Chapter 33
Laplace transform and Green's function

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Here We discuss the solution of the partial differential equation using the Laplace transformation

33.1 Under damped oscillator

We suppose that a damped harmonic oscillator is subjected to an external force $f(t)$ with finite duration and is at rest before the onset of the force. The displacement satisfies a differential equation of the form,

$$L_t x(t) = x''(t) + 2\gamma x'(t) + \omega_0{}^2 x(t) = f(t),$$

where m is a mass, γ is the damping factor, and ω_0 is the natural angular frequency. The initial condition is given by

$$x(0) = x_0, \qquad x'(0) = v_0.$$

We apply the Laplace transform to the above differential equation,

$$s^2 X(s) - sx(0) - x'(0) + 2\gamma[sX(s) - x(0)] + \omega_0{}^2 X(s) = F(s)$$

or

$$(s^2 + 2\gamma s + \omega_0{}^2) X(s) = sx_0 + v_0 + 2\gamma x_0 + F(s)$$

or

$$X(s) = \frac{x_0(s+2\gamma) + v_0}{s^2 + 2\gamma s + \omega_0{}^2} + \frac{F(s)}{s^2 + 2\gamma s + \omega_0{}^2}$$

where

$$X(s) = L[x(t)], \qquad F(s) = L[f(t)]$$

Using the formula (by Mathematica),

$$G(t) = -L^{-1}[\frac{1}{s^2 + 2\gamma s + \omega_0{}^2}] = -\frac{e^{-\gamma t}\sin(\omega_d t)}{\omega_d}$$

$$L^{-1}[\frac{s + 2\gamma}{s^2 + 2\gamma s + \omega_0{}^2}] = e^{-\gamma t}[\cos(\omega_d t) + \frac{\gamma \sin(\omega_d t)}{\omega_d}]$$

we have

$$x(t) = x_0 e^{-\gamma t}[\cos(\omega_d t) + \frac{\gamma \sin(\omega_d t)}{\omega_d}] + v_0 \frac{e^{-\gamma t}\sin(\omega_d t)}{\omega_d} - \int_0^t G(t-\tau) f(\tau) d\tau$$

Note that $G(t)$ is the Green's function for the under-damped oscillator

$$L_x G(t,\tau) = -\delta(t-\tau)$$

and ω_d is defined as

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2}\ .$$

33.2 Diffusion with constant boundary condition

We suppose that $\psi(x,t)$ satisfies a diffusion equation

$$\frac{\partial \psi(x,t)}{\partial t} = D\frac{\partial^2 \psi(x,t)}{\partial x^2}$$

with the boundary condition given by

$$\begin{aligned} &\psi(x=0, t \geq 0) = \psi_0 \Theta(t) \\ &\psi(x>0, t \leq 0) = 0 \end{aligned}.$$

where ψ_0 is constant. We apply the Laplace transform to the differential equation and the boundary condition

$$s\Psi(x,s) - \psi(x, t=0) = D\frac{\partial^2 \Psi(x,s)}{\partial x^2} \qquad (1)$$

where $\Psi(x,s)$ is the Laplace transform of $\psi(x,t)$,

$$\Psi(x,s) = L[\psi(x,t)],$$

and

$$\Psi(x=0,s) = \frac{\psi_0}{s}. \qquad (2)$$

From Eq.(1), we have

$$\frac{\partial^2 \Psi(x,s)}{\partial x^2} - \frac{s}{D}\Psi(x,s) = 0$$

since $\psi(x,t=0) = 0$ for x>0. We solve the differential equation for $\Psi(x,s)$.

$$\Psi(x,s) = \frac{\psi_0}{s}\exp(-\sqrt{\frac{s}{D}}x)$$

Using the Mathematica, we have

$$\psi(x,t) = L^{-1}[\Psi(x,s)] = \psi_0 Erfc[\frac{x}{2\sqrt{Dt}}]$$

where Erfc is the complementary error function.

