Chapter 9
Complex functions
Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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Baron **Augustin-Louis Cauchy** (21 August 1789 – 23 May 1857; French pronunciation was a French mathematician who was an early pioneer of analysis. He started the project of formulating and proving the theorems of infinitesimal calculus in a rigorous manner. He also gave several important theorems in complex analysis and initiated the study of permutation groups in abstract algebra. A profound mathematician, Cauchy exercised a great influence over his contemporaries and successors. His writings cover the entire range of mathematics and mathematical physics.

http://en.wikipedia.org/wiki/Augustin-Louis_Cauchy

9.1 Function of a complex

$$z = x + iy .$$

z is a complex number and both x and y are real.

Suppose we write $f(z)$ - that is a function of z - by this we mean that for each value of z - $f(z)$ can take a complex value.

$f(z)$ is a function of two real variables x and y.

$$f(z) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real functions.

$$f(z) = z^3 = (x+iy)^3$$

$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$\begin{cases} u = x^3 - 3xy^2 \\ v = 3x^2y - y^3 \end{cases}$$

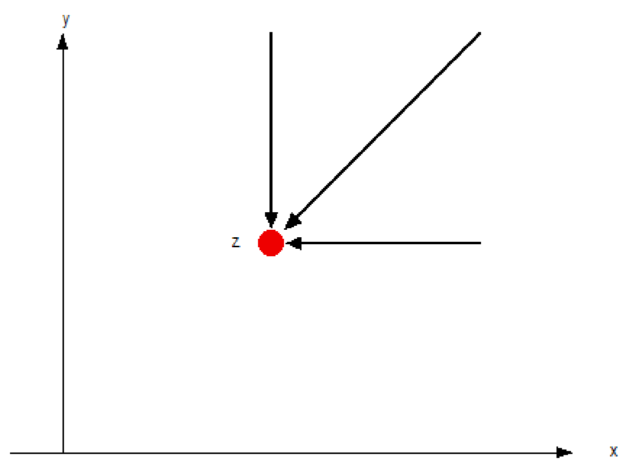
(a)

In our study of complex functions, we will consider those sorts of functions which turn up in physical problems.
-i.e., our functions will have a “smoothness requirement”.
-i.e., they are differentiable.

(1) How do we define the derivative of a complex function?
It seems like it should be

$$\frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

But Δz has two paths - i.e., $\Delta z = \Delta x + i\Delta y$
-thus $\Delta z \to 0$ can assume in an infinite number of ways.
$\Delta z = \Delta r e^{i\alpha}$ and $\Delta z \to 0$ is obtained when $\Delta r \to 0$ but α can have any value.



((**Example**))

Say $f(z) = x + 2iy$ and let us compute directly

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} \qquad \text{for } z = 0,$$

and two different α values:

(a) $\alpha = 0$, $\Delta z = \Delta r$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta r \to 0} \frac{\{(x+\Delta r)+2iy\}-(x+2iy)}{\Delta r} = 1$$

(b) $\alpha = \pi/2$, $\Delta z = \Delta r e^{i\pi/2} = i\Delta r$ and $\Delta x = 0$.

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta r \to 0} \frac{\{x+2i(y+\Delta r)\}-(x+2iy)}{i\Delta r} = 2$$

Thus for $f(z) = x + 2iy$, we do not get a unique result. We say that this $f(z)$ is not differentiable.

For a function of z to be differentiable at $z = z_0$, we need $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ to exist (i.e. finite) and the result should be independent of any Δz.

9.2 Cauchy-Riemann conditions

Let's see what restrictions are imposed on $f(z)$ of it is to be differentiable.

$$f(z) = u(x,y) + iv(x,y)$$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y)-u(x,y)]+i[v(x+\Delta x, y+\Delta y)-v(x,y)]}{\Delta x + i\Delta y}$$

We get unique result no matter what α is in $\Delta z = \Delta r e^{i\alpha}$.

For α = 0, $\Delta x \neq 0$, $\Delta y = 0$

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad (1)$$

For α = π/2, $\Delta x = 0$, $\Delta y \neq 0$

$$\frac{df}{dz} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad (2)$$

And both results must be the same

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{(Cauchy-Riemann condition)}$$

or

$$u_x = v_y$$
$$v_x = -u_y$$

The Cauchy-Riemann conditions are necessary for the existence of a derivative of $f(z)$.

$\Downarrow$

If df/dz exists, the Cauchy-Riemann conditions must hold. Conversely, if the Cauchy-Riemann conditions are satisfied and the partial derivatives of *u* and *v* are continuous, then df/dz exists.

$$\delta f = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$\frac{\delta f}{\delta z} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y}{\delta x + i\delta y} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{\delta y}{\delta x}}{1 + i\frac{\delta y}{\delta x}}$$

Note that

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right),$$

using the Cauchy-Riemann conditions.

Then we have

$$\frac{\delta f}{\delta z} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \qquad (1)$$

On the other hand

$$\left(\frac{\delta f}{\delta z}\right)_{\substack{\delta y=0 \\ \delta x\to 0}} = \frac{\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y}{i\delta y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = -i\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

(2)

$$\left(\frac{\delta f}{\delta z}\right)_{\substack{\delta x=0\\ \delta y\to 0}} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right), \qquad (3)$$

Eqs. (1) ,(2), and (3) show that $\lim_{\delta z\to 0}\frac{\delta f}{\delta z}$ is independent of the direction of approach in the complex plane as long as the partial derivatives are continuous.

((Note))

Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial y \partial x} \\ \dfrac{\partial^2 v}{\partial x \partial y} = -\dfrac{\partial^2 u}{\partial y^2} \end{cases} \quad \Rightarrow \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u = 0 \qquad \text{Laplace eq. for 2D}$$

Similarly

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y \partial x} \\ \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y^2} \end{cases} \quad \Rightarrow \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)v = 0 \qquad \text{Laplace eq. for 2D}$$

Therefore, both u and v are harmonic function that satisfy the Laplace's equation. Furthermore, comparing the 2D gradients

$$\begin{aligned} \nabla u &= (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = (\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}) \\ \nabla v &= (\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}) = (-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}) \end{aligned},$$

Then we have

$$\nabla u \cdot \nabla v = 0 .$$

The lines of constant u (level curves) are orthogonal to the lines of constant v, anywhere that $f'(z) \neq 0$. If u represents a potential function, then v represents the corresponding stream function (lines of force), or vice versa.

((**Example**))

$$f(z) = z^2 = u + iv = x^2 - y^2 + i2xy$$

or

$$u = x^2 - y^2, \quad v = 2\,xy.$$

We make a ContourPlot of u = const, a StreamPlot of ∇u, a ContourPlot of v = const and a StreamPlot of ∇v by using the Mathematica.

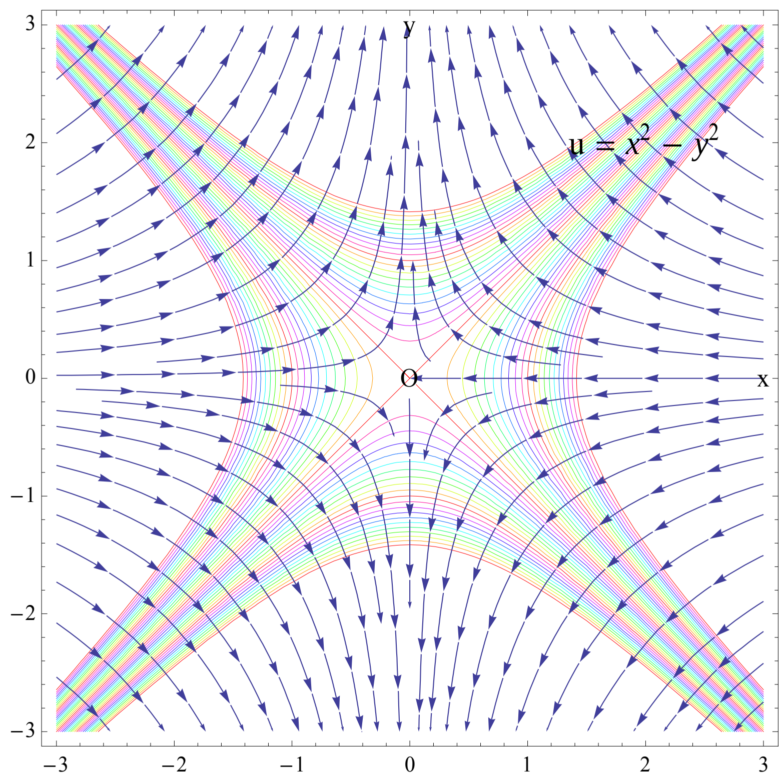


Fig. A ContourPlot of $u = x^2 - y^2$ = constant and a StreamPlot of ∇u .

