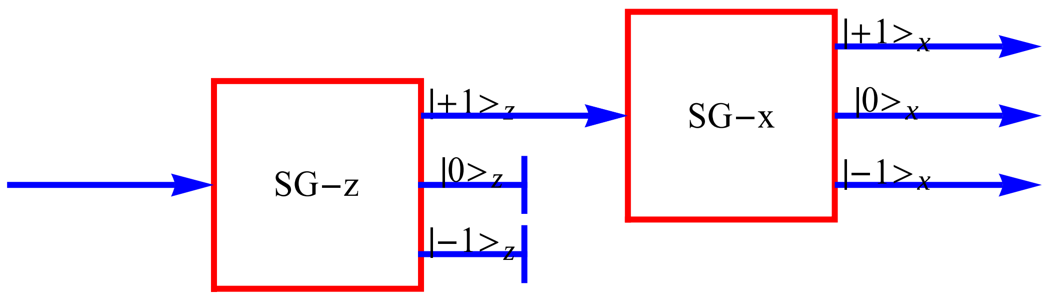
$$P_1 = \left|{}_y\left\langle 1 | 0 \right\rangle\right|^2 = \left|\left\langle 0 | 1 \right\rangle_y\right|^2 = \frac{1}{2}$$

The probability for finding the state $\left|-1\right\rangle$ is

$$P_{-1} = \left|{}_y\left\langle 1 | -1 \right\rangle\right|^2 = \left|\left\langle -1 | 1 \right\rangle_y\right|^2 = \frac{1}{4}$$

---

2. ((Townsend 3.16)) A spin-1 particle exists an SG$_z$ device in a state with $S_z = \hbar$. The beam then enters an SG$_x$ device. What is the probability that the measurement of S_x yields the value 0, $+\hbar$, and -$\hbar$?



$$|0\rangle_x = -\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|-1\rangle$$

The probability for finding the state $|1\rangle_x$ is

$$P_1 = \left|{}_x\langle 0|1\rangle\right|^2 = \left|\langle 1|0\rangle_x\right|^2 = \frac{1}{2}$$

The probability for finding the state $|0\rangle_x$ is

$$P_1 = \left|{}_x\langle 0|0\rangle\right|^2 = \left|\langle 0|0\rangle_x\right|^2 = 0$$

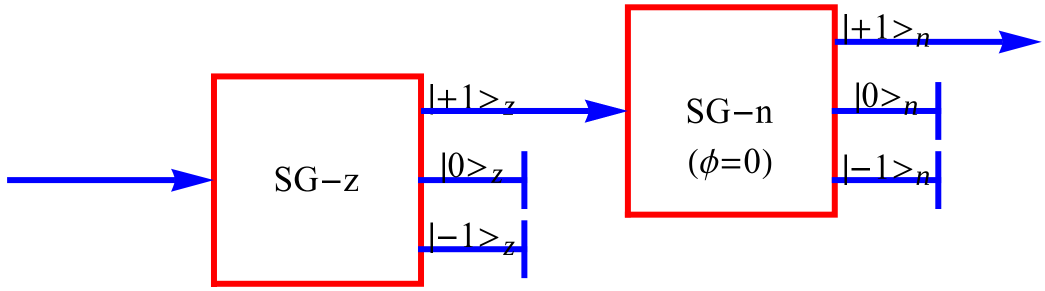
The probability for finding the state $|-1\rangle_x$ is

$$P_{-1} = \left|{}_x\langle 0|-1\rangle\right|^2 = \left|\langle -1|0\rangle_x\right|^2 = \frac{1}{2}$$

---

3. ((Shankhar)) A beam of spin 1 particles, moving along the *y* axis, is incident on two collinear SG apparatuses, the first with *B* along the *z* axis and the second with *B* along the z' axis, which lies in the *x*-*z* plane at an angle *θ* relative to the *z* axis. Both apparatuses transmit only the uppermost beams. What fraction leaving the first will pass the second?

