Chapter 3
Damped simple harmonics

3.1 Solution of the differential equation

We now consider the simple harmonics with damping

$$x''(t) + 2\beta x'(t) + \omega_0{}^2 x(t) = 0$$

with the initial conditions

$$x'(0) = v_o \text{ and } x(0) = x_0$$

The solution of this differential equation depends is classed into three types,

(1) underdamping: $\beta^2 - \omega_0^2 < 0$.
(2) critical damping $\beta^2 - \omega_0^2 = 0$.
(3) overdamping $\beta^2 - \omega_0^2 > 0$.

The displacement $x(t)$ and the velocity $v(t)$ are given by

$$x(t) = e^{-\beta t}[x_0 \cos(\omega_1 t) + (\frac{\beta x_0 + v_0}{\omega_1})\sin(\omega_1 t)]$$

and

$$v(t) = \frac{dx(t)}{dt} = e^{-\beta t}\{v_0 \cos(\omega_1 t) - \frac{[\beta v_0 + x_0(\beta^2 + \omega_1{}^2)]}{\omega_1}\sin(\omega_1 t)\}$$

where

$$\omega_1 = \sqrt{\omega_0{}^2 - \beta^2}\,.$$

In the limit of $\beta \to \omega_0$ (the critical damping), we have

$$x(t) = e^{-\beta t}[x_0 + t(v_0 + x_0\beta)]$$

and

$$v(t) = e^{-\beta t}[v_0 - \beta(v_0 + \beta x_0)t]$$

4.2 Mathematica

- **second-order differential equation for a simple harmonics with damping**

```
Clear["Global`*"]
```

```
eq1 = {x''[t] + 2 β x'[t] + ω0^2 x[t] == 0, x'[0] == v0, x[0] == x0}
```

$$\left\{\omega 0^2\, x[t] + 2\beta\, x'[t] + x''[t] == 0,\ x'[0] == v0,\ x[0] == x0\right\}$$

```
eq2 = DSolve[eq1, x[t], t] // Simplify
```

$$\left\{\left\{x[t] \to \frac{1}{2\sqrt{\beta^2 - \omega 0^2}}\, e^{-t\left(\beta+\sqrt{\beta^2-\omega 0^2}\right)} \left(\left(-1 + e^{2t\sqrt{\beta^2-\omega 0^2}}\right) v0 + x0\left(\left(-1 + e^{2t\sqrt{\beta^2-\omega 0^2}}\right)\beta + \left(1 + e^{2t\sqrt{\beta^2-\omega 0^2}}\right)\sqrt{\beta^2 - \omega 0^2}\right)\right)\right\}\right\}$$

```
x[t_] = Simplify[x[t] /. eq2[[1]] /. {β^2 - ω0^2 → -ω1^2}, ω1 > 0];
```

```
x1[t_] = Exp[β t] x[t] // ExpToTrig // Simplify
```

$$\frac{x0\,\omega 1\,\mathrm{Cos}[t\,\omega 1] + (v0 + x0\,\beta)\,\mathrm{Sin}[t\,\omega 1]}{\omega 1}$$

```
x11[t_] = Exp[-β t] x1[t]
```

$$\frac{e^{-t\beta}\,(x0\,\omega 1\,\mathrm{Cos}[t\,\omega 1] + (v0 + x0\,\beta)\,\mathrm{Sin}[t\,\omega 1])}{\omega 1}$$

```
Limit[x11[t] /. ω1 → Sqrt[ω0^2 - β^2], β → ω0] // Simplify
```

$$e^{-t\,\omega 0}\,(x0 + t\,(v0 + x0\,\omega 0))$$

```
v11[t_] = D[x11[t], t] // Simplify
```

$$\frac{e^{-t\beta}\left(v0\,\omega 1\,\mathrm{Cos}[t\,\omega 1] - \left(v0\,\beta + x0\left(\beta^2 + \omega 1^2\right)\right)\mathrm{Sin}[t\,\omega 1]\right)}{\omega 1}$$

