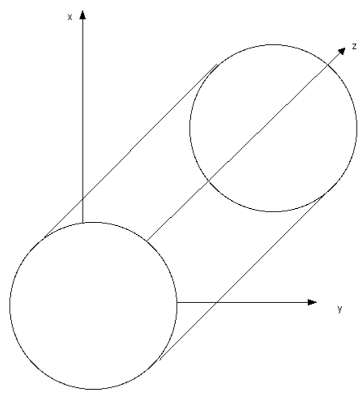
Chapter 34S Waveguides

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
(Date: December 04, 2010)

34S.1 Boundary condition



$$D_1^{\perp} - D_2^{\perp} = \sigma_f$$
$$B_1^{\perp} - B_2^{\perp} = 0$$
$$\mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$
$$\mathbf{H}_1^{//} - \mathbf{H}_2^{//} = \mathbf{K}_f \times \mathbf{n}$$

or

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$$
$$B_1^{\perp} - B_2^{\perp} = 0$$
$$\mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$$
$$\frac{1}{\mu_1}\mathbf{B}_1^{//} - \frac{1}{\mu_2}\mathbf{B}_2^{//} = \mathbf{K}_f \times \mathbf{n}$$

Boundary surface is made of perfect conductor: ***E*** = 0 and ***B*** = 0 inside the materials

$$B_1^{\perp} = B_2^{\perp} = 0$$
$$\mathbf{E}_1^{//} = \mathbf{E}_2^{//} = 0$$

The tangential component of ***E*** is equal to zero. The normal component of ***B*** is equal to zero. ⊥ means perpendicular to the boundary surface.

34S.2 The interior of wave guide

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

We now assume that

$$\mathbf{E} = \mathrm{Re}[\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}]$$

$$\mathbf{B} = \mathrm{Re}[\tilde{\mathbf{B}}_0 e^{i(kz\mathbf{r}-\omega t)}]$$

(1) $\nabla \cdot \mathbf{E} = 0$

$$\nabla \cdot \mathbf{E} \to \nabla \cdot [\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}] = e^{i(kz-\omega t)}\nabla \cdot \tilde{\mathbf{E}}_0 + \tilde{\mathbf{E}}_0 \cdot \nabla e^{i(kz-\omega t)}$$

or

$$e^{i(kz-\omega t)}\nabla \cdot \tilde{\mathbf{E}}_0 + ike^{i(kz-\omega t)}\mathbf{e}_z \cdot \tilde{\mathbf{E}}_0 = 0$$

or

$$\nabla \cdot \tilde{\mathbf{E}}_0 + ik\mathbf{e}_z \cdot \tilde{\mathbf{E}}_0 = 0$$

(2) $\nabla \cdot \mathbf{B} = 0$

Similarly we have

$$\nabla \cdot \tilde{\mathbf{B}}_0 + ik\mathbf{e}_z \cdot \tilde{\mathbf{B}}_0 = 0$$

(3) $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

$$\nabla \times [\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}] = e^{i(kz-\omega t)}\nabla \times \tilde{\mathbf{E}}_0 - \tilde{\mathbf{E}}_0 \times [\nabla e^{i(kz-\omega t)}] = e^{i(kz-\omega t)}(\nabla \times \tilde{\mathbf{E}}_0 - ik\tilde{\mathbf{E}}_0 \times \mathbf{e}_z)$$

$$-\frac{\partial}{\partial t}[\tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}] = i\omega\tilde{\mathbf{B}}_0 e^{i(kz-\omega t)}$$

Then we have

$$\nabla \times \tilde{\mathbf{E}}_0 - ik\tilde{\mathbf{E}}_0 \times \mathbf{e}_z = i\omega \tilde{\mathbf{B}}_0$$

(4) $$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

Similarly we have

$$\nabla \times \tilde{\mathbf{B}}_0 - ik\tilde{\mathbf{B}}_0 \times \mathbf{e}_z = -i\frac{\omega}{c^2}\tilde{\mathbf{E}}_0$$

Hereafter we assume that $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are independent of z.

$$\tilde{\mathbf{E}}_0 = (E_x, E_y, E_z)$$
$$\tilde{\mathbf{B}}_0 = (B_x, B_y, B_z)$$

where these components are in general complex numbers. These components are independent of z. Then we have

$$\begin{aligned}
&\frac{\partial E_y}{\partial y} - ikE_y = i\omega B_x \\
&-\frac{\partial E_y}{\partial x} + ikE_x = i\omega B_y \\
&\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z
\end{aligned} \qquad (1)$$

$$\begin{aligned}
&\frac{\partial B_z}{\partial y} - ikB_y = -i\frac{\omega}{c^2}E_x \\
&-\frac{\partial B_z}{\partial x} + ikB_x = -i\frac{\omega}{c^2}E_y \\
&\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\frac{\omega}{c^2}E_z
\end{aligned} \qquad (2)$$

$$\begin{aligned}
&\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0 \\
&\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0
\end{aligned} \qquad (3)$$

From Eqs.(1) and (2), we have

$$
\begin{aligned}
&kE_x - \omega B_y = -i\frac{\partial E_z}{\partial x} \\
&\frac{\omega}{c^2}E_x - kB_y = i\frac{\partial B_z}{\partial y}
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&kE_y + \omega B_x = -i\frac{\partial E_z}{\partial y} \\
&\frac{\omega}{c^2}E_y + kB_x = -i\frac{\partial B_z}{\partial x}
\end{aligned}
\tag{5}
$$

From Eq.(4) and (5), we have

$$
\begin{aligned}
E_x &= \frac{i}{\kappa_c{}^2}(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}) \qquad & B_x &= \frac{i}{\kappa_c{}^2}[-\frac{\omega}{c^2}\frac{\partial E_z}{\partial y} + k\frac{\partial B_z}{\partial x}] \\
E_y &= \frac{i}{\kappa_c{}^2}(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}) \qquad & B_y &= \frac{i}{\kappa_c{}^2}[\frac{\omega}{c^2}\frac{\partial E_z}{\partial x} + k\frac{\partial B_z}{\partial y}]
\end{aligned}
$$

where

$$
\frac{\omega^2}{c^2} - k^2 = \kappa_c{}^2
$$

What is the solution of E_z and B_z? From

$$
\nabla\cdot\tilde{\mathbf{E}}_0 + ik\mathbf{e}_z\cdot\tilde{\mathbf{E}}_0 = 0
$$

we have

$$
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0\,.
$$

From

$$
\nabla\cdot\tilde{\mathbf{B}}_0 + ik\mathbf{e}_z\cdot\tilde{\mathbf{B}}_0 = 0
$$

we have

$$
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0
$$

Using these equations we reach the final results

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \kappa_c{}^2 E_z = 0$$

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \kappa_c{}^2 B_z = 0$$

((Definition))

TE (transverse electric);	$E_z = 0$
TM (transverse magnetic);	$B_z = 0$
TEM	$B_z = 0$ and $E_z = 0$

TEM waves cannot occur in a hollow wave guide.

34S.3 TE wave in rectangular wave guide

$E_z = 0$. B_z is an independent variable.