((**Mathematica**))

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InverseLaplaceTransform[1/s Exp[-a Sqrt[s]], s, t] //
 Simplify[#, a > 0] &

Erfc[a/(2 Sqrt[t])]
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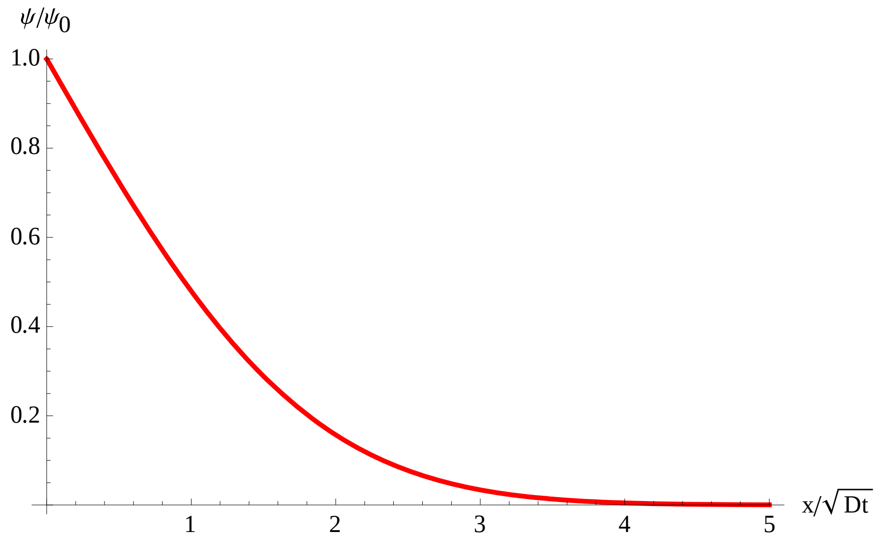

Fig. Diffusion with constant boundary value.

33.3 Example: Green's function for the homogeneous diffusion operator

We consider the solution of the Green's function satisfying

$$\frac{\partial G(x,t)}{\partial t} - D\frac{\partial^2 G(x,t)}{\partial x^2} = \delta(x-\xi)\delta(t-\tau) .$$

Assuming that G =0 for $t<\tau$, we apply the Laplace transform to the above differential equation,

$$sG(x,s) - D\frac{\partial^2 G(x,s)}{\partial x^2} = e^{-s\tau}\delta(x-\xi) ,$$

or

$$\frac{\partial^2 G_1(x-\xi,s)}{\partial x^2} - k^2 G_1(x-\xi,s) = -\delta(x-\xi) ,$$

where

$$k = \sqrt{\frac{s}{D}} , \qquad G(x-\xi,s) = \frac{e^{-s\tau}}{D} G_1(x-\xi,s) ,$$

$G_1(x,s)$ is the Green's function (modified Helmholtz), which satisfies

$$G_1(x-\xi, s) = \frac{1}{2k}\exp(-k|x-\xi|),$$

or

$$G(x-\xi, s) = \frac{e^{-s\tau}}{D}\frac{1}{2\sqrt{\frac{s}{D}}}\exp(-\sqrt{\frac{s}{D}}|x-\xi|) = \frac{1}{2\sqrt{Ds}}\exp(-s\tau-\sqrt{\frac{s}{D}}|x-\xi|).$$

The inverse Laplace transform leads to the Green's function,

$$\begin{aligned} G(x-\xi, t-\tau) &= L^{-1}[\frac{1}{2\sqrt{Ds}}\exp(-s\tau-\sqrt{\frac{s}{D}}|x-\xi|)] \\ &= \frac{1}{2\sqrt{D}}L^{-1}[\frac{1}{\sqrt{s}}\exp(-s\tau-a\sqrt{s})] \\ &= \frac{1}{\sqrt{4\pi D(t-\tau)}}\exp[-\frac{(x-\xi)^2}{4D(t-\tau)}]\Theta(t-\tau) \end{aligned}$$

where

$$a = \frac{|x-\xi|}{\sqrt{D}},$$

$$L^{-1}[\frac{1}{\sqrt{s}}\exp(-a\sqrt{s})] = \frac{1}{\sqrt{\pi t}}\exp(-\frac{a^2}{4t}),$$

$$L^{-1}[\frac{1}{\sqrt{s}}\exp(-s\tau-a\sqrt{s})] = \frac{1}{\sqrt{\pi(t-\tau)}}\exp[-\frac{a^2}{4(t-\tau)}].$$

((**Note**)) See Chapter 17 1D Green's functions.