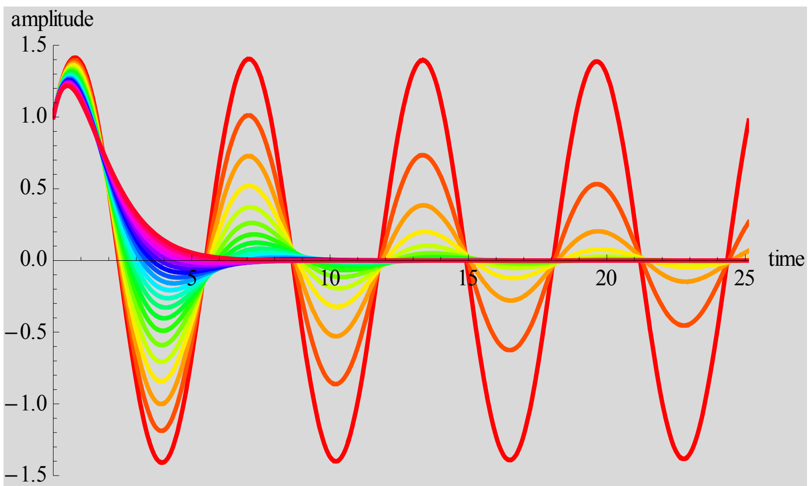
```
Limit[v11[t] /. ω1 → Sqrt[ω0^2 - β^2], β → ω0] // Simplify
```

$$e^{-t\,\omega 0}\left(v0 - t\,v0\,\omega 0 - t\,x0\,\omega 0^2\right)$$

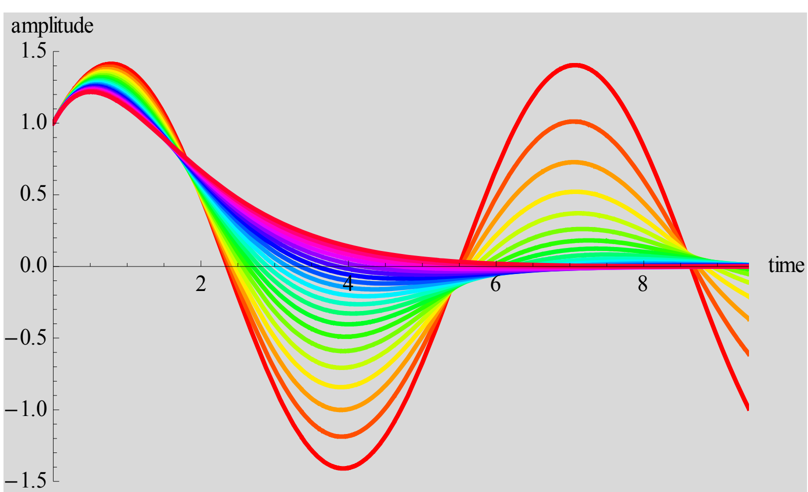
3.3 Time dependence of $x(t)$; the case of underdamping

We assume that $x_0 = 1$, $v_0 = 1$, and $\omega_0 = 1$. $\beta = \omega_0$ is the condition for the critical damping. Since $\omega_0 = 1$, the underdamping occurs for $\beta<1$. We make a plot of $x(t)$ as a function of t, where β is changed as a parameter (the case of underdamping for $\beta<1$).
((**Mathematica**))

```
x2[t_] = x11[t] /. ω1 → Sqrt[ω0^2 - β^2] // Simplify;

v2[t_] = v11[t] /. ω1 → Sqrt[ω0^2 - β^2] // Simplify;

rule1 = {ω0 → 1, x0 → 1, v0 → 1};

x3[t_, β_] = x2[t] /. rule1 // Simplify;

v3[t_, β_] = D[x3[t, β], t] /. rule1 // Simplify;

Plot[Evaluate[Table[x3[t, β], {β, 0.001, 1, 0.05}]], {t, 0, 8 π},
 PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 20}],
 AxesLabel → {"time", "amplitude"}, PlotRange → {{0, 8 π}, {-1.5, 1.5}},
 Background → LightGray]
```




```
Plot[Evaluate[Table[x3[t, β], {β, 0.001, 1, 0.05}]], {t, 0, 3 π},
 PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 20}],
 AxesLabel → {"time", "amplitude"}, PlotRange → {{0, 3 π}, {-1.5, 1.5}},
 Background → LightGray]
```



