Chapter 40 Coherent state

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40.1 Displacement operator

$$[\hat{a},\hat{a}^{+}]=\hat{1}$$

The displacement operator

$$\hat{D}_{\alpha}=\exp(\alpha\hat{a}^{+}-\alpha^{*}\hat{a})=\exp(-\frac{1}{2}\left|\alpha\right|^{2})\exp(\alpha\hat{a}^{+})\exp(-\alpha^{*}\hat{a})$$

(Baker-Hausdorff's relation)

Suppose that $\alpha^{*}=\alpha$. Then we have

$$\hat{D}_{\alpha}=\exp(\alpha\hat{a}^{+}-\alpha^{*}\hat{a})=\exp[\alpha(\hat{a}^{+}-\hat{a})],$$

where

$$\hat{a}^{+}-\hat{a}=-i\sqrt{\frac{2}{m\hbar\omega_{0}}}\hat{p},$$

or

$$\hat{D}_{\alpha}=\exp[-\frac{i}{\hbar}\frac{\sqrt{2}\alpha}{\beta}\hat{p}],$$

with

$$\beta=\sqrt{\frac{m\omega_{0}}{\hbar}}, \qquad a=\frac{\sqrt{2}\alpha}{\beta}.$$

This operator coincides with the translation operator

$$\hat{T}(a)=\exp[-\frac{i}{\hbar}\hat{p}a],$$

where

$$\hat{T}(a)|x\rangle = |x+a\rangle$$

or

$$\langle x|\hat{T}^{+}(a) = \langle x+a|$$

Note that

$$\langle x|\hat{D}_{\alpha}|\psi\rangle = \langle x|\hat{D}_{-\alpha}{}^{+}|\psi\rangle = \left\langle x - \frac{\sqrt{2}\alpha}{\beta}\middle|\psi\right\rangle = \psi(x - \frac{\sqrt{2}\alpha}{\beta})$$

40.2 Properties of $\hat{D}_{\alpha}$

$$\hat{D}_{\alpha}{}^{+} = \hat{D}_{-\alpha} = \exp(-\alpha\hat{a}^{+} + \alpha^{*}\hat{a}) = \exp(-\frac{1}{2}|\alpha|^{2})\exp(-\alpha\hat{a}^{+})\exp(\alpha^{*}\hat{a})$$

$$\hat{D}_{\alpha} = \exp(\alpha\hat{a}^{+} - \alpha^{*}\hat{a}) = \exp(-\frac{1}{2}|\alpha|^{2})\exp(\alpha\hat{a}^{+})\exp(-\alpha^{*}\hat{a})$$

$$\hat{D}_{\alpha}{}^{+}\hat{a}\hat{D}_{\alpha} = \alpha\hat{1} + \hat{a}$$

$$\hat{D}_{\alpha}{}^{+}\hat{a}^{+}\hat{D}_{\alpha} = \alpha^{*}\hat{1} + \hat{a}^{*}$$

$$[\exp(\alpha\hat{a}^{+}),\hat{a}] = -\exp(\alpha\hat{a}^{+})\alpha$$

$$[\exp(\alpha\hat{a}),\hat{a}^{+}] = \exp(\alpha\hat{a})\alpha$$

$$\hat{D}_{\alpha}{}^{+}\hat{N}\hat{D}_{\alpha} = \hat{N} + |\alpha|^{2}\hat{1} + \alpha^{*}\hat{a} + \alpha\hat{a}^{+}$$

with

$$\hat{N} = \hat{a}^{+}\hat{a}$$

$$\hat{D}_{\alpha}{}^{+}\hat{a}\hat{a}\hat{D}_{\alpha} = \hat{a}\hat{a} + \alpha^{2}\hat{1} + 2\alpha\hat{a} = (\hat{a} + \alpha\hat{1})^{2}$$

$$\hat{D}_{\alpha}{}^{+}\hat{a}^{+}\hat{a}^{+}\hat{D}_{\alpha} = \hat{a}^{+}\hat{a}^{+} + (\alpha^{*})^{2}\hat{1} + 2\alpha^{*}\hat{a}^{+} = (\hat{a}^{+} + \alpha^{*}\hat{1})^{2}$$