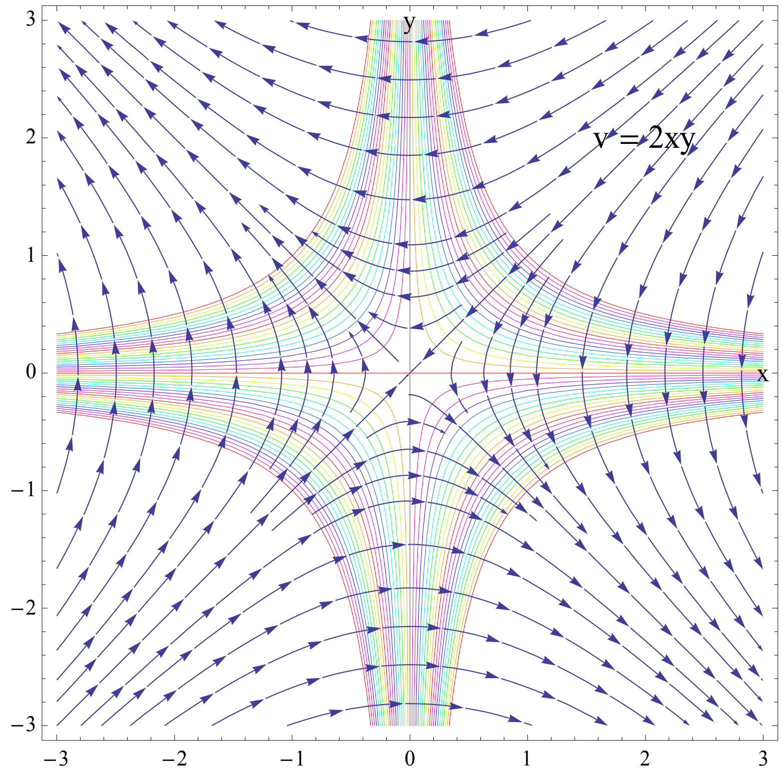


Fig. A ContourPlot of $v = 2xy$ = constant and a StreamPlot of ∇v .

9.3 Example

James J. Kelly
Graduate Mathematical Physics
Level curves of $f(z)=z^2=u+i v$ are shown.

```
f = (x + ⅈ y)^3 // Expand
```

$x^3 + 3\,i\,x^2\,y - 3\,x\,y^2 - i\,y^3$

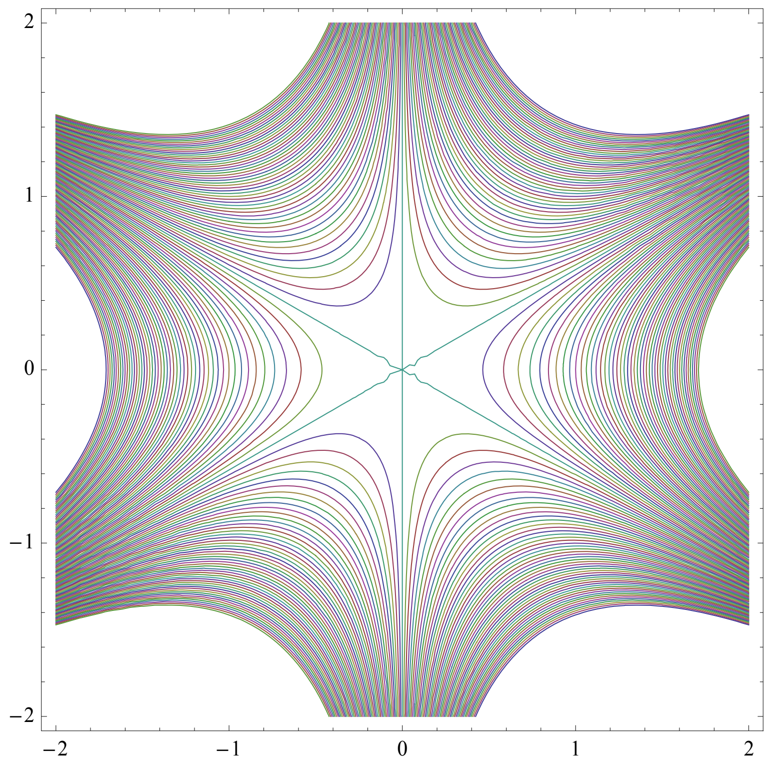
```
u = Simplify[Re[f], {x ∈ Reals, y ∈ Reals}]
```

$x^3 - 3\,x\,y^2$

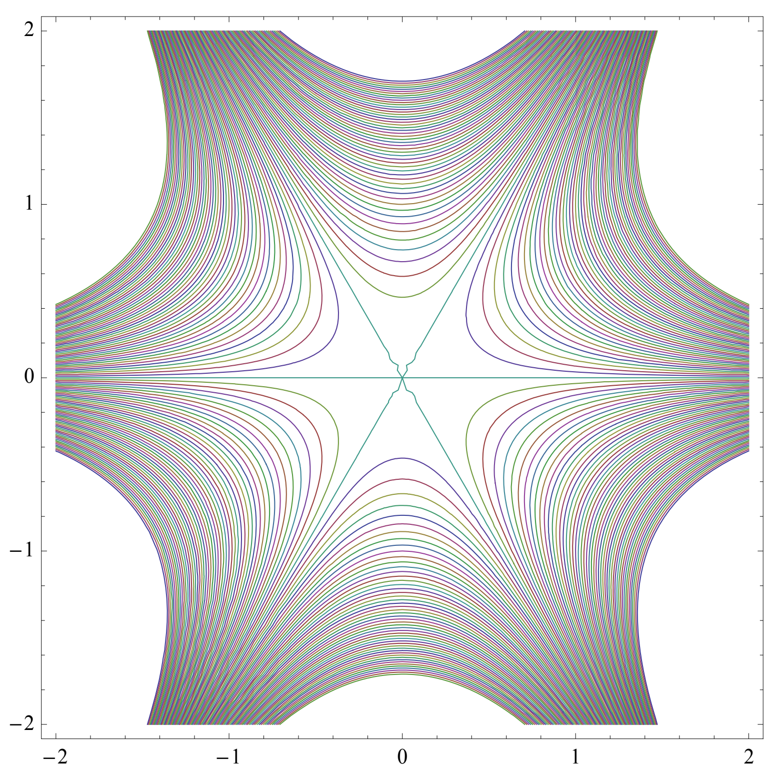
```
v = Simplify[Im[f], {x ∈ Reals, y ∈ Reals}]
```

$3\,x^2\,y - y^3$

```
ContourPlot[Evaluate[Table[u == k, {k, -5, 5, 0.1}]],
 {x, -2, 2}, {y, -2, 2}]
```



```
ContourPlot[Evaluate[Table[v == k, {k, -5, 5, 0.1}]],
 {x, -2, 2}, {y, -2, 2}]
```