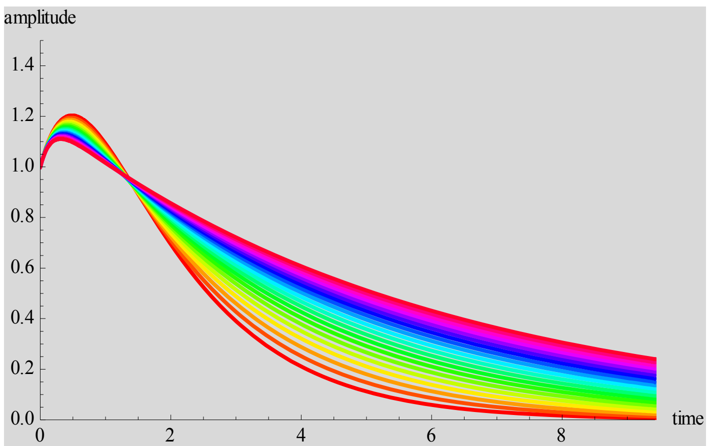
$x(t)$ vs t (β = 0.001 - 1.0, underdamping)

3.3 Time dependence of $x(t)$; the case of overdamping

We assume that $x_0 = 1$, $v_0 = 1$, and $\omega_0 = 1$. We make a plot of $x(t)$ as s function of t, where β is changed as a parameter (β>1 for the case of overdamping).

((**Mathematica**))

```
Plot[Evaluate[Table[x3[t, β], {β, 1.1, 3, 0.1}]], {t, 0, 3 π},
 PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 20}],
 AxesLabel → {"time", "amplitude"}, PlotRange → {{0, 3 π}, {0, 1.5}},
 Background → LightGray]
```



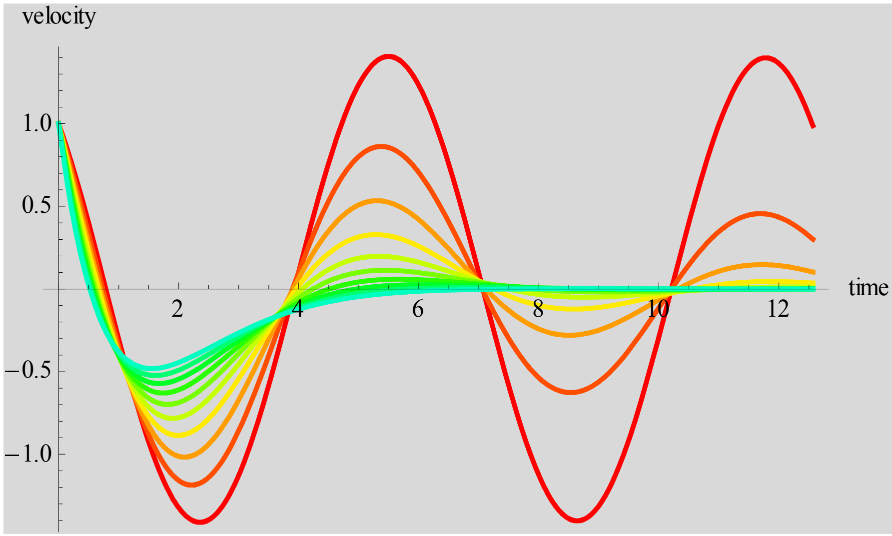
$x(t)$ vs t (β = 1.1 - 3.0, overdamping)

3.4 Time dependence of $v(t)$; the case of underdamping

We assume that $x_0 = 1$, $v_0 = 1$, and $\omega_0 = 1$. We make a plot of $v(t)$ as a function of t where β is changed as a parameter (β<1 for the case of underdamping).

((**Mathematica**))

```
Plot[Evaluate[Table[v3[t, β], {β, 0.001, 1, 0.1}]], {t, 0, 4 π},
 PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 10}],
 AxesLabel → {"time", "velocity"}, Prolog → AbsoluteThickness[2],
 Background → LightGray]
```