In general

$$\hat{D}_{\alpha}{}^{+} f(\hat{a},\hat{a}^{+})\hat{D}_{\alpha} = f(\hat{a} + \alpha\hat{1}, \hat{a}^{+} + \alpha^{*}\hat{1})$$

where *f* is any function of $\hat{a}$ and $\hat{a}^{+}$ with a power series expansion.

$$\hat{D}_\alpha \hat{D}_\beta = \exp(\frac{\alpha\beta^* - \alpha^*\beta}{2})\hat{D}_{\alpha+\beta}$$

$$[\hat{D}_\alpha, \hat{D}_\beta] = 0$$

only when

$$\alpha\beta^* - \alpha^*\beta = 2|\alpha||\beta|i\sin(\theta_\alpha - \theta_\beta) = 0$$

or

$$\theta_\alpha = \theta_\beta$$

$$\hat{D}_\beta \hat{D}_\alpha = \exp(-\frac{\alpha\beta^* - \alpha^*\beta}{2})\hat{D}_{\alpha+\beta}$$

$$\hat{D}_\alpha \hat{D}_\beta = \exp(\alpha\beta^* - \alpha^*\beta)\hat{D}_\beta \hat{D}_\alpha$$

40.3 Coherent state $|\alpha\rangle$

((**Simple harmonics**))

$$\hat{a} | n >= \sqrt{n} | n-1 >$$

$$\hat{a}^+ | n >= \sqrt{n+1} | n+1 >$$

$$\hat{N} | n >= n | n >$$

$$| n >= \frac{1}{\sqrt{n!}}(\hat{a}^+)^n | 0 >$$

$$[\hat{N}, \hat{a}] = -\hat{a}$$

$$[\hat{N}, \hat{a}^+] = \hat{a}^+$$

$$|\alpha\rangle = \hat{D}_\alpha|0\rangle = \exp(-\frac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a})|0\rangle$$

Here

$$\exp(-\alpha^{*}\hat{a})|0\rangle = |0\rangle$$

then we have

$$|\alpha\rangle = \hat{D}_{\alpha}|0\rangle = \exp(-\frac{1}{2}|\alpha|^{2})\exp(\alpha\hat{a}^{+})|0\rangle$$

This is the definition of $|\alpha\rangle$.

$$\hat{a}|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^{2})\hat{a}\exp(\alpha\hat{a}^{+})|0\rangle .$$

Note that

$$[\exp(\alpha\hat{a}^{+}),\hat{a}] = -\exp(\alpha\hat{a}^{+})\alpha$$

or

$$\exp(\alpha\hat{a}^{+})(\alpha\hat{1}+\hat{a}) = \hat{a}\exp(\alpha\hat{a}^{+})$$

Thus

$$\hat{a}|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^{2})\exp(\alpha\hat{a}^{+})(\alpha\hat{1}+\hat{a})|0\rangle$$
$$= \alpha\exp(-\frac{1}{2}|\alpha|^{2})\exp(\alpha\hat{a}^{+})|0\rangle = \alpha|\alpha\rangle$$

or

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

which means that $|\alpha\rangle$ is the eigenket of $\hat{a}$ with the eigenvalue α.

40.4 The explicit expression of $|\alpha\rangle$ in terms of $|n\rangle$

$$|\alpha\rangle = \hat{D}_{\alpha}|0\rangle = \exp(-\frac{1}{2}|\alpha|^{2})\exp(\alpha\hat{a}^{+})|0\rangle$$

$$|n> = \frac{1}{\sqrt{n!}}(\hat{a}^{+})^{n}|0>,$$

$$\left|\alpha\right\rangle = \exp(-\frac{1}{2}\left|\alpha\right|^2)\sum_{n=0}^{\infty}\frac{\alpha^n(\hat{a}^+)^n}{n!}\left|0\right\rangle = \exp(-\frac{1}{2}\left|\alpha\right|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle,$$