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \kappa_c{}^2 B_z = 0$$

with the boundary condition; $B^{\perp} = 0$

$$E_x = \frac{i\omega}{\kappa_c{}^2}\frac{\partial B_z}{\partial y} \qquad B_x = \frac{ik}{\kappa_c{}^2}\frac{\partial B_z}{\partial x}$$

$$E_y = \frac{-i\omega}{\kappa_c{}^2}\frac{\partial B_z}{\partial x} \qquad B_y = \frac{ik}{\kappa_c{}^2}\frac{\partial B_z}{\partial y}$$

$$\mathbf{E}\cdot\mathbf{B} = E_x B_x + E_y B_y + E_z B_z = E_x B_x + E_y B_y + 0\cdot B_z = 0$$

This means that ***E*** is perpendicular to ***B***.

$$\mathbf{B}_t = \frac{ik}{\kappa_c{}^2}\nabla B_z$$

$$\mathbf{E}_t = -\frac{\omega}{k}(\mathbf{e}_z \times \mathbf{B}_t)$$

We now assume that

$$B_z = X(x)Y(y) \qquad \text{(separation variable)}$$

Then we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(x)}{Y(x)} = -\kappa_c^{\ 2}$$

or

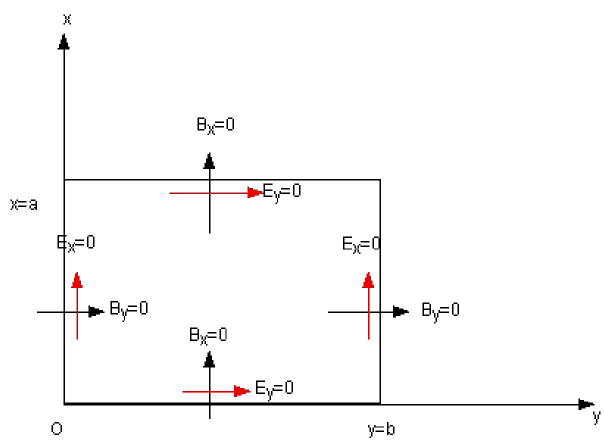
$$\frac{X''(x)}{X(x)} = -\kappa_x^{\ 2}$$
$$\frac{Y''(y)}{Y(y)} = -\kappa_y^{\ 2}$$

with

$$\kappa_x^{\ 2} + \kappa_y^{\ 2} = \kappa_c^{\ 2} = \frac{\omega^2}{c^2} - k^2$$

Then

$$X(x) = A_1 \cos(\kappa_x x) + B_1 \sin(\kappa_x x)$$
$$Y(y) = A_1' \cos(\kappa_y y) + B_1' \sin(\kappa_y y)$$



At x = 0 and x = a, E_y = 0 or B_x = 0

$$(\, B_x = \frac{ik}{\kappa_c^{\ 2}} \frac{\partial B_z}{\partial x}\,, \qquad E_y = \frac{-i\omega}{\kappa_c^{\ 2}} \frac{\partial B_z}{\partial x}\,)$$

$$\frac{\partial B_z}{\partial x} = 0 \qquad \text{or} \qquad \frac{\partial X(x)}{\partial x} = 0 \qquad (x = 0 \text{ and } a)$$

From this boundary condition, we have

$B_1 = 0$, and $\sin(\kappa_x a) = 0$,

$$X(x) = A_1 \cos(\kappa_x a)$$

with

$$\kappa_x = \frac{m\pi}{a}$$

At $y = 0$ and $y = b$,

$E_x = 0$ and $B_y = 0$

$$\left(E_x = \frac{i\omega}{\kappa_c^{\ 2}} \frac{\partial B_z}{\partial y}, \qquad B_y = \frac{ik}{\kappa_c^{\ 2}} \frac{\partial B_z}{\partial y} \right),$$

or

$$\frac{\partial B_z}{\partial y} = 0$$

$$Y(y) = A_1' \cos(\kappa_y b)$$

with

$$\kappa_y = \frac{n\pi}{b}$$

Then we have the TE_{mn} mode

$$B_z = B_0 \cos(\kappa_x x) \cos(\kappa_y y)$$

wit

$$k = \sqrt{\frac{\omega^2}{c^2} - \kappa_c^{\ 2}},$$

with

$$\frac{\omega_{mn}}{c} = \kappa_c = \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \ \ (\omega_{mn}\text{: cut-off angular frequency}).$$

If $\omega < \omega_{mn}$ the wave number k is imaginary. It means exponentially attenuated fields.

TE_{10}, TE_{01}, TE_{20}, TE_{11}, TE_{02}, ……

34S.4 TE_{10} mode

$$\kappa_x = \frac{\pi}{a}$$
$$\kappa_y = 0$$

$$\tilde{E}_x = 0$$
$$\tilde{E}_y = \frac{i\omega}{\left(\frac{\pi}{a}\right)} B_0 \sin(\frac{\pi x}{a})$$
$$\tilde{E}_z = 0$$

$$\tilde{B}_x = \frac{-ik}{\left(\frac{\pi}{a}\right)} B_0 \sin(\frac{\pi x}{a})$$
$$\tilde{B}_y = 0$$
$$\tilde{B}_z = B_0 \cos(\frac{\pi}{a} x)$$

$$E_y = \mathrm{Re}[\frac{\omega}{\left(\frac{\pi}{a}\right)} B_0 i \frac{e^{i\pi x/a} - e -^{i\pi x/a}}{2i} e^{i(kz-\omega t)}] = \mathrm{Re}[\frac{a\omega}{2\pi} B_0 \{e^{i(kz+\frac{\pi x}{a}-\omega t)} - e^{i(kz-\frac{\pi x}{a}-\omega t)}\}]$$

$$B_z = \mathrm{Re}[B_0 \cos(\frac{\pi x}{a}) e^{i(kz\mathbf{r}-\omega t)}] = \mathrm{Re}[\frac{B_0}{2}\{e^{i(kz+\frac{\pi x}{a}-\omega t)} + e^{i(kz-\frac{\pi x}{a}-\omega t)}\}]$$

$$B_x = \mathrm{Re}[\frac{-k}{\left(\frac{\pi}{a}\right)} B_0 i \frac{e^{i\pi x/a} - e -^{i\pi x/a}}{2i} e^{i(kz-\omega t)}] = \mathrm{Re}[\frac{ak}{2\pi} B_0 \{-e^{i(kz+\frac{\pi x}{a}-\omega t)} + e^{i(kz-\frac{\pi x}{a}-\omega t)}\}]$$

The wave vector is given by $(\pm\frac{\pi}{a}, 0, k)$. The magnitude of the wave vector is

$$\sqrt{k^2 + \frac{\pi^2}{a^2}},$$

which is equal to $\frac{\omega}{c}$. This means that the plane wave propagates with the velocity c and wave vector $(\pm\frac{\pi}{a},0,k)$. ***E*** is perpendicular to ***B***. The poynting vector is given by

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0}\mathrm{Re}[\tilde{\mathbf{E}}_0 \times \tilde{\mathbf{B}}_0^{\ *}] = \frac{1}{2\mu_0}[\mathrm{Re}(E_y B_z^{\ *})\mathbf{e}_x - \mathrm{Re}(E_y B_x^{\ *})\mathbf{e}_z]$$

or

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0}\frac{\omega k}{\left(\frac{\pi}{a}\right)^2}|B_0|^2 \sin^2(\frac{\pi x}{a})\mathbf{e}_z$$