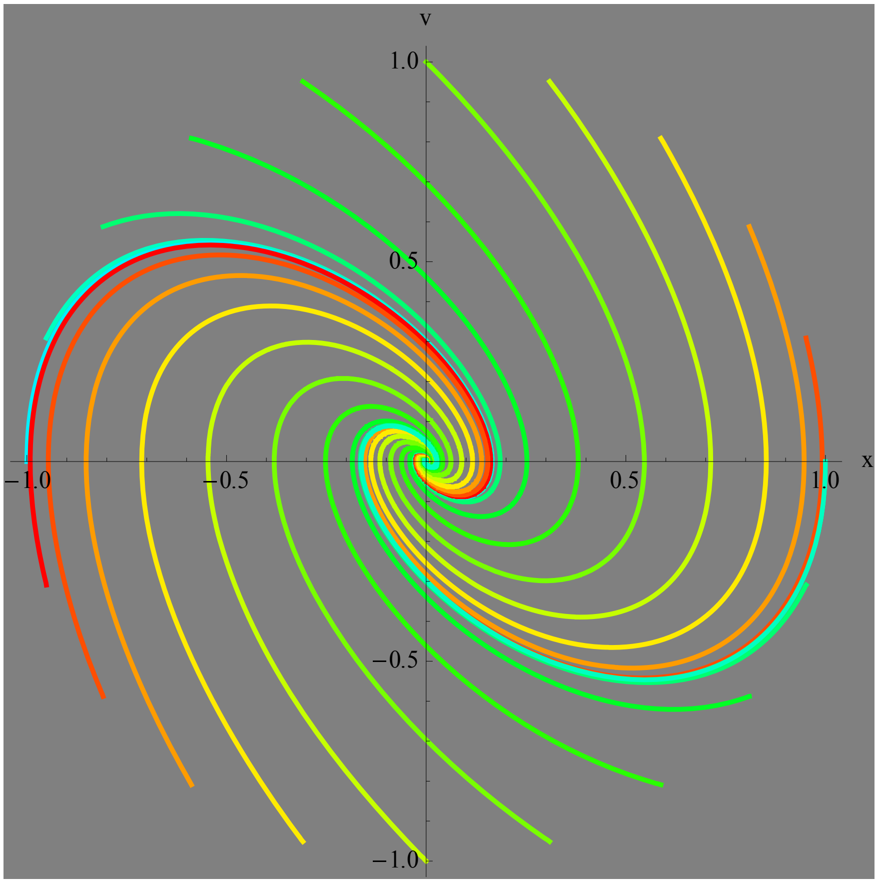


$v(t)$ vs t (β = 0.001 - 1.0, underdamping)

3.5 Phase space of $\{x(t), v(t)\}$ for the case of underdamping

We make a plot of the phase space $\{x(t), v(t)\}$ for β<1. In this case the locus is a spiral and reduces to the fixed point (or point attractor) (the origin) in the limit of $t\rightarrow\infty$.

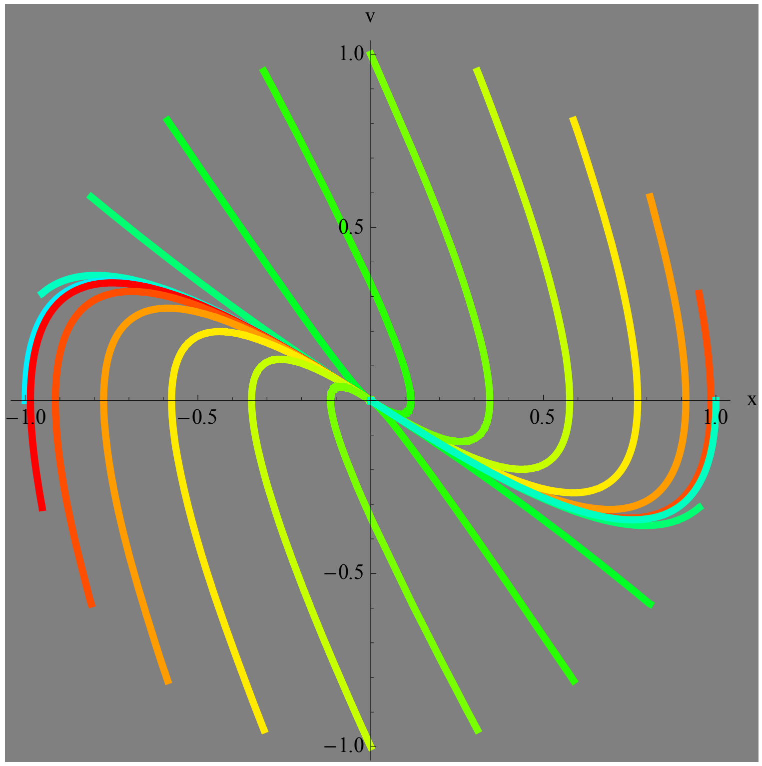
```
ParametricPlot[
 Evaluate[
  Table[{x2[t], v2[t]} /. {β → 0.5, ω0 → 1, x0 → Cos[θ], v0 → Sin[θ]},
   {θ, 0, 2 π, π / 10}]], {t, 0, 4 π},
 PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 10}],
 AxesLabel → {"x", "v"}, Background → Gray, PlotRange → All]
```