or

$$\left\langle n|\alpha\right\rangle = \exp(-\frac{1}{2}\left|\alpha\right|^2)\frac{\alpha^n}{\sqrt{n!}},$$

$$\left|\left\langle n|\alpha\right\rangle\right|^2 = \exp(-\left|\alpha\right|^2)\frac{\left|\alpha\right|^{2n}}{n!},$$

which corresponds to the Poisson distribution function. We now consider the single mode coherent state

$$< N >= \left\langle\alpha\right|\hat{N}\left|\alpha\right\rangle = \left\langle\alpha\right|\hat{a}^+\hat{a}\left|\alpha\right\rangle = \left|\alpha\right|^2,$$

$$< N^2 > \left\langle\alpha\right|\hat{N}^2\left|\alpha\right\rangle = \left\langle\alpha\right|\hat{a}^+\hat{a} + \hat{a}^+\hat{a}^+\hat{a}\hat{a}\left|\alpha\right\rangle = \left|\alpha\right|^2,$$

$$\Delta N = \sqrt{\left\langle\alpha\right|\hat{N}^2\left|\alpha\right\rangle - \left\langle\alpha\right|\hat{N}\left|\alpha\right\rangle^2} = \left|\alpha\right|,$$

or

$$\frac{\Delta N}{< N >} = \frac{1}{\left|\alpha\right|}.$$

40.5 Unitary operator $\hat{R}_\lambda$

The Unitary operator:

$$\hat{R}_\lambda = \exp(i\lambda\hat{a}^+\hat{a})$$

where λ is real.

$$\hat{R}_\lambda{}^+\hat{a}\hat{R}_\lambda = \exp(i\lambda)\hat{a}$$

Note that

$$\exp(\hat{A})\hat{B}\exp(-\hat{A}) = \hat{B} + \frac{[\hat{A},\hat{B}]}{1!} + \frac{[\hat{A},[\hat{A},\hat{B}]]}{2!} + \frac{[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]}{3!} + ...$$

$$\hat{A} = -i\lambda\hat{a}^{+}\hat{a}\,,\ \hat{B} = \hat{a}\,,\ \text{and } [\hat{N},\hat{a}] = -\hat{a}$$

$$[\hat{A},\hat{B}] = -i\lambda[\hat{a}^{+}\hat{a},\hat{a}] = -i\lambda[\hat{N},\hat{a}] = i\lambda\hat{a}$$

$$[\hat{A},[\hat{A},\hat{B}]] = -i\lambda[\hat{a}^{+}\hat{a},i\lambda\hat{a}] = -(i\lambda)^{2}[\hat{N},\hat{a}] = (i\lambda)^{2}\hat{a}$$

$$[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] = -i\lambda[\hat{a}^{+}\hat{a},(i\lambda)^{2}\hat{a}] = -(i\lambda)^{3}[\hat{N},\hat{a}] = (i\lambda)^{3}\hat{a}$$

and so on.

Let us show that

$$\hat{R}_{\lambda}\left|\alpha\right\rangle = c\left|e^{i\lambda}\alpha\right\rangle$$

We note that

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$$

$$\hat{R}_{\lambda}^{+}\hat{a}\hat{R}_{\lambda}\left|\alpha\right\rangle = e^{i\lambda}\hat{a}\left|\alpha\right\rangle = \alpha e^{i\lambda}\left|\alpha\right\rangle$$

or

$$\hat{a}(\hat{R}_{\lambda}\left|\alpha\right\rangle) = e^{i\lambda}\hat{a}\left|\alpha\right\rangle = \alpha e^{i\lambda}(\hat{R}_{\lambda}\left|\alpha\right\rangle)$$

which means that $\hat{R}_{\lambda}\left|\alpha\right\rangle$ is the eigenket of $\hat{a}$ with the eigenvalue $\alpha e^{i\lambda}$. Thus we have

$$\hat{R}_{\lambda}\left|\alpha\right\rangle = c\left|e^{i\lambda}\alpha\right\rangle.$$

$\left|e^{i\lambda}\alpha\right\rangle$ is the coherent state. c is the phase factor.