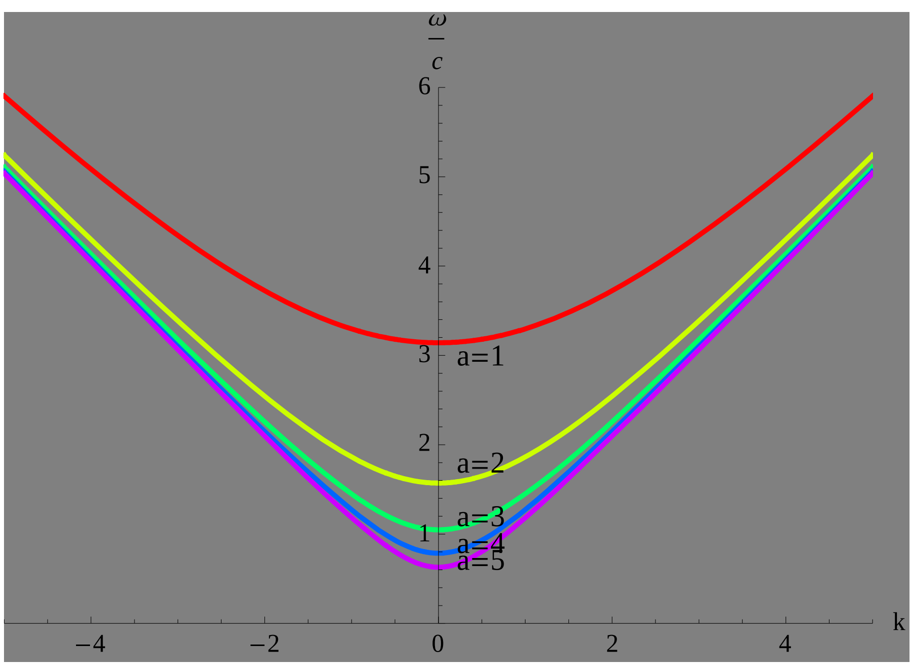
The energy flow is parallel to the z axis.

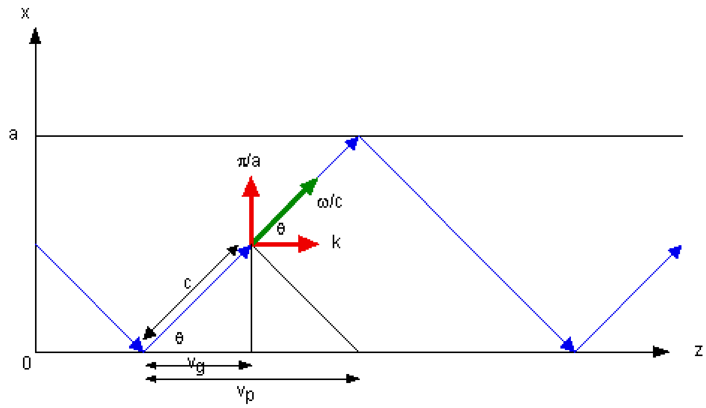
34S.5 Dispesion relation

$$\frac{\omega^2}{c^2} - k^2 = \kappa_c^{\ 2} = \frac{\omega_{10}^{\ 2}}{c^2} = \frac{\pi^2}{a^2}$$

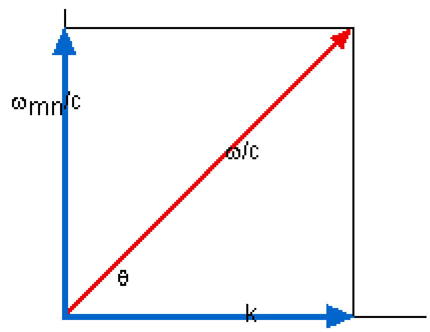
or

$$\frac{\omega}{c} = \sqrt{k^2 + \frac{\pi^2}{a^2}}$$





We define the phase velocity v_p and group velocity v_g.



$$k = \frac{\omega}{c}\cos\theta$$

$$\frac{\omega_{mn}}{c} = \frac{\omega}{c}\sin\theta$$

$$v_p = \frac{c}{\cos\theta},$$

$$v_g = c\sin\theta$$

$$\omega_{mn} = \pi c\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

$$\frac{\omega}{c} = \sqrt{k^2 + \frac{\omega_{mn}{}^2}{c^2}},$$

$$k = \sqrt{\frac{\omega^2 - \omega_{nm}{}^2}{c^2}}$$

The group velocity

$$v_g = \frac{\partial \omega}{\partial k} = \frac{1}{\frac{\partial k}{\partial \omega}} = \frac{1}{\frac{1}{c}\frac{2\omega}{2\sqrt{\omega^2 - \omega_{nm}{}^2}}}$$

$$= \frac{c}{\omega}\sqrt{\omega^2 - \omega_{nm}{}^2} = c\sqrt{1 - \frac{\omega_{nm}{}^2}{\omega^2}} < c$$

The phase velocity

$$v_p = \frac{\omega}{k} = \frac{\omega}{\frac{1}{c}\sqrt{\omega^2 - \omega_{nm}{}^2}} = c\frac{\omega}{\sqrt{\omega^2 - \omega_{nm}{}^2}} > c$$

We also have the relation

$$v_p v_g = c^2$$

34S.6 TM wave in rectangular wave guide

Transverse magnetic mode. $B_z = 0$. E_z is an independent variable.

$$E_x = \frac{ik}{\kappa_c{}^2}\frac{\partial E_z}{\partial x} \qquad B_x = -\frac{i}{\kappa_c{}^2}\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}$$

$$E_y = \frac{ik}{\kappa_c{}^2}\frac{\partial E_z}{\partial y} \qquad B_y = \frac{i}{\kappa_c{}^2}\frac{\omega}{c^2}\frac{\partial E_z}{\partial x}$$

or

$$\mathbf{E}_t = \frac{ik}{\kappa_c{}^2}\nabla E_z$$

$$\mathbf{B}_t = \frac{k}{\omega}(\mathbf{e}_z \times \mathbf{E}_t)$$

E is perpendicular to ***B***.

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \kappa_c{}^2 E_z = 0$$

We now assume that

$$E_z = X(x)Y(y)$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(x)}{Y(x)} = -(\frac{\omega^2}{c^2} - k^2)$$

$$\frac{X''(x)}{X(x)} = -\kappa_x{}^2$$
$$\frac{Y''(y)}{Y(y)} = -\kappa_y{}^2$$

with

$$\kappa_x{}^2 + \kappa_y{}^2 = \kappa_c{}^2 = \frac{\omega^2}{c^2} - k^2$$

$$X(x) = A_1 \cos(\kappa_x x) + B_1 \sin(\kappa_x x)$$
$$Y(y) = A_1' \cos(\kappa_y y) + B_1' \sin(\kappa_y y)$$

The boundary conditrion: $E_z = 0$.