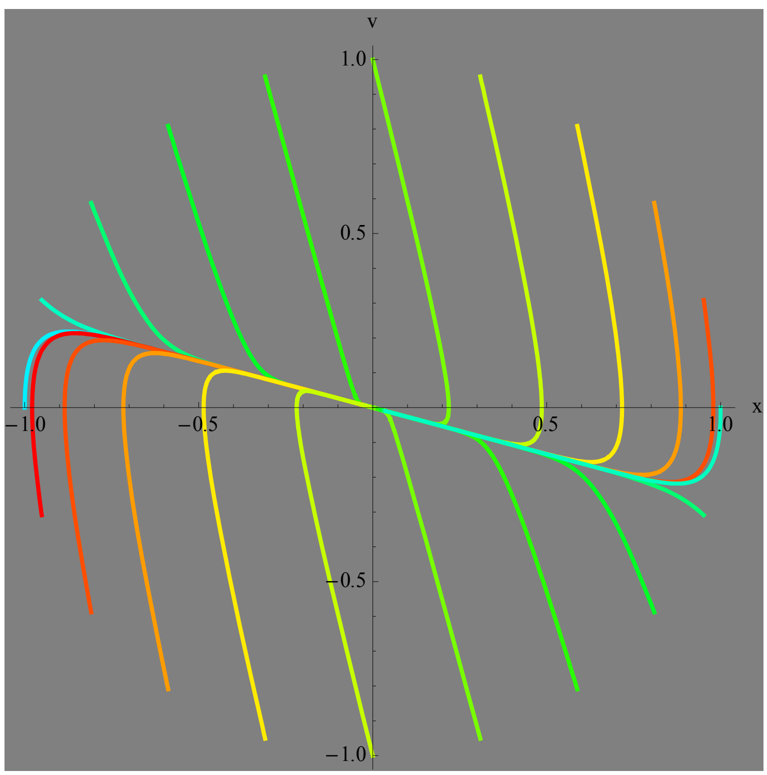
The phase space. $\beta = 0.5$ (underdamping)

3.6 Phase space of $\{x(t), v(t)\}$ for the case of overdamping ($\beta = 1.1$).

There is no spiral. The locus directly converges to the origin (attractor).



The phase space. $\beta = 1.1$ (overdamping)



The phase space. $\beta = 2.0$ (overdamping)

For β<1 (underdamping), the locus staring from a point far away from the origin in the phase space tends to approach to a straight line given by

$$v = -(\beta - \sqrt{\beta^2 - \omega_0{}^2})x \,.$$

The reason is as follows.

$$r = \lim_{t\to\infty} \frac{v(t)}{x(t)} = \frac{-\beta v_0 - x_0\omega_0{}^2 + v_0\sqrt{\beta^2 - \omega_0{}^2}}{v_0 + x_0(\beta + \sqrt{\beta^2 - \omega_0{}^2})}$$

when this ratio is equal to $v_0/x_0 = a$. Then we have

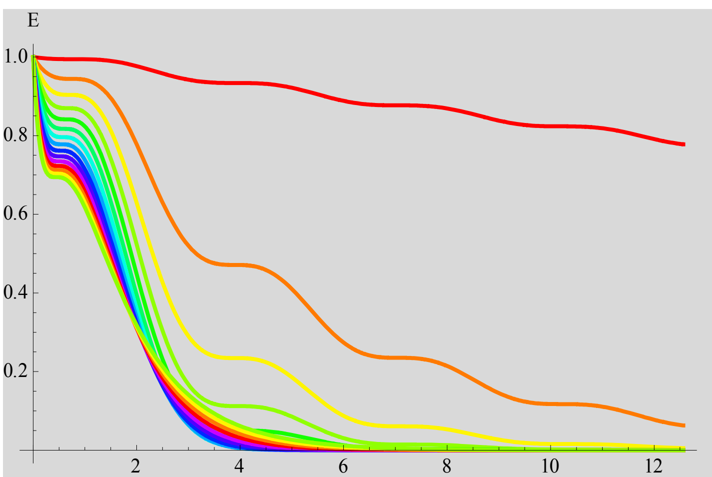
$$a = -\beta + \sqrt{\beta^2 - \omega_0{}^2} \,.$$

3.7 Time dependence of the mechanical energy

The mechanical energy is defined as

$$E(t) = \frac{1}{2}m[v(t)]^2 + \frac{1}{2}k[x(t)]^2 \,.$$

We assume that $x_0 = 1$, $v_0 = 1$, and $\omega_0 = 1$. We make a plot of $E(t)$ as s function of t, where β is changed as a parameter (β= 0.01 - 1.2)).