9.4 Example

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=\frac{\partial u}{\partial (iy)}+i\frac{\partial v}{\partial (iy)}$$

$$f(z)=z^3=(x+iy)^3=(x^3-3xy^2)+i(3x^2y-y^3)$$

$$u=(x^3-3xy^2),\ v=(3x^2y-y^3)$$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=3x^2-3y^2+i6xy$$

or

$$f'(z)=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=-i(-6xy)+3x^2-3y^2=3x^2-3y^2+i6xy$$

((**Laplace equation**))

$$\frac{\partial^2 u}{\partial x^2}=6x,\ \frac{\partial^2 u}{\partial y^2}=-6x$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u=0$$

Can we write $f'(z)$ as a derivative? The answer is yes.

$$f'(z)=2z^2=3(x+iy)^2=3(x^2-y^2+2ixy)$$
$$=3(x^2-y^2)+6ixy$$

9.5 Mathematica

Cauchy-Riemann condition

Cauchy-Rieman conditions

```
SuperStar /: expr_* := expr /. {Complex[a_, b_] :> Complex[a, -b]}
```

```
RC[f_] := Module[{u, v, w, uy, vx, ux, vy}, w = ComplexExpand[f[x + ⅈ y]];
  u = (w + w*) / 2 // Simplify; v = (w - w*) / (2 ⅈ) // Simplify;
  uy = D[u, y] // Simplify; vx = D[v, x] // Simplify;
  ux = D[u, x] // Simplify; vy = D[v, y] // Simplify;
  uxx = D[ux, x] // Simplify; uyy = D[uy, y] // Simplify;
  vxx = D[vx, x] // Simplify; vyy = D[vy, y] // Simplify;
  List[{"u", u}, {"v", v}, {"uy", uy}, {"vx", vx}, {"ux", ux},
   {"vy", vy}, {"uy+vx", uy + vx}, {"ux-vy", ux - vy},
   {"uxx+uyy", uxx + uyy}, {"vxx+vyy", vxx + vyy}]]
```

```
f1 = Function[{z}, z^4]
```

Function[{z}, z^4]

```
RC[f1] // Simplify // TableForm
```

u	$x^4 - 6 x^2 y^2 + y^4$
v	$4 x y (x^2 - y^2)$
uy	$4 y (-3 x^2 + y^2)$
vx	$12 x^2 y - 4 y^3$
ux	$4 (x^3 - 3 x y^2)$
vy	$4 (x^3 - 3 x y^2)$
uy+vx	0
ux-vy	0
uxx+uyy	0
vxx+vyy	0

```
f2 = Function[{z}, z^3]
```

Function[{z}, z^3]

```
RC[f2] // Simplify // TableForm
```

u	$x^3 - 3 x y^2$
v	$3 x^2 y - y^3$
uy	$-6 x y$
vx	$6 x y$
ux	$3 (x^2 - y^2)$
vy	$3 (x^2 - y^2)$
uy+vx	0
ux-vy	0
uxx+uyy	0
vxx+vyy	0

```
f3 = Function[{z}, z^5]
```

$\text{Function}\left[\{z\},\ z^5\right]$

```
RC[f3] // Simplify // TableForm
```

u	$x^5 - 10\,x^3\,y^2 + 5\,x\,y^4$
v	$5\,x^4\,y - 10\,x^2\,y^3 + y^5$
uy	$20\,x\,y\,\left(-x^2 + y^2\right)$
vx	$20\,x\,y\,\left(x^2 - y^2\right)$
ux	$5\,\left(x^4 - 6\,x^2\,y^2 + y^4\right)$
vy	$5\,\left(x^4 - 6\,x^2\,y^2 + y^4\right)$
uy+vx	0
ux−vy	0
uxx+uyy	0
vxx+vyy	0

```
f4 = Function[{z}, 1/z]
```

$\text{Function}\left[\{z\},\ \frac{1}{z}\right]$

```
RC[f4] // Simplify // TableForm
```

u	$\frac{x}{x^2+y^2}$
v	$-\frac{y}{x^2+y^2}$
uy	$-\frac{2\,x\,y}{\left(x^2+y^2\right)^2}$
vx	$\frac{2\,x\,y}{\left(x^2+y^2\right)^2}$
ux	$\frac{-x^2+y^2}{\left(x^2+y^2\right)^2}$
vy	$\frac{-x^2+y^2}{\left(x^2+y^2\right)^2}$
uy+vx	0
ux−vy	0
uxx+uyy	0
vxx+vyy	0

```
f5 = Function[{z}, z z*]

Function[{z}, z z*]

RC[f5] // Simplify // TableForm

u        x^2 + y^2
v        0
uy       2 y
vx       0
ux       2 x
vy       0
uy+vx    2 y
ux-vy    2 x
uxx+uyy  4
vxx+vyy  0

f6 = Function[{z}, z*]

Function[{z}, z*]

RC[f6] // Simplify // TableForm

u        x
v        -y
uy       0
vx       0
ux       1
vy       -1
uy+vx    0
ux-vy    2
uxx+uyy  0
vxx+vyy  0
```

((**Definition**)) Analytic

If $f(z)$ is differentiable at $z = z_0$ and in some small region around z_0, we say that $f(z)$ is **analytic** at $z = z_0$. If $f(z)$ is analytic everywhere in the (finite) complex plane, we call it an **entire** function.

((**Example**)) **Arfken 6-2-8**

Using $f(re^{i\theta}) = R(r,\theta)e^{i\Phi(r,\theta)}$, in which $R(r,\theta)$ and $\Phi(r,\theta)$ are differentiable real functions of r and θ, show that the Cauchy-Riemann condition in polar coordinates become

(a) $\dfrac{\partial R(r,\theta)}{\partial r} = \dfrac{R(r,\theta)}{r}\dfrac{\partial\Theta(r,\theta)}{\partial\theta}$, (b) $\dfrac{1}{r}\dfrac{\partial R(r,\theta)}{\partial\theta} = -R(r,\theta)\dfrac{\partial\Theta(r,\theta)}{\partial r}$

Hint: set up the derivative first with dz radial ($r \to r$+dr) and then with dz tangential ($r \to \theta$+ dθ).

((Mathematica))

Arfken 6 - 2 - 8

(a) Derivative with δr while θ fixed

```
Clear["Global`*"]
```

$$\text{eq1} = \frac{R[r+\delta r,\ \theta]\, e^{i\,\Phi[r+\delta r,\theta]} - R[r,\ \theta]\, e^{i\,\Phi[r,\theta]}}{(r+\delta r)\, e^{i\,\theta} - (r)\, e^{i\,\theta}} \ \text{// Simplify}$$

$$\frac{e^{-i\,\theta}\left(-e^{i\,\Phi[r,\theta]}\,R[r,\ \theta] + e^{i\,\Phi[r+\delta r,\theta]}\,R[r+\delta r,\ \theta]\right)}{\delta r}$$

We apply the L' Hospital theorem to calculate.

$$\text{eq11} = \frac{D\left[e^{-i\,\theta}\left(-e^{i\,\Phi[r,\theta]}\,R[r,\ \theta] + e^{i\,\Phi[r+\delta r,\theta]}\,R[r+\delta r,\ \theta]\right),\ \delta r\right] \text{ /. } \delta r \to 0}{D[\delta r,\ \delta r] \text{ /. } \delta r \to 0}$$

$$e^{-i\,\theta}\left(e^{i\,\Phi[r,\theta]}\,R^{(1,0)}[r,\ \theta] + i\,e^{i\,\Phi[r,\theta]}\,R[r,\ \theta]\,\Phi^{(1,0)}[r,\ \theta]\right)$$