At $x = 0$ and $x = a$, $E_z = 0$

$$X(x) = B_1 \sin(\kappa_x x)$$

with

$$\kappa_x = \frac{m\pi}{a}$$

At $y = 0$ and $y = b$, $E_z = 0$

$$Y(y) = B_1' \sin(\kappa_y y)$$

with

$$\kappa_y = \frac{n\pi}{b}$$

Then we have the TM_{mn} mode

$$E_z = E_0 \sin(\kappa_x x)\sin(\kappa_y y)$$

wit

$$k = \sqrt{\frac{\omega^2 - \omega_{mn}{}^2}{c^2}}\,, \qquad \text{or} \qquad \frac{\omega^2}{c^2} - k^2 = \kappa_c{}^2 = \frac{\omega_{mn}{}^2}{c^2}$$

34S.7 Schematic representation of TE mode in rectangular guide

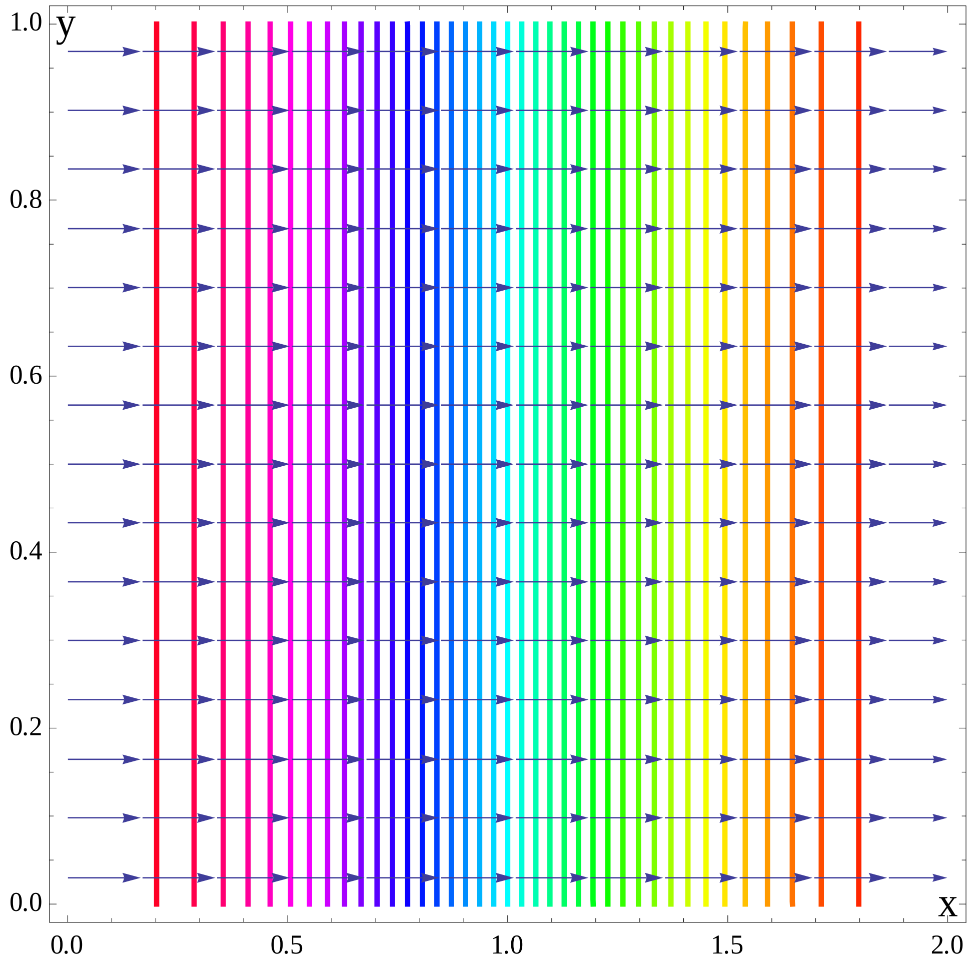
$$B_z = B_0 \cos(\kappa_x x)\cos(\kappa_y y)\cos(kz - \omega t)$$