33.4 One dimensional heat conduction-I: Laplace transform

We consider the differential equation for the 1D heat equation with a given boundary condition.

$$(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2})T(x,t) = 0,$$

with the initial condition given by

$$T(x,t=0) = T_0(x),$$

which is an initial temperature distribution at t = 0. We apply the Laplace transform to this differential equation.

$$sT(x,s) - T(x,t=0) - D\frac{\partial^2}{\partial x^2}T(x,s) = 0.$$

or

$$D\frac{\partial^2}{\partial x^2}T(x,s) - sT(x,s) = -T_0(x),$$

or

$$\frac{\partial^2}{\partial x^2}T(x,s) - k^2 T(x,s) = -\frac{1}{D}T_0(x),$$

where

$$k = \sqrt{\frac{s}{D}},$$

and

$$T(x,s) = L[T(x,t)]. \qquad \text{(Laplace transform)}$$

Using the Green's function (modified Helmholtz), we obtain the solution as

$$T(x,s) = \int_{-\infty}^{\infty} G(x-\xi,s)\frac{1}{D}T_0(\xi)d\xi,$$

where the Green's function $G(x-\xi,s)$ satisfies

$$\frac{\partial^2}{\partial x^2}G(x-\xi,s)-k^2T(x-\xi,s)=-\delta(x-\xi),$$

and the solution is obtained as

$$G(x-\xi,s)=\frac{1}{2k}\exp(-k|x-\xi|).$$

Finally we have

$$\begin{aligned}
T(x,s)&=\int_{-\infty}^{\infty}\frac{1}{2k}\exp[-k|x-\xi|]\frac{1}{D}T_0(\xi)d\xi \\
&=\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\frac{s}{D}}}\exp[-\sqrt{\frac{s}{D}}|x-\xi|]\frac{1}{D}T_0(\xi)d\xi \\
&=\frac{1}{2\sqrt{D}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{s}}\exp[-\sqrt{\frac{s}{D}}|x-\xi|]T_0(\xi)d\xi
\end{aligned}$$

The inverse Laplace transform:

$$T(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}\exp[-\frac{(x-\xi)^2}{4Dt}]T_0(\xi)d\xi$$

where

$$a=\frac{|x-\xi|}{\sqrt{D}}$$

$$L^{-1}[\frac{1}{\sqrt{s}}\exp(-a\sqrt{s})]=\frac{1}{\sqrt{\pi t}}\exp(-\frac{a^2}{4t})$$

33.5 One dimensional heat conduction II: Fourier transform

We consider the differential equation for the 1D heat equation with a given boundary condition.

$$(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2})T(x,t) = 0\,,$$

with the condition given by

$$T(x=0,t) = T_0(t)\,,$$

and $T(x, t = 0) = 0$ for $x>0$. We apply the Fourier transform to this differential equation (over x,).

$$T(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}T(x,t)dx\,,$$

$$T(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}T(k,t)dk\,.$$

We note that

$$\begin{aligned}\int_0^{\infty} e^{-ikx}\frac{\partial^2 T(x,t)}{\partial x^2}dx &= [e^{-ikx}\frac{\partial T(x,t)}{\partial x}]|_0^{\infty} + ik\int_0^{\infty} e^{-ikx}\frac{\partial T(x,t)}{\partial x}dx \\ &= [e^{-ikx}\frac{\partial T(x,t)}{\partial x}]|_0^{\infty} + ik\{[e^{-ikx}T(x,t)|_0^{\infty} + ik\int_0^{\infty} e^{-ikx}T(x,t)dx\}\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{0} e^{-ikx}\frac{\partial^2 T(x,t)}{\partial x^2}dx &= [e^{-ikx}\frac{\partial T(x,t)}{\partial x}]|_{-\infty}^{0} + ik\int_{-\infty}^{0} e^{-ikx}\frac{\partial T(x,t)}{\partial x}dx \\ &= [e^{-ikx}\frac{\partial T(x,t)}{\partial x}]|_{-\infty}^{0} + ik\{[e^{-ikx}T(x,t)|_{-\infty}^{0} + ik\int_{-\infty}^{0} e^{-ikx}T(x,t)dx\}\end{aligned}$$