(b) Derivative with $\delta\theta$ while r fixed

$$\text{eq2} = \frac{R[r,\ \theta+\delta\theta]\, e^{i\,\Phi[r,\theta+\delta\theta]} - R[r,\ \theta]\, e^{i\,\Phi[r,\theta]}}{(r)\, e^{i\,(\theta+\delta\theta)} - (r)\, e^{i\,\theta}} \ \text{// Simplify}$$

$$\frac{e^{-i\,\theta}\left(-e^{i\,\Phi[r,\theta]}\,R[r,\ \theta] + e^{i\,\Phi[r,\delta\theta+\theta]}\,R[r,\ \delta\theta+\theta]\right)}{\left(-1+e^{i\,\delta\theta}\right)\, r}$$

We apply the L' Hospital theorem to calculate.

$$\text{eq22} = \frac{D\left[e^{-i\,\theta}\left(-e^{i\,\Phi[r,\theta]}\,R[r,\ \theta] + e^{i\,\Phi[r,\delta\theta+\theta]}\,R[r,\ \delta\theta+\theta]\right),\ \delta\theta\right] \text{ /. } \delta\theta \to 0}{D\left[\left(-1+e^{i\,\delta\theta}\right)\, r,\ \delta\theta\right] \text{ /. } \delta\theta \to 0} \ \text{// Expand}$$

$$-\frac{i\,e^{-i\,\theta+i\,\Phi[r,\theta]}\,R^{(0,1)}[r,\ \theta]}{r} + \frac{e^{-i\,\theta+i\,\Phi[r,\theta]}\,R[r,\ \theta]\,\Phi^{(0,1)}[r,\ \theta]}{r}$$

$$\text{eq3} = r\,(\text{eq11} - \text{eq22})\, e^{-i\,\Phi[r,\theta]}\, e^{i\,\theta} \ \text{// FullSimplify}$$

$$i\,R^{(0,1)}[r,\ \theta] + r\,R^{(1,0)}[r,\ \theta] - R[r,\ \theta]\left(\Phi^{(0,1)}[r,\ \theta] - i\,r\,\Phi^{(1,0)}[r,\ \theta]\right)$$

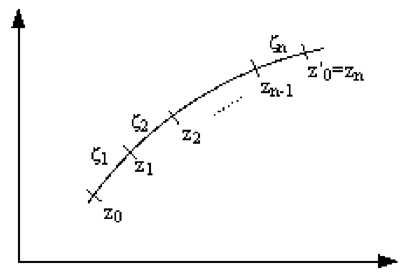
The real part and the imaginary part of eq3=0,

$-R[r,\ \theta]\,\Phi^{(0,1)}[r,\ \theta] + r\,R^{(1,0)}[r,\ \theta]=0$

$R^{(0,1)}[r,\ \theta] + r\,R[r,\ \theta]\,\Phi^{(1,0)}[r,\ \theta]=0$

9.6 Cauchy's integral theorem

Integral of a complex variable over a contour in the complex plane. We divide the contour $z_0 z_0'$ into n intervals by picking n-1 intermediate points (ζ_i) on the contour.



Consider the sum

$$S_n = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}),$$

where ζ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with $|z_j - z_{j-1}| \to 0$ for all j.

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)dz,$$

(contour integral)

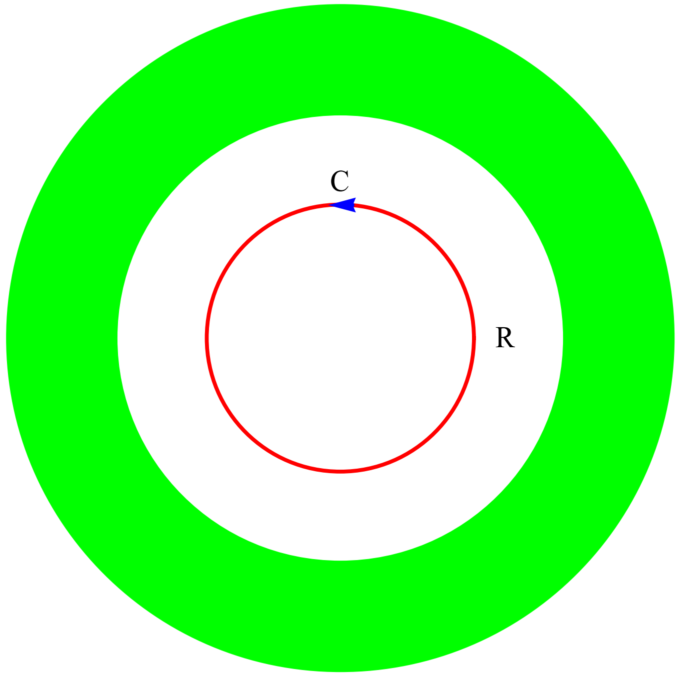
(exists and is independent of the details of choosing points z_j and ζ_j.)

$$\int_{z_1}^{z_2} f(z)dz = \int_{x_1y_1}^{x_2y_2} (u+iv)(dx+idy) = \int_{x_1y_1}^{x_2y_2} (udx - vdy) + i\int_{x_1y_1}^{x_2y_2} (vdx + udy).$$

9.7 Cauchy's integral theorem

If a function $f(z)$ is analytic and its partial derivatives are continuous throughout some simple connected region R, for every closed path C in R, the line integral of $f(z)$ around C is zero:

$$\int_C f(z)dz = \oint f(z)dz = 0.$$



(A) **Stroke's theorem proof**

$$dz = dx + idy\ ,$$

$$f(z) = u(x,u) + iv(x,y)\,,$$

$$\oint f(z)dz = \oint (udx - vdy) + i\oint (vdx + udy)\,.$$

We consider the Stroke's theorem

$$\oint \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{r}$$

with

$$d\mathbf{a} = dxdy\hat{\mathbf{z}}$$

$$\Rightarrow \oint_C (A_x dx + A_y dy) = \int_S (\nabla \times \mathbf{A})_z\, dxdy = \int_S \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dxdy$$

Let $A_x = u$, $A_y = -v$.

$$\text{First term } = \oint_C (udx - vdy) = \int_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

Let $A_x = v$, $A_y = u$.

$$\text{Second term } = \oint_C (vdx + udy) = \int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy$$

Thus

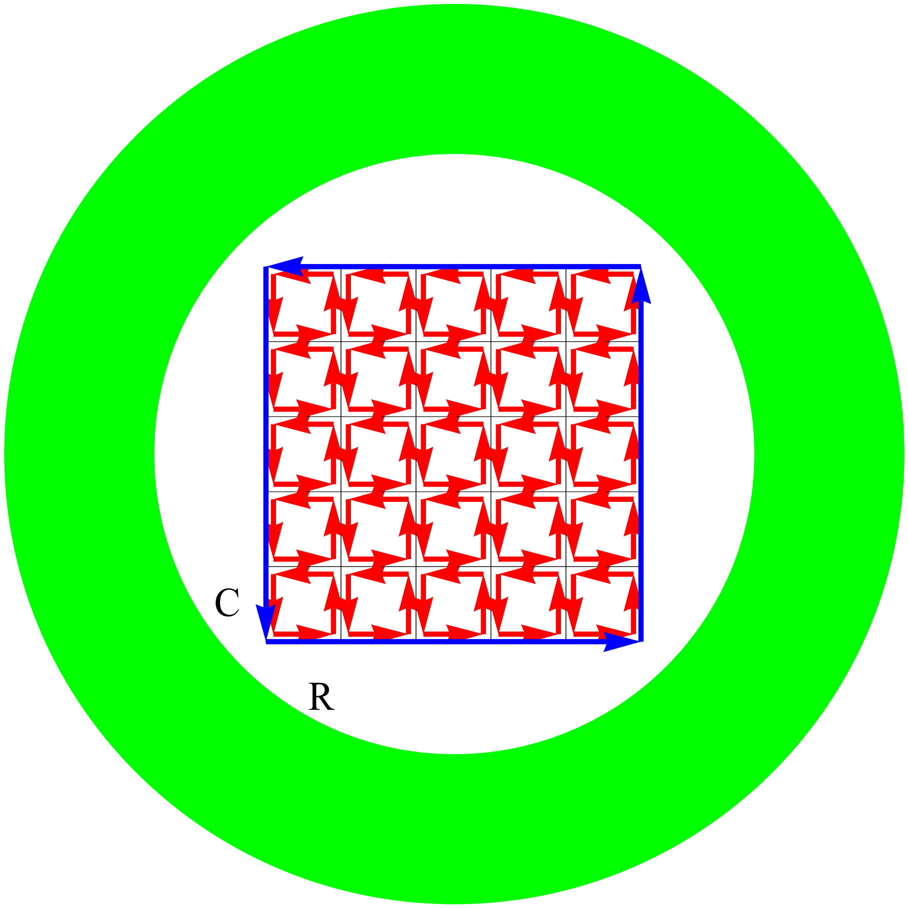
$$\begin{aligned} \oint f(z)dz &= \oint (udx - vdy) + i \oint (vdx + udy) \\ &= -\int_S \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) dxdy + i \int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy = 0 \end{aligned}$$