3.8 Forced oscillation (steady state solution)

$$x''(t) + 2\beta x'(t) + \omega_0{}^2 x(t) = \xi_0 \cos(\omega t)$$

We assume that $x(t)$ can be given by

$$x(t) = \mathrm{Re}[Ae^{i\omega t}]$$

Re denotes a real part. A is in general a complex number. i (= $\sqrt{-1}$) is a pure imaginary

((Note))
Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

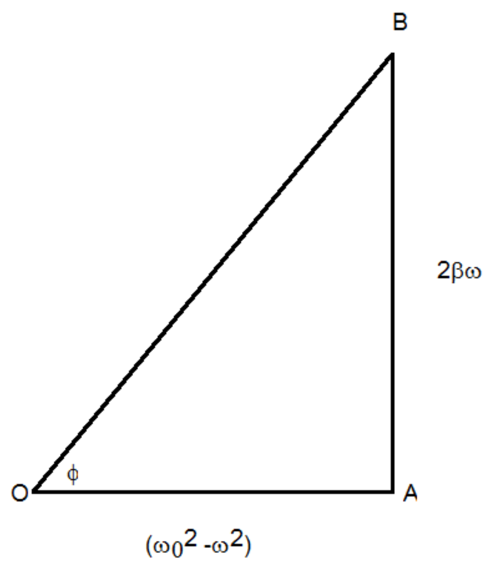
R.P. Feynman: This is the most remarkable formula. This is our jewel (22-10 volume-1, Feynman's lecture on physics.)

Then we have

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^{\ 2}x(t) = \mathrm{Re}[(-\omega^2 + 2\beta i\omega + \omega_0^{\ 2})Ae^{i\omega t}] = \mathrm{Re}[\xi_0 e^{i\omega t}]$$

or

$$A = \frac{\xi_0}{(-\omega^2 + \omega_0^{\ 2}) + 2\beta i\omega} = \frac{\xi_0}{\sqrt{(-\omega^2 + \omega_0^{\ 2})^2 + 4\beta^2\omega^2}} e^{-i\phi}$$



Then x is obtained as

$$x(t) = \mathrm{Re}[Ae^{i\omega t}] = \mathrm{Re}[\frac{\xi_0}{\sqrt{(-\omega^2 + \omega_0^{\ 2})^2 + 4\beta^2\omega^2}} e^{i(\omega t - \phi)}]$$

$$= \frac{\xi_0}{\sqrt{(-\omega^2 + \omega_0^{\ 2})^2 + 4\beta^2\omega^2}} \cos(\omega t - \phi)$$

or

$$\left|\frac{A}{\xi_0}\right| = \frac{1}{\sqrt{(-\omega^2 + {\omega_0}^2)^2 + 4\beta^2\omega^2}} = \frac{1}{{\omega_0}^2\sqrt{(\frac{\omega^2}{{\omega_0}^2} - 1)^2 + 4\frac{\beta^2}{{\omega_0}^2}\frac{\omega^2}{{\omega_0}^2}}}$$

or

$$Y = \left|\frac{A}{\xi_0}\right|{\omega_0}^2 = \frac{1}{\sqrt{(\frac{\omega^2}{{\omega_0}^2} - 1)^2 + 4\frac{\beta^2}{{\omega_0}^2}\frac{\omega^2}{{\omega_0}^2}}} = \frac{1}{\sqrt{(x^2 - 1)^2 + 4\varsigma^2 x^2}}$$

Now we calculate the value of Y as a function of x when a parameter ζ is changed.