Then we have

$$\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial^2 T(x,t)}{\partial x^2}dx = [e^{-ikx}\frac{\partial T(x,t)}{\partial x}]|_{0+}^{\infty} + [e^{-ikx}\frac{\partial T(x,t)}{\partial x}]|_{-\infty}^{0-}] + ik([e^{-ikx}T(x,t)]|_{0+}^{\infty} + [e^{-ikx}T(x,t)]|_{-\infty}^{0-})$$

$$-k^2\int_{-\infty}^{\infty} e^{-ikx}T(x,t)dx$$

$$= -k^2\int_{-\infty}^{\infty} e^{-ikx}T(x,t)dx - ik[T(x=0+,t) - T(x=0-,t)] - [\frac{\partial T(x=0+,t)}{\partial x} - \frac{\partial T(x=0-,t)}{\partial x}]$$

Since

$$T(x=0+,t) - T(x=0-,t) = -T_0(t),$$

and

$$\frac{\partial T(x=0+,t)}{\partial x} - \frac{\partial T(x=0-,t)}{\partial x} = 0,$$

we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial^2 T(x,t)}{\partial x^2}dx = -\frac{1}{\sqrt{2\pi}}k^2\int_{-\infty}^{\infty} e^{-ikx}T(x,t)dx + \frac{ikT_0(t)}{\sqrt{2\pi}}$$
$$= -k^2 T(k,t) + \frac{ikT_0(t)}{\sqrt{2\pi}},$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\frac{\partial T(x,t)}{\partial t}dx = \frac{\partial}{\partial t}T(k,t)$$

Using this we get the differential equation for $T(k, t)$,

$$(\frac{\partial}{\partial t} + Dk^2)T(k,t) = -\frac{iDkT_0(t)}{\sqrt{2\pi}}$$

The solution of the Green's function satisfying

$$(\frac{\partial}{\partial t} + Dk^2)G(k,t-\tau) = -\delta(t-\tau).$$

is given by

$$G(k,t-\tau) = -\exp(-Dk^2 t)\Theta(t-\tau)$$

((Note)) See Chapter 14. for the derivation of $G(k,t-\tau)$.

Then we get

$$T(k,t) = \frac{iDk}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k,t-\tau) T_0(\tau) d\tau$$
$$= -\frac{iDk}{\sqrt{2\pi}} \int_0^t \exp[-Dk^2(t-\tau)] T_0(\tau) d\tau$$

Finally we have

$$T(x,t) = \frac{1}{\sqrt{2\pi}} \frac{(-iD)}{\sqrt{2\pi}} \int_0^t T_0(\tau) d\tau \int_{-\infty}^{\infty} kdk \exp[ikx - Dk^2(t-\tau)]$$
$$= \frac{1}{2} \int_0^t d\tau \frac{xT_0(\tau)}{[4\pi D(t-\tau)^3]^{3/2}} \exp[-\frac{x^2}{4D(t-\tau)}]$$

((**Mathematica**))

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Integrate[k Exp[ⅈ k x - D1 k^2 t], {k, -∞, ∞}] //
 Simplify[#, {x > 0, D1 > 0, t > 0}] &
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$$\frac{i\, e^{-\frac{x^2}{4\,D1\,t}} \sqrt{\pi}\, x}{2\,(D1\,t)^{3/2}}$$

33.6 Three dimensional heat conduction: Green's function

We consider the Green's function given by

$$(\frac{\partial}{\partial t} - D\nabla^2) G(\mathbf{r},t) = -\delta(\mathbf{r})\delta(t)$$