The intial state after passing the first SG_z, is $|1\rangle_z = |1\rangle$



We note that

$$|1\rangle_{\mathbf{n}} = \frac{1+\cos\theta}{2}|1\rangle + \frac{\sin\theta}{\sqrt{2}}|0\rangle + \frac{1-\cos\theta}{2}|-1\rangle ,$$

$$|0\rangle_{\mathbf{n}} = -\frac{\sin\theta}{\sqrt{2}}|1\rangle + \cos\theta|0\rangle + \frac{\sin\theta}{\sqrt{2}}|-1\rangle ,$$

$$|-1\rangle_{\mathbf{n}} = \frac{1-\cos\theta}{2}|1\rangle - \frac{\sin\theta}{\sqrt{2}}|0\rangle + \frac{1+\cos\theta}{2}|-1\rangle ,$$

The probability for finding the state $|1\rangle_n$ is

$$P_1 = \left|{}_n\langle 1|1\rangle\right|^2 = \left|\langle 1|1\rangle_n\right|^2 = \frac{1}{4}(1+\cos\theta)^2 .$$

The probability for finding the state $|0\rangle_n$ is

$$P_0 = \left|{}_n\langle 0|1\rangle\right|^2 = \left|\langle 1|0\rangle_n\right|^2 = \frac{1}{2}\sin^2\theta$$

The probability for finding the state $|-1\rangle_n$ is

$$P_{-1} = \left|{}_n\langle -1|1\rangle\right|^2 = \left|\langle 1|-1\rangle_n\right|^2 = \frac{1}{4}(1-\cos\theta)^2$$

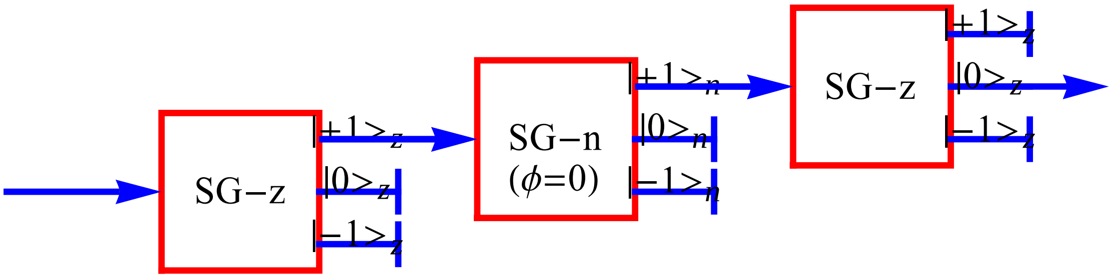
The total probability is

$$P_1 + P_0 + P_{-1} = 1$$

---

4. ((Townsend 3.20)) A beam of spin-1 particle is sent through a series of three Stern-Gerlach measuring devices. The first SG$_z$ device transmits particles with $S_z = \hbar$ and filters out particles with $S_z = 0$ and $S_z = -\hbar$. The second

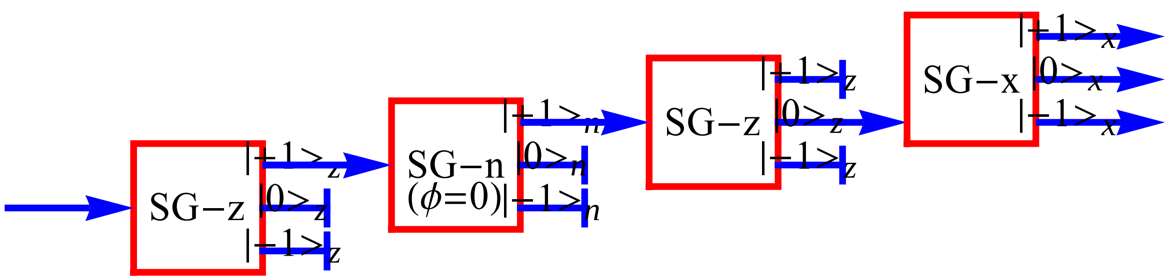
device, an SG_n device, transmits particles with $S_n = \hbar$ and filters out particles with $S_n = 0$ and $S_n = -\hbar$, where the axis *n* makes an angle θ ($0 \le \theta \le \pi/2$) in the *x*-*z* plane with respect to the *z* axis. A last SG_z device transmits particles with $S_z = 0$ and filters out particles with $S_z = \hbar$ and $S_z = -\hbar$.