If $f(z)$ is analytic, the Cauchy-Riemann condition is satisfied.

(B) Cauchy-Goursat Proof

Suppose that $f(z)$ is analytic and its partial derivatives are continuous throughout the region *R*.

$$\oint_C f(z)dz = 0$$



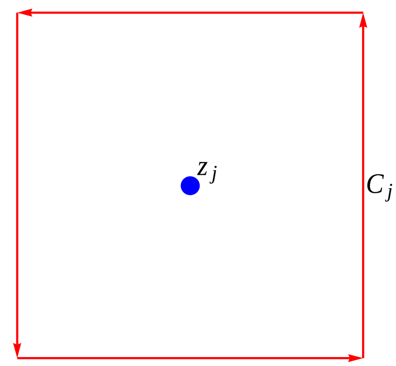
We subdivide the region inside the contour C into a network of small squares. Then

$$\oint_C f(z)dz = \sum_j \oint_{C_j} f(z)dz .$$

All integrals along interior lines cancel out. Now we consider

$$\oint_{C_j} f(z)dz ,$$

with the contour path C_j,



z_j: an interior point of the *j*-th subregion. We construct the function

$$\delta_j(z, z_j) = \frac{f(z) - f(z_j)}{z - z_j} - \left.\frac{df}{dz}\right|_{z=z_j}.$$

We may make for $^\forall \varepsilon > 0$,

$$\left|\delta_j(z, z_j)\right| < \varepsilon$$

where e is an arbitrary chosen small positive quantity.

$$(z - z_j)\delta_j(z, z_j) = f(z) - f(z_j) - \left.\frac{df}{dz}\right|_{z=z_j}(z - z_j)$$

$$\oint_{C_j} f(z)dz = \oint_{C_j}(z - z_j)\delta_j(z, z_j)dz + \oint_{C_j}\{f(z_j) + \left.\frac{df}{dz}\right|_{z=z_j}(z - z_j)\}dz$$

Since $\oint_{C_j} dz = 0$ and $\oint_{C_j} zdz = 0$,

$$\left|\oint_{C_j} f(z)dz\right| = \left|\oint_{C_j}(z - z_j)\delta_j(z, z_j)dz\right| < A\varepsilon \to 0.$$

If a function $f(z)$ is analytic on and within a closed path *C*,

$$\oint_C f(z)dz = 0.$$

((**Note**))

(1)

$$\int_{z_0}^{z_n=z_0'} dz = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \sum_{j=1}^{n} (z_j - z_{j-1}) = z_0' - z_0 .$$

(2)

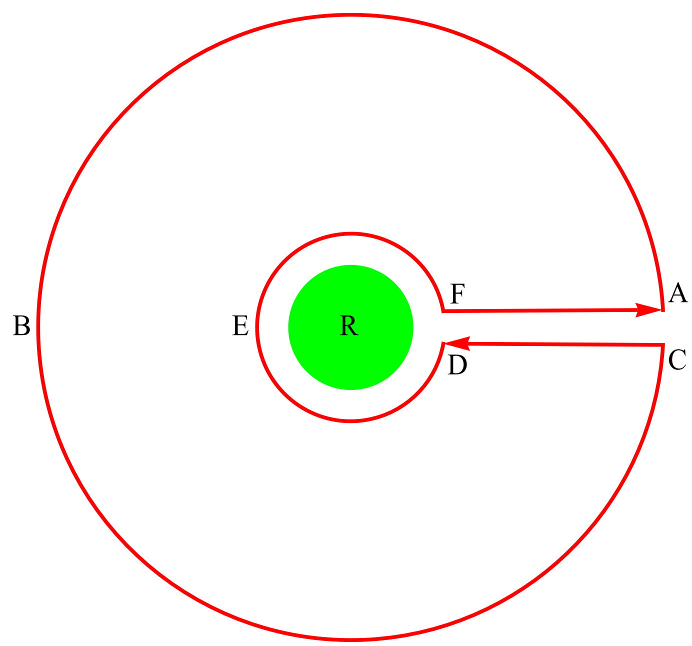
$$\int_{z_0}^{z_n=z_0'} z dz = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \sum_{j=1}^{n} \frac{z_j + z_{j-1}}{2}(z_j - z_{j-1}) = \frac{1}{2}\sum_{j=1}^{n} (z_j^2 - z_{j-1}^2)$$
$$= \frac{1}{2}(z_0'^2 - z_0^2) .$$

Thus we have

$$\oint_C dz = 0 \qquad \text{and} \qquad \oint_C z dz = 0$$

((**Multiply connected region**))

We consider the multiply connected region in which $f(z)$ is not defined (not analytic) in the interior *R*.



We can construct a contour for which the theorem holds. The new contour never crosses the interior *R*.

From the Cauchy's theorem, we have

$$\oint f(z)dz = \int_{ABC} f(z)dz + \int_{CD} f(z)dz + \int_{DEF} f(z)dz + \int_{FA} f(z)dz = 0 .$$

Since

$$\int_{FA} f(z)dz = -\int_{CD} f(z)dz ,$$

then

$$\int_{ABC} f(z)dz = -\int_{DEF} f(z)dz = \int_{FED} f(z)dz .$$

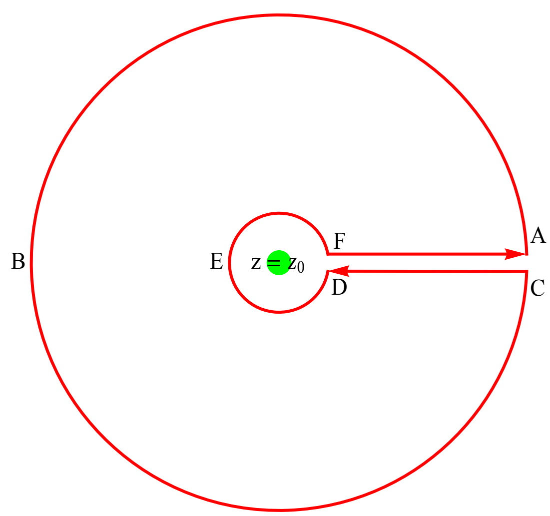
Therefore we obtain

$$\int_{ABC} f(z)dz = \int_{FED} f(z)dz .$$

where the contours C_1 (= A-B-C) and C_2 (= F-E-D) are the counterclockwise.