$$\mathbf{E}_t \propto \nabla B_z$$

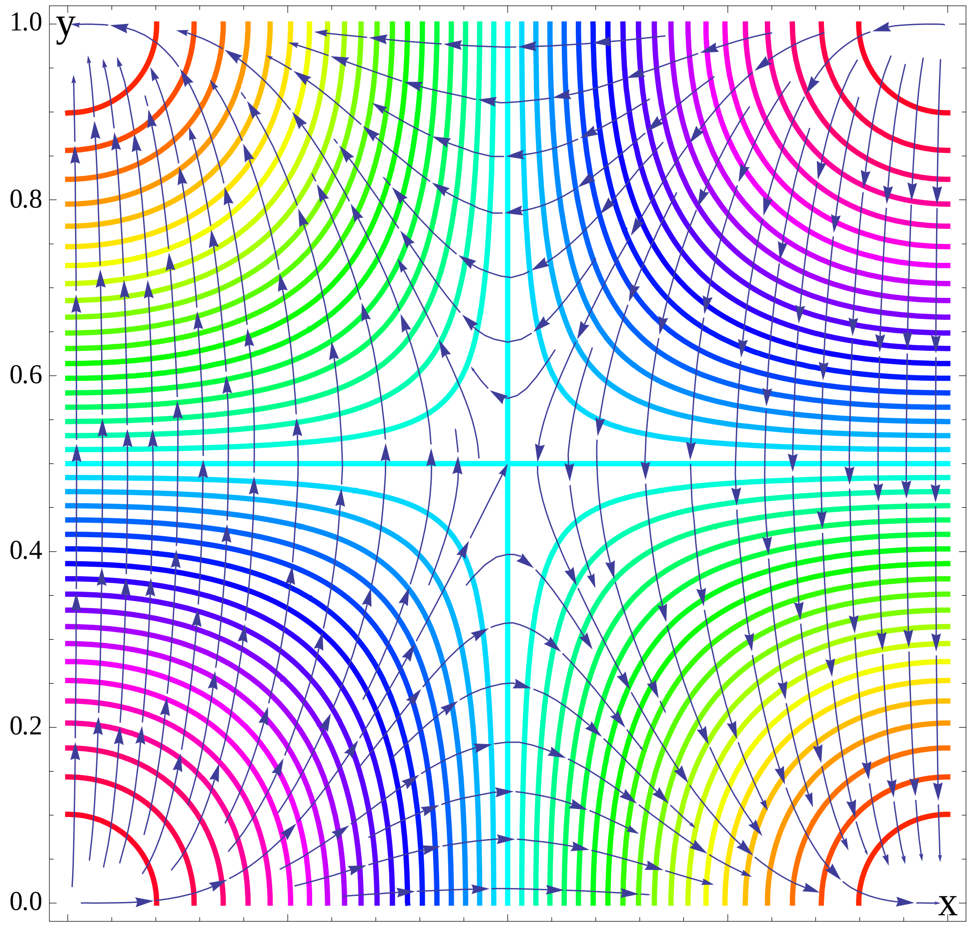
where

$$\kappa_x = \frac{m\pi}{a},$$
$$\kappa_y = \frac{m\pi}{b},$$

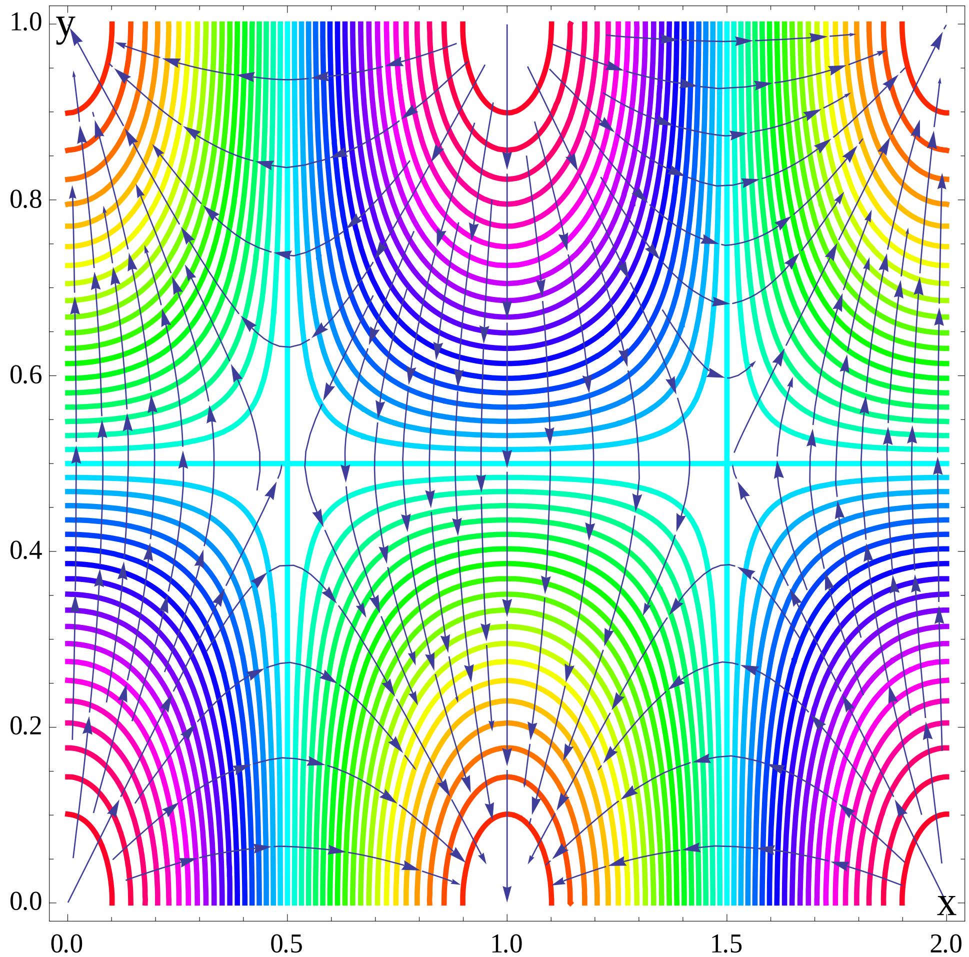
((**Mathematica**))
We use the ContourPlot and StreamPlot.