(a) **What fraction of the particles transmitted by the first SG_z device will survive the third measurement?**

(b) **How must the angle θ of the SG_n device be oriented so as to maximize the number of particles that are transmitted by the final SG_z device? What fraction of the particles survive the third measurements for this value of θ?**

(c) **What fractions of the particles with $S_x = \hbar$, $S_x = 0$, and $S_x = -\hbar$, respectively, survive after the fourth device, SG_x which device transmits particles with $S_x = \hbar$, $S_x = 0$, and $S_x = -\hbar$?**



We note that

$$|1\rangle_n = \frac{1+\cos\theta}{2}|1\rangle + \frac{\sin\theta}{\sqrt{2}}|0\rangle + \frac{1-\cos\theta}{2}|-1\rangle ,$$

$$|0\rangle_n = -\frac{\sin\theta}{\sqrt{2}}|1\rangle + \cos\theta|0\rangle + \frac{\sin\theta}{\sqrt{2}}|-1\rangle ,$$

$$|-1\rangle_n = \frac{1-\cos\theta}{2}|1\rangle - \frac{\sin\theta}{\sqrt{2}}|0\rangle + \frac{1+\cos\theta}{2}|-1\rangle ,$$

$$|1\rangle_x = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle,$$

$$|0\rangle_x = -\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|-1\rangle,$$

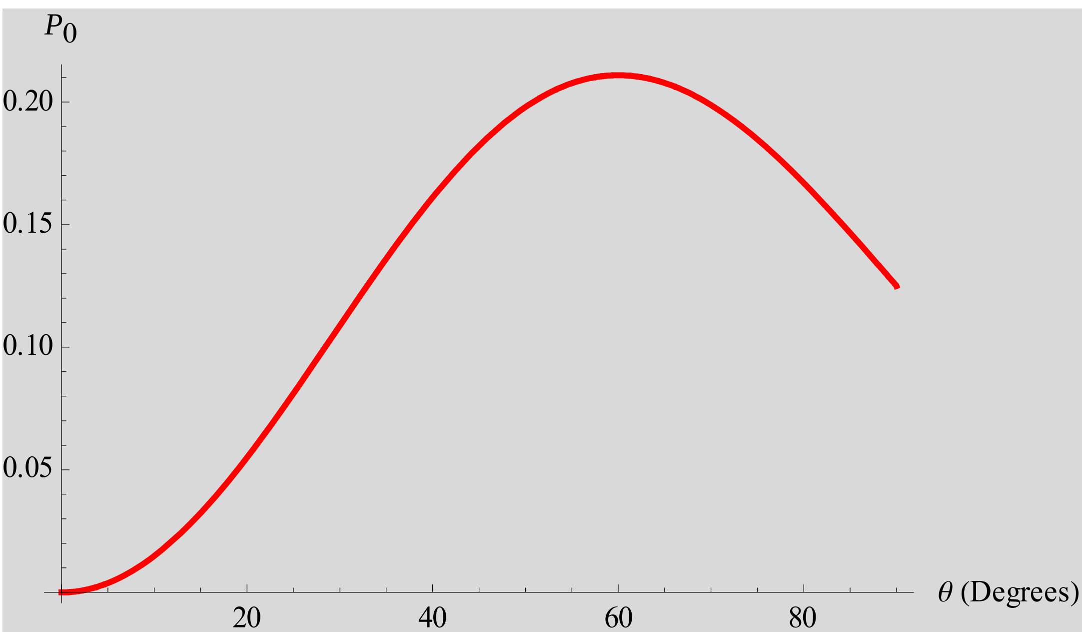
$$|-1\rangle_x = \frac{1}{2}|1\rangle - \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle$$

(a)

The fraction of the particles transmitted by the first SG_Z device will survive the third measurement is

$$P_0 = \left|\langle 1|1\rangle_n\right|^2 \left|{}_n\langle 1|0\rangle\right|^2 = \left|\langle 0|1\rangle_n\right|^2 = \frac{\sin^2\theta(1+\cos\theta)^2}{8}$$

(b)



$$\frac{dP_0}{d\theta} = 2\cos^5\frac{\theta}{2}\sin\frac{\theta}{2}(2\cos\theta - 1).$$

P_0 takes a maximum (= 0.210938) at $\theta = 60°$

(c)

The fractions of the particles with $S_x = \hbar$,

$$P_0 \left| \langle 0 | 1 \rangle_x \right|^2 = \frac{P_0}{2}$$

The fractions of the particles with $S_x = 0$,

$$P_0 \left| \langle 0 | 0 \rangle_x \right|^2 = 0$$

The fractions of the particles with $S_x = -\hbar$,

$$P_0 \left| \langle 0 | -1 \rangle_x \right|^2 = \frac{P_0}{2}$$

5.