9.8 Cauchy's integral formula

$$\oint \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) \qquad \text{Cauchy's integral}$$



We consider a function $f(z)$ that is analytic on a closed contour A-B-C and within the interior by the path A-B-C. $f(z)/(z - z_0)$ is not analytic at $z = z_0$. We apply the

Cauchy’s integral theorem to the contour ABCDEF. In this region $f(z)/(z-z_0)$ is analytic.

$$\int_{ABC}\frac{f(z)}{z-z_0}dz+\int_{CD}\frac{f(z)}{z-z_0}dz+\int_{DEF}\frac{f(z)}{z-z_0}dz+\int_{FA}\frac{f(z)}{z-z_0}dz=0,$$

or

$$\int_{ABC}\frac{f(z)}{z-z_0}dz-\int_{FED}\frac{f(z)}{z-z_0}dz=0.$$

Let $z=z_0+re^{i\theta}$, r is small and will eventually be made to approach zero.
We have

$$\int_{FED}\frac{f(z)}{z-z_0}dz=\int_{FED}\frac{f(z_0+re^{i\theta})}{re^{i\theta}}re^{i\theta}id\theta=\int_{FED}f(z_0+re^{i\theta})id\theta.$$

Taking the limit as $r\rightarrow 0$, we obtain

$$\int_{ABC}\frac{f(z)}{z-z_0}dz=if(z_0)\int_0^{2\pi}d\theta=2\pi if(z_0)\qquad\text{(Residue)}$$

f z_0 is exterior to C_1,

$$\int_{C_1}\frac{f(z)}{z-z_0}dz=0.$$

We have therefore

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{z-z_0}dz=\begin{cases}f(z_0) & z_0:\text{interior}\\ 0 & z_0:\text{exterior}\end{cases}.$$

9.9 Derivatives

Suppose that $f(z)$ is analytic. By the definition of derivative, we have

$$
\begin{aligned}
f'(z_0) &= \lim_{\delta z_0 \to 0} \frac{f(z_0 + \delta z_0) - f(z_0)}{\delta z_0} \\
&= \lim_{\delta z_0 \to 0} \frac{1}{2\pi i \delta z_0} \left(\oint \frac{f(z)dz}{z - (z_0 + \delta z_0)} - \oint \frac{f(z)dz}{z - z_0} \right) \\
&= \lim_{\delta z_0 \to 0} \frac{1}{2\pi i \delta z_0} \oint \frac{\delta z_0 f(z)}{(z - z_0 - \delta z_0)(z - z_0)} dz \\
&= \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2} dz
\end{aligned}
$$

Similarly,

$$
f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz
$$

$$
\lim_{\delta z_0 \to 0} \frac{(z - z_0)^{k+1} - (z - z_0 - \delta z_0)^{k+1}}{\delta z_0 (z - z_0 - \delta z_0)^{k+1} (z - z_0)^{k+1}} = \frac{(k+1)(z - z_0)^k}{(z - z_0)^{2k+2}}
$$

Thus

$$
\begin{aligned}
\lim_{\delta z_0 \to 0} \frac{f^{(k)}(z_0 + \delta z_0) - f^{(k)}(z_0)}{\delta z_0} &= \frac{k!}{2\pi i} \oint \frac{(k+1)(z - z_0)^k}{(z - z_0)^{2k+2}} f(z) dz \\
&= \frac{(k+1)!}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{k+2}} dz
\end{aligned}
$$

The requirement that $f(z)$ be analytic not only gurantees a first derivative but also derivatives of all orderes as well.

<u>Goursat's theorem</u>

The existence of $f''(z)$ shows that $f'(z)$ is continuous."

$f(z)$ is analytic

$\downarrow$

$f'(z)$ is analytic

$\downarrow$

$f''(z)$ is analytic

$\downarrow$

.

.

.

$\downarrow$

$f^{(n)}(z)$ is analytic

All the derivatives of $f(z)$ are analytic within *R*.

Morera's theorem

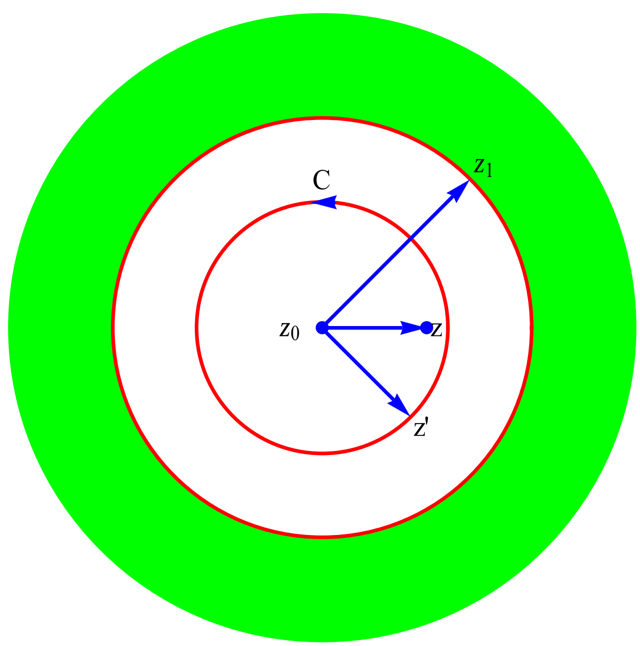
This is the converse of Cauchy's integral theorem.

"If a function $f(z)$ is continuous in a simple connected region *R* and $\oint_C f(z)dz = 0$ for every closed contour *C* within *R*, then $f(z)$ is analytic throughout *R*.

9.10 Taylor expansion and Laurent expansion

(a) Taylor expansion

$f(z)$ is analytic on and within *C*. $z = z_1$ is the nearest point for which $f(z)$ is not analytic. *C* is a circle centered at z_0 with radius $|z' - z_0| (< |z_1 - z_0|)$.



Cauchy integral formula

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{z'-z} \\
&= \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)-(z-z_0)} \\
&= \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)\left[1-\dfrac{(z-z_0)}{(z'-z_0)}\right]} \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_C \frac{f(z')\dfrac{(z-z_0)^n}{(z'-z_0)^n}dz'}{(z'-z_0)} \\
&= \sum_{n=0}^{\infty}(z-z_0)^n\frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)^{n+1}} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(z_0)(z-z_0)^n
\end{aligned}
\tag{1}
$$

where z' is a point on the contour C, and z is any point interior to C.

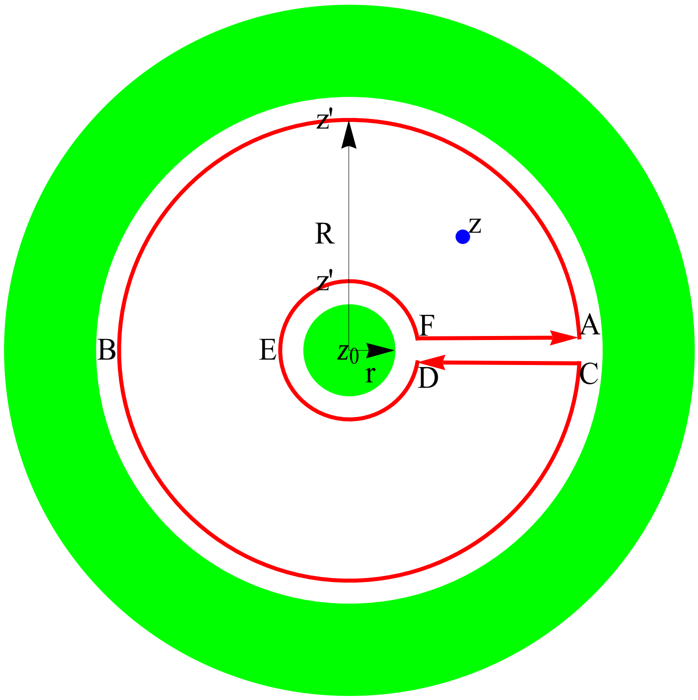
$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n \quad \text{for } |t|<1 \text{ (even for complex number } t\text{).}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z')}{(z'-z_0)^{n+1}}dz'$$

Taylor expansion: $f(z)$ is analytic at $z = z_0$.