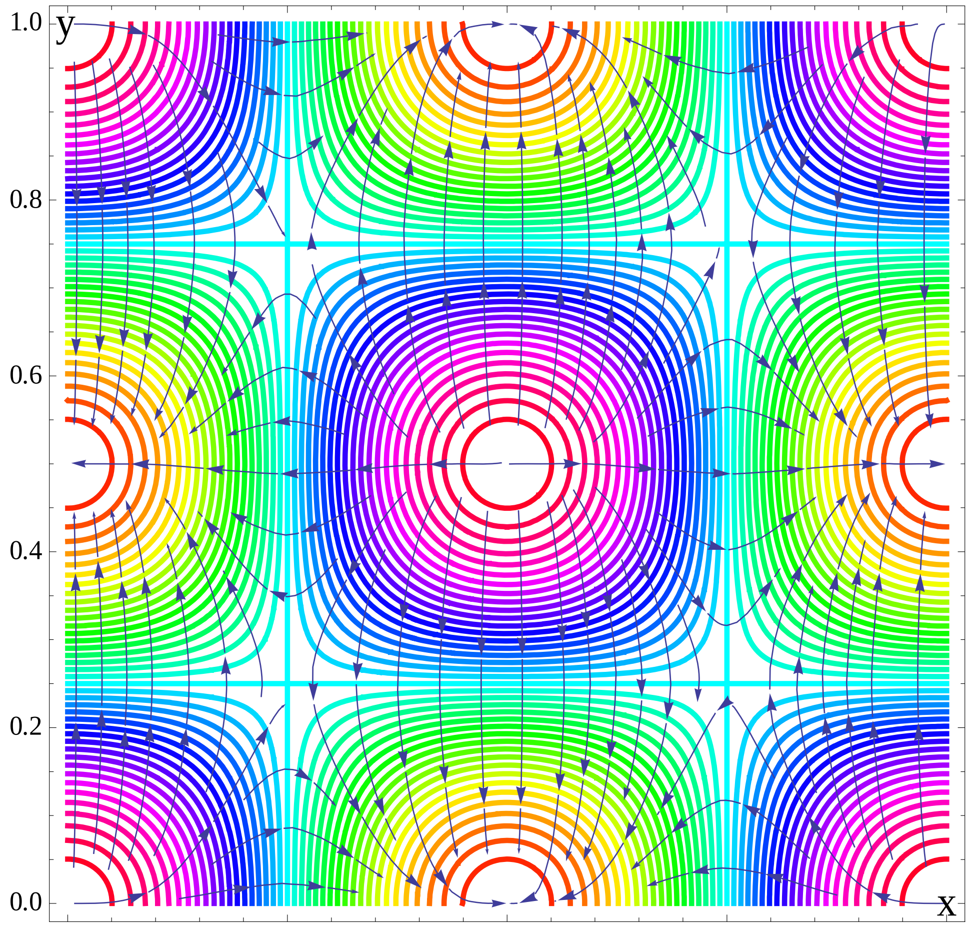
TE_{10} mode



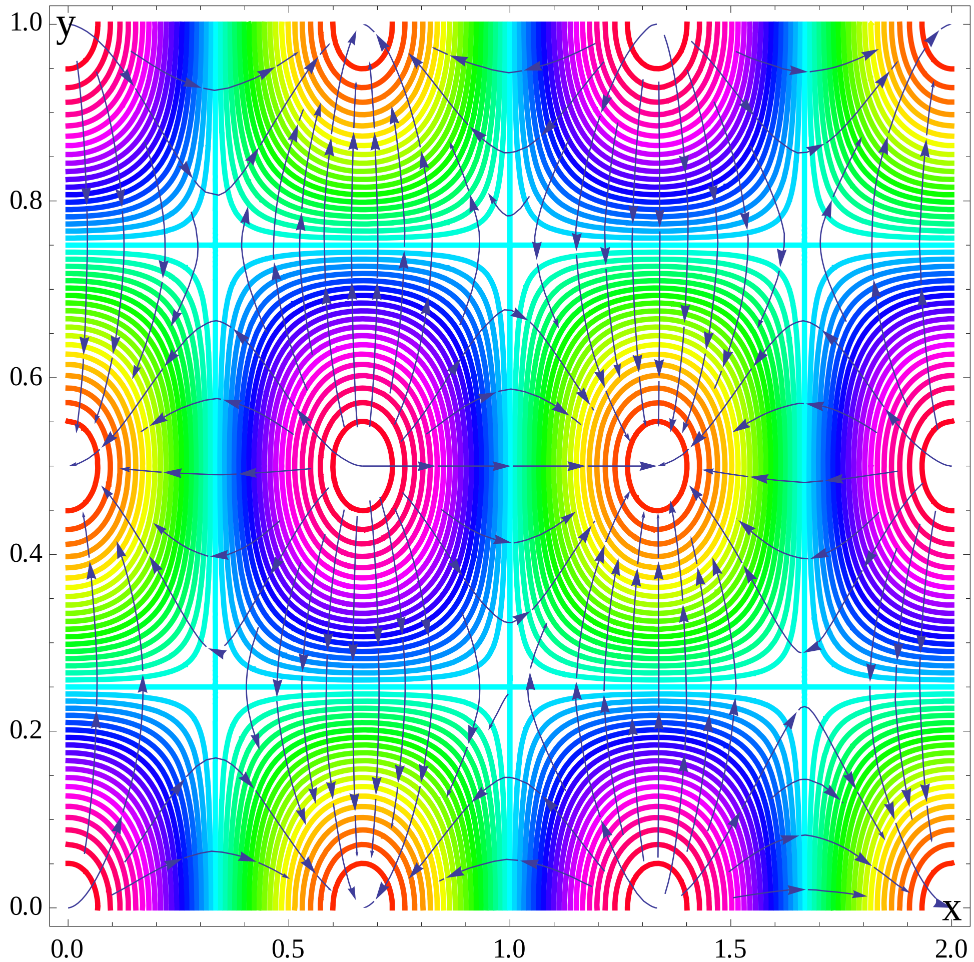
TE$_{11}$ mode



TE_{21} mode



TE_{22} mode



TE_{32} mode

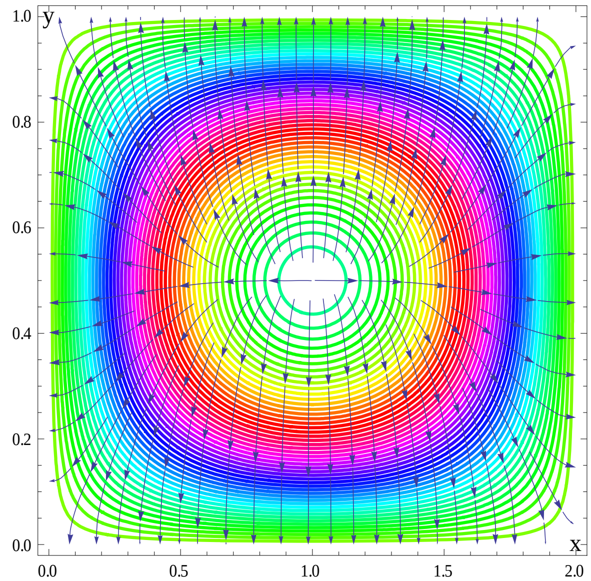
34S.8 Schematic representation of TM mode in rectangular guide

$$E_z = E_0 \sin(\kappa_x x)\sin(\kappa_y y)\cos(kz - \omega t)\,.$$

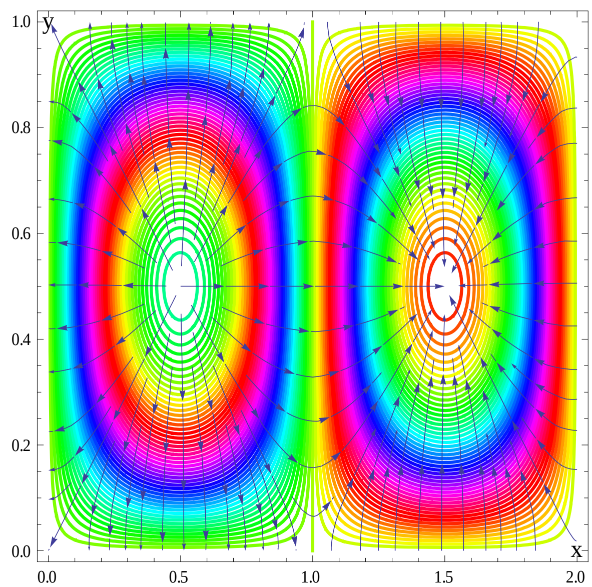
$$\mathbf{B}_t \propto \nabla E_z$$

where

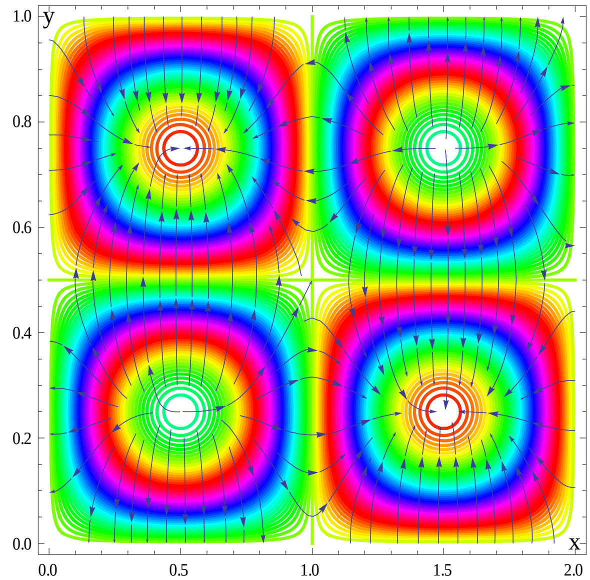
$$\kappa_x = \frac{m\pi}{a},$$
$$\kappa_y = \frac{m\pi}{b},$$



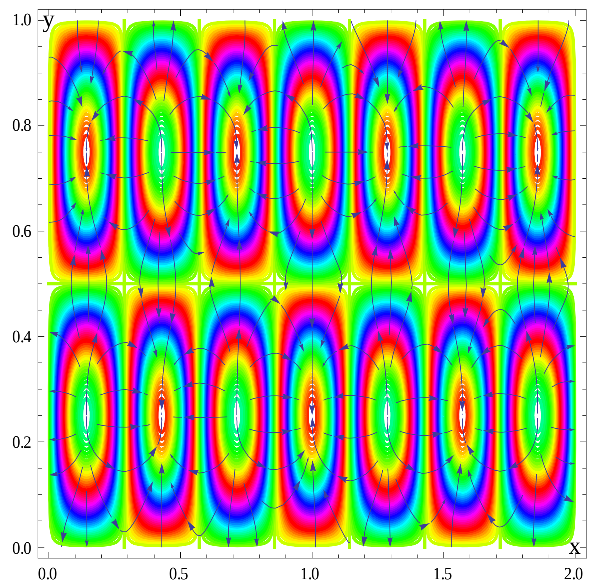
TM$_{11}$ mode



TM$_{21}$ mode



TM_{22} mode



TM_{72} mode

34S.9 The possibility of TEM mode

We now consider the possibility that $E_z = 0$ and $B_z = 0$ (TEM mode). The TEM modes must satisfy

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

$$kB_y = \frac{\omega}{c^2} E_x$$

$$kB_x = -\frac{\omega}{c^2} E_y$$

$$kE_x = \omega B_y$$

$$kE_y = -\omega B_x$$

From the relations

(5)

$$kE_x = \omega B_y, \quad \text{and} \quad kB_y = \frac{\omega}{c^2} E_x$$

we have the dispersion relation

$$\omega = ck \; .$$

This means that the TEM wave propagates with $v_p = \omega/k = c$, the speed of light, just as if it were a plane wave in an unbounded region.

$$E_x = cB_y$$

$$E_y = -cB_x$$

E is perpendicular to **B**; $\quad \mathbf{E} \cdot \mathbf{B} = 0$

$$\mathbf{E} \times \mathbf{B} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ E_x & E_y & 0 \\ B_x & B_y & 0 \end{vmatrix} = (E_x B_y - E_y B_x)\mathbf{e}_z = \frac{1}{c} \mathbf{E}^2 \mathbf{e}_z$$