A spin-1 particle is in the state

$$|\psi\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix} = \frac{1}{\sqrt{14}} \left(|1\rangle + 2|0\rangle + 3i|-1\rangle \right)$$

(a) What are the probabilities that a measurement of S_z will yield the values $\hbar$, 0, or $-\hbar$ for this state?

(b) What is $\langle S_z \rangle$?

(c) What is the probability that a measurement of S_x will yield the values $\hbar$, 0, or $-\hbar$ for this state?

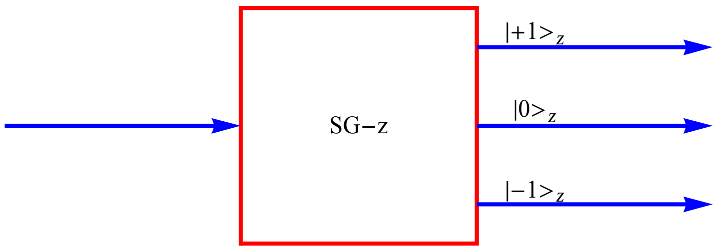
(d) What is $\langle S_x \rangle$ for this state?

$$|1\rangle_x = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-1\rangle,$$

$$|0\rangle_x = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = -\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|-1\rangle,$$

$$\left|-1\right\rangle_x = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1.2 \end{pmatrix} = \frac{1}{2}\left|1\right\rangle - \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{2}\left|-1\right\rangle$$

(a) and (b)



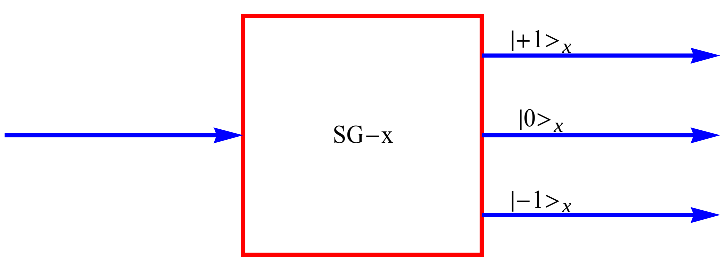
$$P(\left|1\right\rangle) = \left|\left\langle 1\middle|\psi\right\rangle\right|^2 = \frac{1}{14}$$

$$P(\left|0\right\rangle) = \left|\left\langle 0\middle|\psi\right\rangle\right|^2 = \frac{4}{14} = \frac{2}{7}$$

$$P(\left|-1\right\rangle) = \left|\left\langle -1\middle|\psi\right\rangle\right|^2 = \frac{9}{14}$$

$$\left\langle S_z\right\rangle = \hbar P(\left|1\right\rangle) + 0\hbar P(\left|0\right\rangle) - \hbar P(\left|-1\right\rangle)$$
$$= \frac{\hbar}{14} - \frac{9\hbar}{14} = -\frac{8\hbar}{14} = -\frac{4\hbar}{7} = -0.57143\hbar$$

(c) and (d)



$$P(\left|1\right\rangle_x) = \left|{}_x\left\langle 1\middle|\psi\right\rangle\right|^2 = \frac{9+2\sqrt{2}}{28}$$

$$P(\left|0\right\rangle_x) = \left|{}_x\left\langle 0\middle|\psi\right\rangle\right|^2 = \frac{5}{14}$$

$$P(|-1\rangle_x) = |{}_x\langle -1|\psi\rangle|^2 = \frac{9-2\sqrt{2}}{28}$$

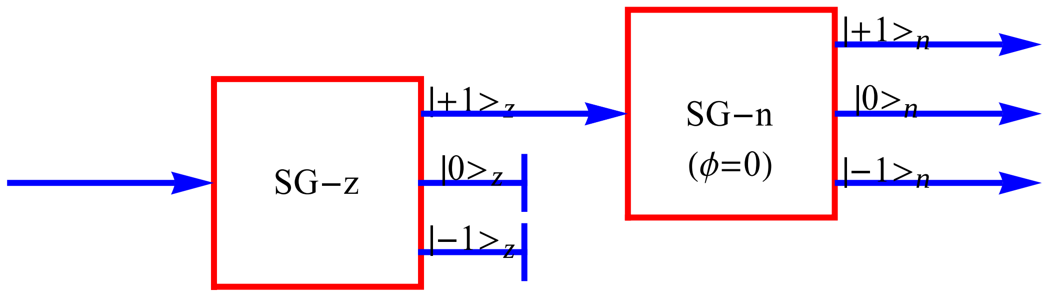
$$\begin{aligned}\langle S_x\rangle &= \hbar P(|1\rangle_x) + 0\hbar P(|0\rangle_x) - \hbar P(|-1\rangle_x) \\ &= \hbar[P(|1\rangle_x) - P(|-1\rangle_x)] \\ &= \frac{\sqrt{2}}{7}\hbar = 0.20231\hbar\end{aligned}$$