(b) Laurentz expansion

$f(z)$ is not analytic in the regions denoted by green.



According to the Cauchy theorem, we have

$$f(z) = \frac{1}{2\pi i}\oint_{ABCDEFA} \frac{f(z')dz'}{z'-z} = \frac{1}{2\pi i}\left[\int_{ABC} \frac{f(z')dz'}{z'-z} + \int_{CD} \frac{f(z')dz'}{z'-z} + \int_{DEF} \frac{f(z')dz'}{z'-z} + \int_{FA} \frac{f(z')dz'}{z'-z}\right]$$
$$= \frac{1}{2\pi i}\left[\int_{ABC} \frac{f(z')dz'}{z'-z} - \int_{FED} \frac{f(z')dz'}{z'-z}\right]$$

On the path (ABC), we have

$$|z'-z_0| > |z-z_0|$$

On the path (FED), we have

$$|z'-z_0| < |z-z_0|$$

Then

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_{ABC}\frac{f(z')dz'}{z'-z} &= \frac{1}{2\pi i}\oint_{ABC}\frac{f(z')dz'}{(z'-z_0)-(z-z_0)} \\
&= \frac{1}{2\pi i}\oint_{ABC}\frac{f(z')dz'}{(z'-z_0)\left[1-\dfrac{(z-z_0)}{(z'-z_0)}\right]} \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{C}\frac{f(z')\dfrac{(z-z_0)^n}{(z'-z_0)^n}dz'}{(z'-z_0)} \\
&= \sum_{n=0}^{\infty}(z-z_0)^n\frac{1}{2\pi i}\oint_{C}\frac{f(z')dz'}{(z'-z_0)^{n+1}} \\
&= \sum_{n=0}^{\infty}a_n(z-z_0)^n
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{1}{2\pi i}\oint_{FED}\frac{f(z')dz'}{z'-z} &= \frac{1}{2\pi i}\oint_{FED}\frac{f(z')dz'}{-(z'-z_0)+(z-z_0)} \\
&= \frac{1}{2\pi i}\oint_{FED}\frac{f(z')dz'}{(z-z_0)\left[1-\dfrac{(z'-z_0)}{(z-z_0)}\right]} \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{FED}\frac{f(z')\dfrac{(z'-z_0)^n}{(z-z_0)^n}dz'}{(z-z_0)} \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint_{FED}\frac{f(z')(z'-z_0)^n dz'}{(z-z_0)^{n+1}} \\
&= \sum_{n=1}^{\infty}\frac{1}{2\pi i}(z-z_0)^{-n}\oint_{FED}f(z')(z'-z_0)^{n-1}dz'
\end{aligned}
$$

These two series are combined into one series (Laurent series)

$$
f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}b_n(z-z_0)^{-n},
$$

with

$$
a_n=\frac{1}{2\pi i}\oint_{ABC}\frac{f(z')}{(z'-z_0)^{n+1}}dz'.
$$

$$b_n = \frac{1}{2\pi i}\oint_{FED} f(z')(z'-z_0)^{n-1}dz'$$

---

(1) Analytic at $z = z_0$

If $f(z)$ is analytic in the vicinity of z_0, we can perform a Taylor series expansion.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \, .$$

(2) Isolated singular point at $z = z_0$

We say we have an isolated singular point at $z = z_0$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_n}{(z - z_0)^n} + \ldots.$$

(i) As long as one of the b's are nonzero, z_0 is a **singular point**.

(ii) If b_n does not vanish, but all b_j with $j>n$ vanish, then $f(z)$ has a **pole of order *n***.

(iii) If $b_1 \neq 0$ and all other b_j's = 0 the pole is **a single pole**.

(iv) Isolated essential singularity (all the b_j's) normally does not occur in physics.

9.11 Examples of Taylor and Lorentz expansions (Mathematica)

9.11.1 Example-1

```
Clear["Global`*"]

Series[Exp[z], {z, 0, 10}]
```

$$1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}+\frac{z^5}{120}+\frac{z^6}{720}+\frac{z^7}{5040}+\frac{z^8}{40\,320}+\frac{z^9}{362\,880}+\frac{z^{10}}{3\,628\,800}+O[z]^{11}$$

```
Series[Cos[z], {z, 0, 10}]
```

$$1-\frac{z^2}{2}+\frac{z^4}{24}-\frac{z^6}{720}+\frac{z^8}{40\,320}-\frac{z^{10}}{3\,628\,800}+O[z]^{11}$$

```
Series[Sin[z], {z, 0, 10}]
```

$$z-\frac{z^3}{6}+\frac{z^5}{120}-\frac{z^7}{5040}+\frac{z^9}{362\,880}+O[z]^{11}$$

```
Series[Cot[z], {z, 0, 10}] // Normal
```

$$\frac{1}{z}-\frac{z}{3}-\frac{z^3}{45}-\frac{2\,z^5}{945}-\frac{z^7}{4725}-\frac{2\,z^9}{93\,555}$$

```
Series[Tan[z], {z, 0, 10}] // Normal
```

$$z+\frac{z^3}{3}+\frac{2\,z^5}{15}+\frac{17\,z^7}{315}+\frac{62\,z^9}{2835}$$

```
Series[Csc[z]^2, {z, 0, 10}] // Normal
```

$$\frac{1}{3}+\frac{1}{z^2}+\frac{z^2}{15}+\frac{2 z^4}{189}+\frac{z^6}{675}+\frac{2 z^8}{10\,395}+\frac{1382 z^{10}}{58\,046\,625}$$

```
Series[Sec[z]^2, {z, 0, 10}] // Normal
```

$$1+z^2+\frac{2 z^4}{3}+\frac{17 z^6}{45}+\frac{62 z^8}{315}+\frac{1382 z^{10}}{14\,175}$$

```
Series[1/Sin[z], {z, 0, 10}] // Normal
```

$$\frac{1}{z}+\frac{z}{6}+\frac{7 z^3}{360}+\frac{31 z^5}{15\,120}+\frac{127 z^7}{604\,800}+\frac{73 z^9}{3\,421\,440}$$

```
Series[z/Sin[z], {z, 0, 10}] // Normal
```

$$1+\frac{z^2}{6}+\frac{7 z^4}{360}+\frac{31 z^6}{15\,120}+\frac{127 z^8}{604\,800}+\frac{73 z^{10}}{3\,421\,440}$$

```
Series[z^-3 Sin[z], {z, 0, 10}] // Normal
```

$$-\frac{1}{6}+\frac{1}{z^2}+\frac{z^2}{120}-\frac{z^4}{5040}+\frac{z^6}{362\,880}-\frac{z^8}{39\,916\,800}+\frac{z^{10}}{6\,227\,020\,800}$$

$$\text{Series}\left[\frac{1}{\text{Cos}[z]}, \{z, 0, 10\}\right] \text{ // Normal}$$

$$1+\frac{z^2}{2}+\frac{5 z^4}{24}+\frac{61 z^6}{720}+\frac{277 z^8}{8064}+\frac{50521 z^{10}}{3628800}$$

$$\text{Series}\left[\text{Csc}[z]^2 \text{Log}[1-z], \{z, 0, 10\}\right] \text{ // Normal}$$

$$-\frac{1}{2}-\frac{1}{z}-\frac{2 z}{3}-\frac{5 z^2}{12}-\frac{17 z^3}{45}-\frac{17 z^4}{60}-\frac{229 z^5}{945}-\frac{1531 z^6}{7560}-\frac{502 z^7}{2835}-\frac{5903 z^8}{37800}-\frac{21872 z^9}{155925}-\frac{158707 z^{10}}{1247400}$$