$$x = \frac{\omega}{\omega_0}$$

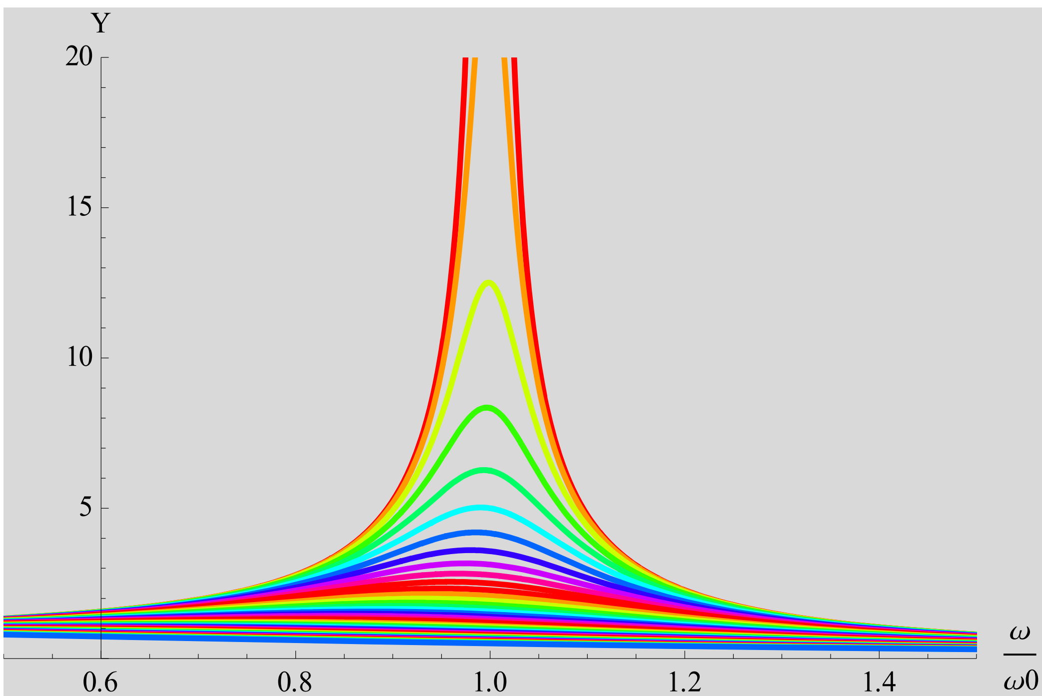
$$\varsigma = \frac{\beta}{\omega_0}$$

((**Mathematica**))

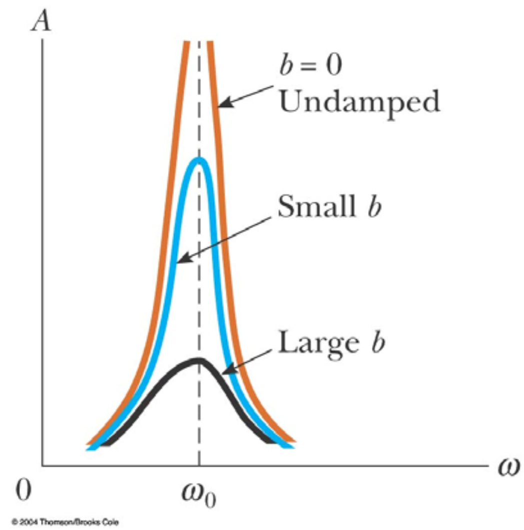
$$Y = \frac{1}{\sqrt{(x^2-1)^2 + 4\,\zeta^2 x^2}}$$

$$\frac{1}{\sqrt{(-1+x^2)^2 + 4\,x^2\,\zeta^2}}$$

```
Plot[Evaluate[Table[Y, {ζ, 0, 1, 0.02}]], {x, 0.5, 1.5},
 PlotRange → {{0.5, 1.5}, {0, 20}},
 PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}],
 Background → LightGray, AxesLabel → {"ω/ω0", "Y"}]
```



Y vs $\frac{\omega}{\omega_0}$ where $\zeta (= \frac{\beta}{\omega_0})$ is changed as a parameter. $\zeta = 0 - 1.0$.



3.9. Energy consideration in the forced oscillation

We start from

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^{\,2} x(t) = \xi_0 \cos(\omega t).$$

Multiplying $\dot{x}(t)$ on both sides, we have

$$\dot{x}'(t)\ddot{x}(t) + 2\beta[\dot{x}(t)]^2 + \omega_0^{\,2} x(t)\dot{x}(t) = \xi_0 \cos(\omega t)\dot{x}(t),$$

or

$$\frac{d}{dt}\{\frac{1}{2}[\dot{x}(t)]^2 + \frac{\omega_0^{\,2}}{2}[x(t)]^2\} + 2\beta[\dot{x}(t)]^2 = \xi_0 \cos(\omega t)\dot{x}(t).$$