27S.11 Feynman's thinking SG experiment

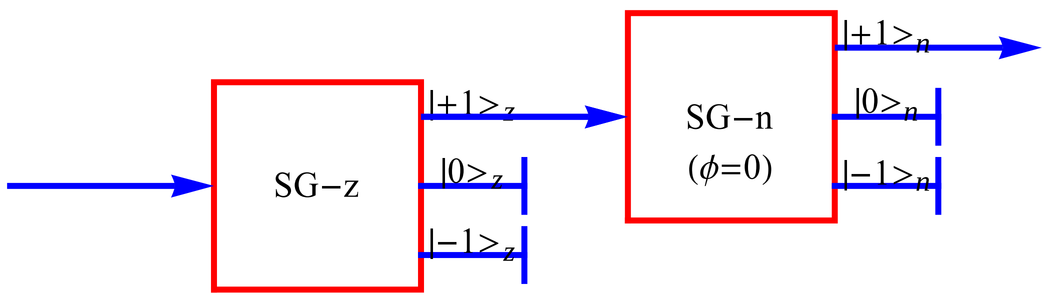
Richard Phillips Feynman (pronounced /ˈfaɪnmən/, May 11, 1918 – February 15, 1988) was an American physicist known for his work in the path integral formulation of quantum mechanics, the theory of quantum electrodynamics and the physics of the superfluidity of supercooled liquid helium, as well as in particle physics (he proposed the parton model). For his contributions to the development of quantum electrodynamics, Feynman, jointly with Julian Schwinger and Sin-Itiro Tomonaga, received the Nobel Prize in Physics in 1965. He developed a widely used pictorial representation scheme for the mathematical expressions governing the behavior of subatomic particles, which later became known as Feynman diagrams. During his lifetime, Feynman became one of the best-known scientists in the world.
http://en.wikipedia.org/wiki/Richard_Feynman

http://www.atomicarchive.com/Bios/FeynmanPhoto.shtml

1. Experiment-1

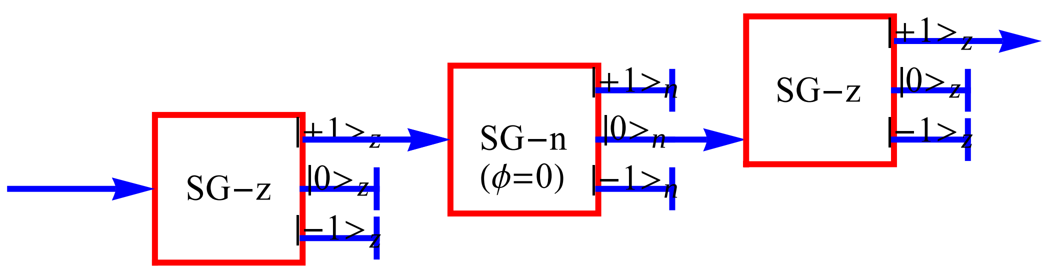


2. Experiment-2

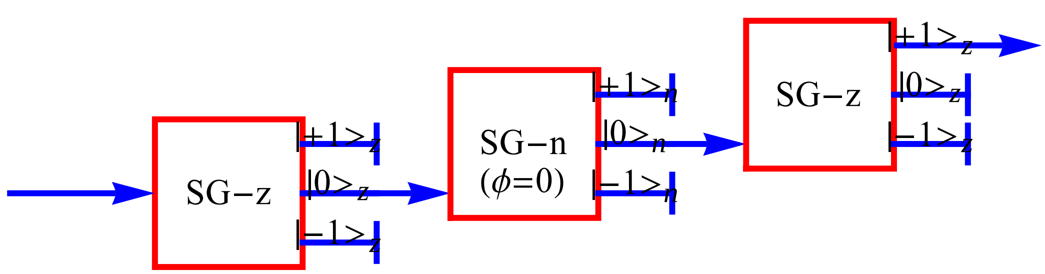


Two Stern-Gerlach type filters in series; the second is tilted at the angle θ from the z axis in the x-z plane.