40.6 Orthogonality

The coherent state ($\left|\alpha\right\rangle$) do not form an orthogonal state.

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle,$$

$$\hat{a}\left|\beta\right\rangle = \beta\left|\beta\right\rangle,$$

$$\left|\alpha\right\rangle = \hat{D}_\alpha\left|0\right\rangle = \exp(-\frac{1}{2}\left|\alpha\right|^2)\exp(\alpha\hat{a}^+)\exp(-\alpha^*\hat{a})\left|0\right\rangle$$
$$= \exp(-\frac{1}{2}\left|\alpha\right|^2)\exp(\alpha\hat{a}^+)\left|0\right\rangle$$

$$\left\langle\beta\right| = \left\langle 0\right|\hat{D}_\beta{}^+ = \left\langle 0\right|\hat{D}_{-\beta} = \left\langle 0\right|\exp(-\frac{1}{2}\left|\beta\right|^2)\exp(\beta^*\hat{a}),$$

$$\left\langle\beta\middle|\alpha\right\rangle = \exp(-\frac{1}{2}\left|\alpha\right|^2)\exp(-\frac{1}{2}\left|\beta\right|^2)\left\langle 0\right|\exp(\beta^*\hat{a})\exp(\alpha\hat{a}^+)\left|0\right\rangle.$$

Note that

$$\exp(\beta^*\hat{a})\exp(\alpha\hat{a}^+)\exp(-\beta^*\hat{a})) = \exp(\alpha\hat{a}^+)\exp(\alpha\beta^*),$$

or

$$\exp(\beta^*\hat{a})\exp(\alpha\hat{a}^+) = \exp(\alpha\beta^*)\exp(\alpha\hat{a}^+)\exp(\beta^*\hat{a}),$$

$$\left\langle\beta\middle|\alpha\right\rangle = \exp(-\frac{\left|\alpha\right|^2+\left|\beta\right|^2}{2}+\alpha\beta^*)\left\langle 0\right|\exp(\alpha\hat{a}^+)\exp(\beta^*\hat{a})\left|0\right\rangle,$$

or

$$\left\langle\beta\middle|\alpha\right\rangle = \exp(-\frac{\left|\alpha\right|^2+\left|\beta\right|^2}{2}+\alpha\beta^*).$$

We also note that

$$\left|\alpha-\beta\right|^2 = (\alpha-\beta)(\alpha-\beta)^* = \left|\alpha\right|^2+\left|\beta\right|^2-(\alpha\beta^*+\alpha^*\beta),$$

$$\left|\left\langle\beta\middle|\alpha\right\rangle\right|^2 = \exp(-\left|\alpha\right|^2-\left|\beta\right|^2+\alpha\beta^*+\alpha^*\beta) = \exp(-\left|\alpha-\beta\right|^2).$$

The distance $\left|\alpha-\beta\right|^2$ measures the degree to which the two eigenstates are approximately orthogonal.

40.7 Closure relation

$$\hat{K} = \frac{1}{\pi}\int |\alpha\rangle d^2\alpha \langle\alpha| = \hat{1}.$$

The integration is extended over the entire α plane with a real element of area. Let us give a proof for this.

$$|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

$$\langle\alpha| = \exp(-\frac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{(\alpha^*)^n}{\sqrt{n!}}\langle n|.$$

We calculate $\hat{K}$ defined by

$$\begin{aligned}\hat{K} &= \frac{1}{\pi}\int |\alpha\rangle d^2\alpha \langle\alpha| \\ &= \frac{1}{\pi}\int \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \exp(-|\alpha|^2) d^2\alpha \sum_{m=0}^{\infty}\frac{(\alpha^*)^m}{\sqrt{m!}}\langle m|\end{aligned}$$