34S.10 Maxwell's equation in the cylindrical coordinates

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

We assume that

$$\mathbf{E} = \mathrm{Re}[\tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}]$$

$$\mathbf{B} = \mathrm{Re}[\tilde{\mathbf{B}}_0 e^{i(kz\mathbf{r}-\omega t)}]$$

$$\nabla \cdot \tilde{\mathbf{E}}_0 + ik\mathbf{e}_z \cdot \tilde{\mathbf{E}}_0 = 0$$

$$\nabla \cdot \tilde{\mathbf{B}}_0 + ik\mathbf{e}_z \cdot \tilde{\mathbf{B}}_0 = 0$$

$$\nabla \times \tilde{\mathbf{E}}_0 - ik\tilde{\mathbf{E}}_0 \times \mathbf{e}_z = i\omega \tilde{\mathbf{B}}_0$$

$$\nabla \times \tilde{\mathbf{B}}_0 - ik\tilde{\mathbf{B}}_0 \times \mathbf{e}_z = -i\frac{\omega}{c^2}\tilde{\mathbf{E}}_0$$

Hereafter we assume that $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are independent of z.

$$\tilde{\mathbf{E}}_0 = (E_\rho, E_\phi, E_z)$$
$$\tilde{\mathbf{B}}_0 = (B_\rho, B_\phi, B_z)$$

where these components are in general complex numbers. Then we have

$$ikE_z + \frac{1}{\rho}E_\rho + \frac{\partial}{\partial\rho}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\phi = 0$$
$$ikB_z + \frac{1}{\rho}B_\rho + \frac{\partial}{\partial\rho}B_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}B_\phi = 0$$
$$-ikE_\phi + \frac{1}{\rho}\frac{\partial}{\partial\phi}E_z = i\omega B_\rho$$
$$ikE_\rho - \frac{\partial}{\partial\rho}E_z = i\omega B_\phi$$
$$\frac{1}{\rho}E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho + \frac{\partial}{\partial\rho}E_\phi = i\omega B_z$$
$$-ikB_\phi + \frac{1}{\rho}\frac{\partial}{\partial\phi}B_z = -i\frac{\omega}{c^2}E_\rho$$
$$ikB_\rho - \frac{\partial}{\partial\rho}B_z = -i\frac{\omega}{c^2}E_\phi$$
$$\frac{1}{\rho}B_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}B_\rho + \frac{\partial}{\partial\rho}B_\phi = -i\frac{\omega}{c^2}E_z$$

or

$$ikE_z + \frac{1}{\rho}E_\rho + \frac{\partial}{\partial\rho}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\phi = 0$$

$$ikB_z + \frac{1}{\rho}B_\rho + \frac{\partial}{\partial\rho}B_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}B_\phi = 0$$

$$\frac{1}{\rho}E_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}E_\rho + \frac{\partial}{\partial\rho}E_\phi = i\omega B_z$$

$$\frac{1}{\rho}B_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}B_\rho + \frac{\partial}{\partial\rho}B_\phi = -i\frac{\omega}{c^2}E_z$$

From

$$-ikB_\phi + \frac{1}{\rho}\frac{\partial}{\partial\phi}B_z = -i\frac{\omega}{c^2}E_\rho$$

$$ikE_\rho - \frac{\partial}{\partial\rho}E_z = i\omega B_\phi$$

we have

$$E_\rho = \frac{i}{\rho\kappa_c{}^2}\left(\omega\frac{\partial B_z}{\partial\phi} + k\rho\frac{\partial E_z}{\partial\rho}\right)$$

$$B_\phi = \frac{i}{\rho\kappa_c{}^2}\left(k\frac{\partial B_z}{\partial\phi} + \rho\frac{\omega}{c^2}\frac{\partial E_z}{\partial\rho}\right)$$

From

$$-ikE_\phi + \frac{1}{\rho}\frac{\partial}{\partial\phi}E_z = i\omega B_\rho$$

$$ikB_\rho - \frac{\partial}{\partial\rho}B_z = -i\frac{\omega}{c^2}E_\phi$$

we have

$$E_\phi = -\frac{i}{\rho\kappa_c{}^2}\left(\rho\omega\frac{\partial B_z}{\partial\rho} - k\frac{\partial E_z}{\partial\phi}\right)$$

$$B_\rho = -\frac{i}{\rho\kappa_c{}^2}\left(-k\rho\frac{\partial B_z}{\partial\rho} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial\phi}\right)$$

From

$$ikE_z + \frac{1}{\rho}E_\rho + \frac{\partial}{\partial \rho}E_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}E_\phi = 0$$
$$ikB_z + \frac{1}{\rho}B_\rho + \frac{\partial}{\partial \rho}B_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}B_\phi = 0$$

we have

$$[\frac{\partial^2 B_z}{\partial^2 \phi} + \rho(\frac{\partial B_z}{\partial \rho} + \rho\frac{\partial^2 B_z}{\partial^2 \rho})] + \rho^2 \kappa_c{}^2 B_z = 0$$
$$[\frac{\partial^2 E_z}{\partial^2 \phi} + \rho(\frac{\partial E_z}{\partial \rho} + \rho\frac{\partial^2 E_z}{\partial^2 \rho})] + \rho^2 \kappa_c{}^2 E_z = 0$$

34S.11 TE$_{nm}$ modes

$E_z = 0$. B_z is an independent variable.

$$E_\rho = \frac{i\omega}{\rho\kappa_c{}^2}\frac{\partial B_z}{\partial \phi}$$
$$B_\phi = \frac{ik}{\rho\kappa_c{}^2}\frac{\partial B_z}{\partial \phi}$$

$$E_\phi = \frac{-i\omega}{\kappa_c{}^2}\frac{\partial B_z}{\partial \rho}$$
$$B_\rho = \frac{ik}{\kappa_c{}^2}\frac{\partial B_z}{\partial \rho}$$

$$\mathbf{B}_t = B_\rho \mathbf{e}_\rho + B_\phi \mathbf{e}_\phi = \frac{ik}{\kappa_c{}^2}[\frac{\partial B_z}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial B_z}{\partial \phi}\mathbf{e}_\phi] = \frac{ik}{\kappa_c{}^2}\nabla_t B_z$$

$$\mathbf{E}_t = E_\rho \mathbf{e}_\rho + E_\phi \mathbf{e}_\phi = -\frac{i\omega}{\kappa_c{}^2}[-\frac{1}{\rho}\frac{\partial B_z}{\partial \phi}\mathbf{e}_\rho + \frac{\partial B_z}{\partial \rho}\mathbf{e}_\phi] = -\frac{\omega}{k}(\mathbf{e}_z \times \mathbf{B}_t)$$

$$\rho^2 \kappa_c{}^2 B_z + \frac{\partial^2 B_z}{\partial^2 \phi} + \rho(\frac{\partial B_z}{\partial \rho} + \rho\frac{\partial^2 B_z}{\partial^2 \rho}) = 0$$

Suppose that

$$\frac{\partial^2 B_z}{\partial^2 \phi} = -n^2 B_z \qquad (n = 0, 1, 2, 3, \ldots)$$

$$\rho^2 \frac{\partial^2 B_z}{\partial^2 \rho} + \rho \frac{\partial B_z}{\partial \rho} + (\rho^2 k_c{}^2 - n^2) B_z = 0$$

where

$$\frac{\omega^2}{c^2} - k^2 = \kappa_c{}^2 \text{ (dispersion relation)}$$

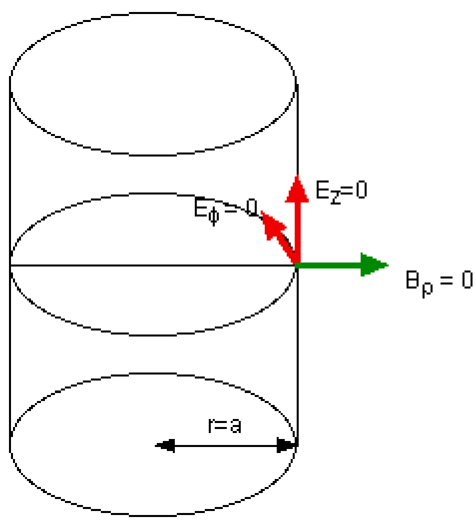
When $\kappa_c \rho = \xi$

$$\xi^2 \frac{\partial^2 B_z}{\partial^2 \xi} + \xi \frac{\partial B_z}{\partial \xi} + (\xi^2 - n^2) B_z = 0$$

(Bessel function)

$$B_z = A_1 J_n(\xi) + B_1 N_n(\xi)$$

$B_1 = 0$ since $N_n(\xi)$ is infinite at $\xi = 0$.