3. Experiment-3



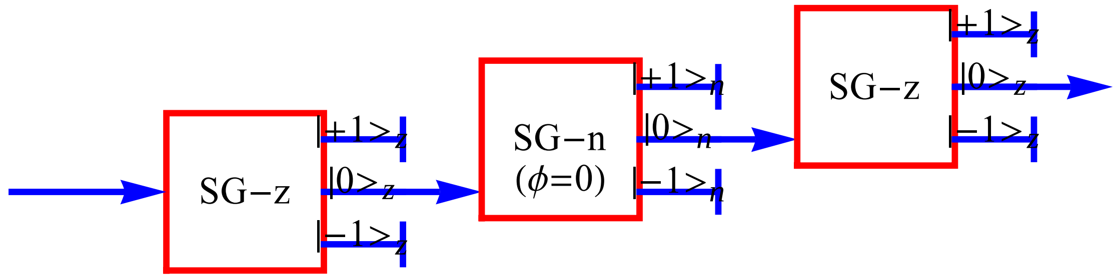
4. Experiment 4



The probability that an atom that comes out of SG_z (the first) will also go through both $SG_{\mathbf{n}}$ and SGz (the second) is

$$P_4 = \left| {}_z\langle 1|0\rangle_{\mathbf{n}} \right|^2 \left| {}_{\mathbf{n}}\langle 0|0\rangle_z \right|^2$$

5. Experiment-5



The probability that an atom that comes out of SG_z (the first) will also go through both $SG_{\mathbf{n}}$ and SGz (the second) is

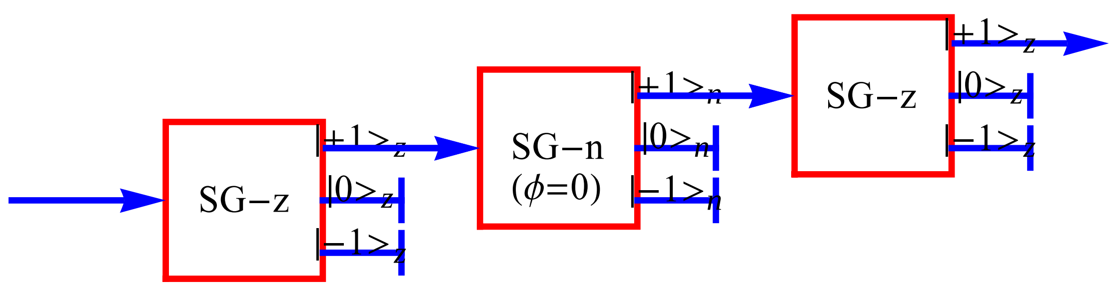
$$P_5 = \left| {}_z\langle 0|0\rangle_{\mathbf{n}} \right|^2 \left| {}_{\mathbf{n}}\langle 0|0\rangle_z \right|^2$$

Then we have

$$\frac{P_4}{P_5} = \frac{\left| {}_z\langle 1|0\rangle_{\mathbf{n}} \right|^2}{\left| {}_z\langle 0|0\rangle_{\mathbf{n}} \right|^2} = \frac{\left| \langle 1|0\rangle_{\mathbf{n}} \right|^2}{\left| \langle 0|0\rangle_{\mathbf{n}} \right|^2} = \frac{1}{2}\tan^2\theta$$

This ratio does not depend on which state is selected by the first SG_z.

6. Experiment-6



REFERENCES

1. R.P. Feynman, R.,B. Leighton, and M. Sands, *The Feynman Lectures in Physics*, 6th edition (Addison Wesley, Reading Massachusetts, 1977).

2. J.J. Sakurai, *Modern Quantum Mechanics*, Revised Edition (Addison-Wesley, Reading Massachusetts, 1994).
3. J.S. Townsend, *A Modern Approach to Quantum Mechanics* (McGraw-Hill, Inc., New York, 1992).
4. B. Friedrich and D. Herschbach, Phys. Today December 2003, p.53. "Stern and Gerlach: How a bad cigar helped reorient atomic physics."