We apply the Fourier transform to this equation,

$$G(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{r}) G(\mathbf{k},t) ,$$

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{r}) ,$$

and

$$\begin{aligned}
(\frac{\partial}{\partial t} - D\nabla^2) G(\mathbf{r},t) &= \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} (\frac{\partial}{\partial t} - D\nabla^2) \exp(i\mathbf{k}\cdot\mathbf{r}) G(\mathbf{k},t) \\
&= \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{k} [\frac{\partial}{\partial t} G(\mathbf{k},t) + Dk^2 G(\mathbf{k},t)] \exp(i\mathbf{k}\cdot\mathbf{r}) \\
&= -\frac{1}{(2\pi)^3} \delta(t) \int d^3\mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{r})
\end{aligned}$$

or

$$\frac{\partial}{\partial t} G(\mathbf{k},t) + Dk^2 G(\mathbf{k},t) = -\frac{1}{(2\pi)^{3/2}} \delta(t)$$

The solution for G(***k***,t) is given by

$$G(\mathbf{k},t) = -\frac{1}{(2\pi)^{3/2}} \exp(-Dk^2 t) \Theta(t)$$

Using the inverse Fourier transform, we have

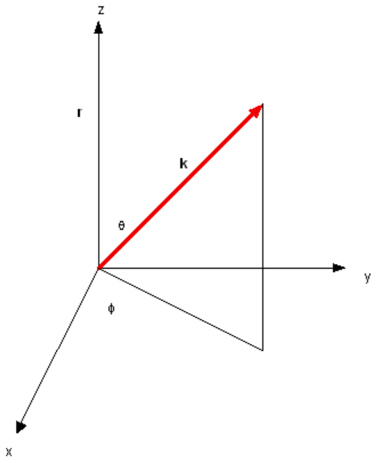
$$\begin{aligned}
G(\mathbf{r},t) &= -\frac{1}{(2\pi)^3} \Theta(t) \int d^3\mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{r}) \exp(-Dk^2 t) \\
&= -\frac{1}{(2\pi)^3} \Theta(t) \int d^3\mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{r}) \exp(-Dk^2 t)
\end{aligned}$$

We calculate the integral *I*

$$I = \int d^3\mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{r})\exp(-Dk^2 t)$$
$$= 2\pi \int_0^\infty dk k^2 \exp(-Dk^2 t) \int_0^\pi d\theta \sin\theta e^{ikr\cos\theta}$$
$$= 2\pi \int_0^\infty dk k^2 \exp(-Dk^2 t)(\frac{e^{ikr} - e^{-ikr}}{ikr})$$
$$= \frac{2\pi}{ir} \int_{-\infty}^\infty dk k \exp(ikr - Dk^2 t)$$
$$= \frac{2\pi}{ir} \frac{i\sqrt{\pi} r}{2(Dt)^{3/2}} \exp(-\frac{r^2}{4Dt})$$
$$= \pi \frac{\sqrt{\pi}}{(Dt)^{3/2}} \exp(-\frac{r^2}{4Dt})$$

((Note))

$\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$, $d\mathbf{k} = k^2 dk \sin\theta d\theta d\phi$



Then we get

$$G(\mathbf{r},t) = -\Theta(t)\frac{1}{(4\pi Dt)^{3/2}}\exp(-\frac{r^2}{4Dt})$$

REFERENCES

Susan M. Lea Mathematics for Physicists (Thomson Brooks/Cole, 2004).

P.K. Kythe, P. Puri, and M.R. Schaferkötter, Partial Differential Equations and Mathematica (CRC Press, New York, 1997).

W. Appel Mathematics for Physics and Physicists (Princeton University Press, Princeton and Oxford, 2007).

B.R. Kusse and E.A. Westwig, *Mathematical Physics; Applied Mathematics for Scientists and Engineers*, 2nd edition (Wiley-VCH Verlag GmbH & Co. KGaA, Winheim, 2006).

J.J. Kelly, *Graduate Mathematical Physics with Mathematica Supplement* (Wiley-VCH Verlag GmbH & Co. KGaA, Winheim, 2006).