Boundary condition: surface of cylinder

$$E_z = 0, \qquad E_\phi = 0 \text{ (or } \frac{\partial B_z}{\partial \rho} = 0)$$

$E_z = 0$ is always satisfied because of TE mode. The boundary condition:

$$\frac{d}{d\xi}J_n(\xi)=0 \text{ at } \xi=\kappa_c a.$$

Then we have a solution for B_z,

$$B_z = B_0 J_n(\kappa_c\rho)\cos(n\theta+\varphi)$$

where B_0 and φ_0 are constants.

Without loss of generality, we can rotate the angular reference axis to make φ equal to zero.

$$B_z = B_0 J_n(\kappa_c\rho)\cos(n\theta).$$

The values of $\kappa_c a$ are as follows.

n =0

$\kappa_c\rho$ = 3.83171 (m = 1)
7.01559 (m = 2)
10.1735 (m = 3)
13.3237 (m = 4)
16.4706 (m = 5)
19.6159 (m = 6).

n =1

$\kappa_c\rho$ = 1.84118 (m = 1)
5.33144 (m = 2)
8.53632 (m = 3)
11.706 (m = 4)
14.8636 (m = 5)

n =2

$\kappa_c\rho$ = 3.05424 (m = 1)
6.70613 (m = 2)
9.96947 (m = 3)
13.1704 (m = 4)
17.7887 (m = 5)

n = 3

$\kappa_c\rho$ = 4.20119 (m = 1)
8.01524 (m = 2)
11.346 (m = 3)
14.5858 (m = 4)
17.7887 (m = 5)

$n = 4$

$\kappa_c \rho$ = 5.31755 (m = 1)

9.2824 (m = 2)

12.6819 (m = 3)

15.9641 (m = 4)

19.196 (m = 5)

$n = 5$

$\kappa_c \rho$ = 6.41562 (m = 1)

10.5199 (m = 2)

13.9872 (m = 3)

17.3128 (m = 4)

34S.12 $\quad$ TM$_{nm}$ mode

$B_z = 0$. E_z is an independent variable.

$$E_\rho = \frac{ik}{\kappa_c^2}\frac{\partial E_z}{\partial \rho}$$

$$B_\phi = \frac{i}{\kappa_c^2}\frac{\omega}{c^2}\frac{\partial E_z}{\partial \rho}$$

$$E_\phi = \frac{ik}{\rho\kappa_c^2}\frac{\partial E_z}{\partial \phi}$$

$$B_\rho = -\frac{i}{\rho\kappa_c^2}\frac{\omega}{c^2}\frac{\partial E_z}{\partial \phi}$$

$$\mathbf{E}_t = E_\rho \mathbf{e}_\rho + E_\phi \mathbf{e}_\phi = \frac{ik}{\kappa_c^2}\left(\frac{\partial E_z}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial E_z}{\partial \phi}\mathbf{e}_\phi\right) = \frac{ik}{\kappa_c^2}\nabla_t E_z$$

$$\mathbf{e}_z \times \mathbf{E}_t = \frac{ik}{\kappa_c^2}\left(\frac{\partial E_z}{\partial \rho}\mathbf{e}_z \times \mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial E_z}{\partial \phi}\mathbf{e}_z \times \mathbf{e}_\phi\right) = \frac{ik}{\kappa_c^2}\left(-\frac{1}{\rho}\frac{\partial E_z}{\partial \phi}\mathbf{e}_\rho + \frac{\partial E_z}{\partial \rho}\mathbf{e}_\phi\right)$$

$$\mathbf{B}_t = B_\rho \mathbf{e}_\rho + B_\phi \mathbf{e}_\phi = \frac{i\omega}{\kappa_c^2 c^2}\left[-\frac{1}{\rho}\frac{\partial E_z}{\partial \phi}\mathbf{e}_\rho + \frac{\partial E_z}{\partial \rho}\mathbf{e}_\phi\right] = \frac{\omega}{kc^2}(\mathbf{e}_z \times \mathbf{E}_t)$$

$$\left[\frac{\partial^2 E_z}{\partial^2 \phi} + \rho\left(\frac{\partial E_z}{\partial \rho} + \rho\frac{\partial^2 E_z}{\partial^2 \rho}\right)\right] + \rho^2\kappa_c^2 E_z = 0$$

Suppose that

$$\frac{\partial^2 E_z}{\partial^2 \phi} = -n^2 E_z \ \ (n = 0, 1, 2, 3, \ldots)$$

$$\rho^2 \frac{\partial^2 E_z}{\partial^2 \rho} + \rho \frac{\partial E_z}{\partial \rho} + (\rho^2 k_c{}^2 - n^2) E_z = 0$$

where

$$\frac{\omega^2}{c^2} - k^2 = \kappa_c{}^2 \ \ \text{(dispersion relation)}$$

When $\kappa_c \rho = \xi$

$$\xi^2 \frac{\partial^2 E_z}{\partial^2 \xi} + \xi \frac{\partial E_z}{\partial \xi} + (\xi^2 - n^2) E_z = 0$$

Then we have a solution for B_z,

$$E_z = E_0 J_n(\kappa_c \rho) \cos(n\theta)$$

where E_0 is constants.

Boundary condition:

$E_z = 0$ at $\rho = \mathrm{a}$.

$$J_n(\kappa_c a) = 0$$

or $\kappa_c a$ is a zero point of the Bessel function $J_n(\xi)$.

n =0

$\kappa_c \rho$ = 2.40483 (m = 1)
5.52008 (m = 2)
8.65373 (m = 3)
11.7915 (m = 4)
14.9309 (m = 5)
18.0711 (m = 6).

n =1

$\kappa_c \rho$ = 3.83171 (m = 1)
7.01559 (m = 2)
10.1735 (m = 3)
13.3237 (m = 4)

16.4706 (m = 5)

n =2

$\kappa_c\rho$ = 5.13562 (m = 1)
8.41724 (m = 2)
11.6198 (m = 3)
14.796 (m = 4)
17.5958 (m = 5)

n = 3

$\kappa_c\rho$ = 6.38016 (m = 1)
9.76102 (m = 2)
13.0152 (m = 3)
16.2235 (m = 4)
19.4094 (m = 5)

n = 4

$\kappa_c\rho$ = 7.58834 (m = 1)
11.0647(m = 2)
14.3725(m = 3)
17.616(m = 4)