We assume that

$$\alpha = |\alpha| e^{i\theta}$$

then we have

$$\alpha^n(\alpha^*)^m = |\alpha|^{n+m} e^{i(n-m)\theta}$$

$$J = \int d^2\alpha \exp(-|\alpha|^2)|\alpha|^{n+m}\exp[i(n-m)\theta]$$

with

$$d^2\alpha = |\alpha| d\theta d|\alpha|$$

Then

$$J = \iint |\alpha| d\theta d|\alpha| \exp(-|\alpha|^2)|\alpha|^{n+m}\exp[i(n-m)\theta]$$

where $0 \le |\alpha| < \infty$ and $0 \le \theta \le 2\pi$. Note that

$$\int_0^{2\pi} e^{i(n-m)\theta} d\theta = 2\pi\delta_{n,m}$$

$$J = 2\pi\delta_{n,m}\int_0^\infty d|\alpha|\exp(-|\alpha|^2)|\alpha|^{2n} = \pi n!\,\delta_{n,m}$$

Thus

$$\hat{K} = \frac{1}{\pi}\sum_{n.m}|n\rangle\langle m|\frac{\pi n!}{n!}\delta_{n,m} = \sum_n |n\rangle\langle n| = \hat{1},$$

Using the closure relation, we have

$$|\alpha\rangle = \int \frac{d^2\beta}{\pi}|\beta\rangle\langle\beta|\alpha\rangle,$$

where

$$\langle\beta|\alpha\rangle^2 = \exp(-|\alpha-\beta|^2)\ .$$

This means that a coherent state forms an complete set and that the simultaneous measurement of $\hat{a}_1$ and $\hat{a}_2$, represented by the projection operator $|\alpha\rangle\langle\alpha|$ is not an exact measurement but instead an approximate measurement with a finite error measurement.

40.8 $|x\rangle$-representation of the coherent state

If we multiply the left-hand side of

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$

by $\langle q|$ and use

$$\hat{a} = \frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i\hat{p}}{m\omega_0}\right),$$

with

$$\beta = \sqrt{\frac{m\omega_0}{\hbar}}\ .$$

Then we have

$$\langle x|\hat{a}|\alpha\rangle = \alpha\langle x|\alpha\rangle$$

or

$$\langle x|\frac{\beta}{\sqrt{2}}\left(\hat{x}+\frac{i\hat{p}}{m\omega_0}\right)|\alpha\rangle=\alpha\langle x|\alpha\rangle,$$

or

$$\frac{\hbar}{m\omega_0}\frac{\partial}{\partial x}\langle x|\alpha\rangle+(x-\sqrt{2}\frac{\alpha}{\beta})\langle x|\alpha\rangle=0.$$

The solution for $\langle x|\alpha\rangle$ can be solved as

$$\langle x|\alpha\rangle=\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}\exp[-\frac{m\omega_0}{2\hbar}(x-\sqrt{2}\frac{\alpha}{\beta})^2],$$

where

$$\int_{-\infty}^{\infty}\left|\langle x|\alpha\rangle\right|^2 dx=1.$$

Note that

$$<x>=\int_{-\infty}^{\infty}x\left|\langle x|\alpha\rangle\right|^2 dx=\sqrt{2}\frac{\alpha}{\beta},$$

$$<(\Delta x)^2>=\int_{-\infty}^{\infty}(x-<x>)^2\left|\langle x|\alpha\rangle\right|^2 dx=\frac{\hbar}{2m\omega_0},$$

$$<p>=\frac{\hbar}{i}\int_{-\infty}^{\infty}\langle x|\alpha\rangle\frac{\partial}{\partial x}\langle x|\alpha\rangle dx=0,$$

$$<(\Delta p)^2>=\left(\frac{\hbar}{i}\right)^2\int_{-\infty}^{\infty}\langle x|\alpha\rangle\frac{\partial^2}{\partial x^2}\langle x|\alpha\rangle dx=\frac{m\hbar\omega_0}{2}.$$

Then we have

$$\Delta x\Delta p=\sqrt{<(\Delta x)^2>}\sqrt{<(\Delta p)^2>}=\sqrt{\frac{\hbar}{2m\omega_0}}\sqrt{\frac{m\hbar\omega_0}{2}}=\frac{\hbar}{2}.$$

The wavefunction $\langle x|\alpha\rangle$ is indeed the minimum uncertainty wave-packet with stationary quantum uncertainty.