This equation can be rewritten as

$$\{\frac{1}{2}[\dot{x}(t)]^2 + \frac{\omega_0^{\,2}}{2}[x(t)]^2 - \frac{1}{2}[\dot{x}(t=0)]^2 - \frac{\omega_0^{\,2}}{2}[x(t=0)]^2 + \int_0^t 2\beta[\dot{x}(t_1)]^2 dt_1 = \int_0^t \xi_0 \cos(\omega t_1)\dot{x}(t_1) dt_1$$

Here we introduce the instantaneous energy $\varepsilon(t)$ which is defined by

$$\varepsilon(t) = \frac{1}{2}[\dot{x}(t)]^2 + \frac{\omega_0^{\,2}}{2}[x(t)]^2.$$

We take an average of the above equation over a one period T,

$$\frac{1}{T}\{\varepsilon(t=T)-\varepsilon(t=0)\}+\frac{1}{T}\int_0^T 2\beta[\dot{x}(t_1)]^2 dt_1-\frac{1}{T}\int_0^T \xi_0\cos(\omega t_1)\dot{x}(t_1)dt_1=0 .$$

We now calculate the second and third terms using our steady-state solution

$$x(t)=\mathrm{Re}[Ae^{i\omega t}]$$
$$\dot{x}(t)=\mathrm{Re}[A(i\omega)e^{i\omega t}],$$

with

$$A=\frac{\xi_0}{(-\omega^2+\omega_0{}^2)+2\beta i\omega}=\frac{\xi_0[(-\omega^2+\omega_0{}^2)-2\beta i\omega]}{(-\omega^2+\omega_0{}^2)^2+4\beta^2\omega^2}=\chi'-i\chi''$$

and

$$iA=\frac{i\xi_0}{(-\omega^2+\omega_0{}^2)+2\beta i\omega}=\frac{\xi_0[i(-\omega^2+\omega_0{}^2)+2\beta\omega]}{(-\omega^2+\omega_0{}^2)^2+4\beta^2\omega^2}$$

where χ' and χ'' are the real part and imaginary part of *A*.

The calculation of the second term:

$$\begin{aligned}\frac{1}{T}\int_0^T 2\beta[x'(t_1)]^2 dt_1 &= \frac{2\beta}{T}\int_0^T \frac{[A(i\omega)e^{i\omega t_1}+A^*(-i\omega)e^{-i\omega t_1}][A(i\omega)e^{i\omega t_1}+A^*(-i\omega)e^{-i\omega t_1}]}{4}dt_1\\
&=\frac{2\beta}{4T}[2A(i\omega)A^*(-i\omega)]\\
&=\frac{\beta}{T}|A|^2\omega^2\\
&=\frac{\beta}{2\pi}|A|^2\omega^3\\
&=\frac{\beta}{2\pi}\xi_0{}^2\frac{\omega^3}{(-\omega^2+\omega_0{}^2)^2+4\beta^2\omega^2}\\
&=\frac{\xi_0\omega^2}{4\pi}\chi''\end{aligned}$$

The calculation of the third term:

$$
\begin{aligned}
\frac{1}{T}\int_0^T \xi_0 \cos(\omega t_1) x'(t_1) dt_1 &= \frac{\xi_0}{T}\int_0^T \frac{(e^{i\omega t_1} + e^{-i\omega t_1})[A(i\omega)e^{i\omega t_1} + A^*(-i\omega)e^{-i\omega t_1}]}{4} dt_1 \\
&= \frac{\xi_0}{4T}[A(i\omega) + A^*(-i\omega)] \\
&= \frac{\xi_0}{2T}\mathrm{Re}[A(i\omega)] \\
&= \frac{\xi_0 \omega}{2T} \frac{\xi_0 2\beta\omega}{(-\omega^2 + \omega_0{}^2)^2 + 4\beta^2\omega^2} \\
&= \frac{\beta}{2\pi}\xi_0{}^2 \frac{\omega^3}{(-\omega^2 + \omega_0{}^2)^2 + 4\beta^2\omega^2} \\
&= \frac{\xi_0 \omega^2}{4\pi}\chi''
\end{aligned}
$$

Then it is found that the second term is equal to the third term. These terms are proportional to χ'' (imaginary part of *A*). The energy absorbed by the system from the external force is dissipated through the resistive damping. Then we have

$$\varepsilon(t = T) = \varepsilon(t = 0) .$$

The sum of the kinetic energy and the potential energy is a periodic function of *t* with a period of *T*.