$$\text{Series}\left[\frac{z}{\text{Sin}[z]-\text{Tan}[z]}, \{z, 0, 10\}\right] \text{ // Normal}$$

$$\frac{1}{2}-\frac{2}{z^2}+\frac{11 z^2}{120}+\frac{157 z^4}{15120}+\frac{641 z^6}{604800}+\frac{1417 z^8}{13305600}+\frac{1402631 z^{10}}{130767436800}$$

$$\text{Series}[\text{Sinh}[z], \{z, 0, 10\}]$$

$$z+\frac{z^3}{6}+\frac{z^5}{120}+\frac{z^7}{5040}+\frac{z^9}{362880}+O[z]^{11}$$

$$\text{Series}[\text{Cosh}[z], \{z, 0, 10\}]$$

$$1+\frac{z^2}{2}+\frac{z^4}{24}+\frac{z^6}{720}+\frac{z^8}{40320}+\frac{z^{10}}{3628800}+O[z]^{11}$$

$$\text{Series}\left[\frac{\text{Csc}[z]}{z}, \{z, 0, 10\}\right]$$

$$\frac{1}{z^2}+\frac{1}{6}+\frac{7 z^2}{360}+\frac{31 z^4}{15120}+\frac{127 z^6}{604800}+\frac{73 z^8}{3421440}+\frac{1414477 z^{10}}{653837184000}+O[z]^{11}$$

$$\text{Series}\left[\frac{\text{Sec}[z]}{z}, \{z, 0, 10\}\right]$$

$$\frac{1}{z}+\frac{z}{2}+\frac{5 z^3}{24}+\frac{61 z^5}{720}+\frac{277 z^7}{8064}+\frac{50521 z^9}{3628800}+O[z]^{11}$$

9.11.2 Example-2

```
Clear["Global`*"]
```

$$\text{Series}\left[\frac{1}{1+z}, \{z, 0, 10\}\right]$$

$$1 - z + z^2 - z^3 + z^4 - z^5 + z^6 - z^7 + z^8 - z^9 + z^{10} + O[z]^{11}$$

$$\text{Series}[\text{Log}[1+z], \{z, 0, 10\}]$$

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \frac{z^6}{6} + \frac{z^7}{7} - \frac{z^8}{8} + \frac{z^9}{9} - \frac{z^{10}}{10} + O[z]^{11}$$

$$\text{Series}\left[\frac{1}{z\,(z-2)^3}, \{z, 2, 10\}\right] \text{ // Normal}$$

$$-\frac{1}{16} + \frac{1}{2\,(-2+z)^3} - \frac{1}{4\,(-2+z)^2} + \frac{1}{8\,(-2+z)} + \frac{1}{32}\,(-2+z) - \frac{1}{64}\,(-2+z)^2 + \frac{1}{128}\,(-2+z)^3 - \frac{1}{256}\,(-2+z)^4 + \frac{1}{512}\,(-2+z)^5 - \frac{(-2+z)^6}{1024} + \frac{(-2+z)^7}{2048} - \frac{(-2+z)^8}{4096} + \frac{(-2+z)^9}{8192} - \frac{(-2+z)^{10}}{16\,384}$$

```
Series[Exp[ⅈ z]/(z^2 + 1)^2, {z, ⅈ, 10}] // Normal
```

$$\frac{9}{16\,e} - \frac{1}{4\,e\,(-i+z)^2} - \frac{i}{2\,e\,(-i+z)} + \frac{23\,i\,(-i+z)}{48\,e} - \frac{67\,(-i+z)^2}{192\,e} - \frac{37\,i\,(-i+z)^3}{160\,e} + \frac{1663\,(-i+z)^4}{11\,520\,e} + \frac{6983\,i\,(-i+z)^5}{80\,640\,e} - \frac{5431\,(-i+z)^6}{107\,520\,e} - \frac{5237\,i\,(-i+z)^7}{181\,440\,e} + \frac{942\,659\,(-i+z)^8}{58\,060\,800\,e} + \frac{213\,359\,i\,(-i+z)^9}{23\,654\,400\,e} - \frac{38\,020\,573\,(-i+z)^{10}}{7\,664\,025\,600\,e}$$

```
Series[1/(z^4 + a^4), {z, (1 + ⅈ)/Sqrt[2] a, 5}] // Normal //
  PowerExpand
```

$$\frac{3}{8\,a^4} - \frac{\frac{1}{4}+\frac{i}{4}}{\sqrt{2}\,a^3\left(-\frac{(1+i)\,a}{\sqrt{2}}+z\right)} - \frac{\left(\frac{5}{16}-\frac{5\,i}{16}\right)\left(-\frac{(1+i)\,a}{\sqrt{2}}+z\right)}{\sqrt{2}\,a^5} - \frac{5\,i\left(-\frac{(1+i)\,a}{\sqrt{2}}+z\right)^2}{32\,a^6} + \frac{\left(\frac{1}{64}+\frac{i}{64}\right)\left(-\frac{(1+i)\,a}{\sqrt{2}}+z\right)^3}{\sqrt{2}\,a^7} + \frac{7\left(-\frac{(1+i)\,a}{\sqrt{2}}+z\right)^4}{128\,a^8} - \frac{\left(\frac{15}{256}-\frac{15\,i}{256}\right)\left(-\frac{(1+i)\,a}{\sqrt{2}}+z\right)^5}{\sqrt{2}\,a^9}$$

```
Series[(1 - z)^-ⅈ, {z, 0, 5}] // Normal // Simplify
```

$$\frac{1}{12}\left(12+12\,i\,z-(6-6\,i)\,z^2-(6-2\,i)\,z^3-5\,z^4-(4+i)\,z^5\right)$$

```
Series[z^4/(z^2 - c^2)^4, {z, c, 5}] // Normal
```

$$\frac{3}{256\,c^4}+\frac{1}{16\,(-c+z)^4}+\frac{1}{8\,c\,(-c+z)^3}+\frac{1}{32\,c^2\,(-c+z)^2}-\frac{1}{32\,c^3\,(-c+z)}-\frac{(-c+z)^2}{256\,c^6}+\frac{(-c+z)^3}{256\,c^7}-\frac{11\,(-c+z)^4}{4096\,c^8}+\frac{3\,(-c+z)^5}{2048\,c^9}$$

```
Series[z^4/(z^2 - c^2)^4, {z, -c, 5}] // Normal
```

$$\frac{3}{256\,c^4}+\frac{1}{16\,(c+z)^4}-\frac{1}{8\,c\,(c+z)^3}+\frac{1}{32\,c^2\,(c+z)^2}+\frac{1}{32\,c^3\,(c+z)}-\frac{(c+z)^2}{256\,c^6}-\frac{(c+z)^3}{256\,c^7}-\frac{11\,(c+z)^4}{4096\,c^8}-\frac{3\,(c+z)^5}{2048\,c^9}$$

```
Series[Cot[π z]/(z - a)^2, {z, a, 1}] // Normal
```

$$\pi^2\,\mathrm{Cot}[a\,\pi]+\frac{\mathrm{Cot}[a\,\pi]}{(-a+z)^2}+\pi^2\,\mathrm{Cot}[a\,\pi]^3+\frac{-\pi-\pi\,\mathrm{Cot}[a\,\pi]^2}{-a+z}+(-a+z)\left(-\frac{\pi^3}{3}-\frac{4}{3}\,\pi^3\,\mathrm{Cot}[a\,\pi]^2-\pi^3\,\mathrm{Cot}[a\,\pi]^4\right)$$

9.11.3 Example-3

```
Clear["Global`*"]
```

```
Series[z/(Exp[z] - 1), {z, 0, 6}]
```

$$1-\frac{z}{2}+\frac{z^2}{12}-\frac{z^4}{720}+\frac{z^6}{30240}+O[z]^7$$

```
Series[1/(e^z - 1), {z, 0, 10}]
```

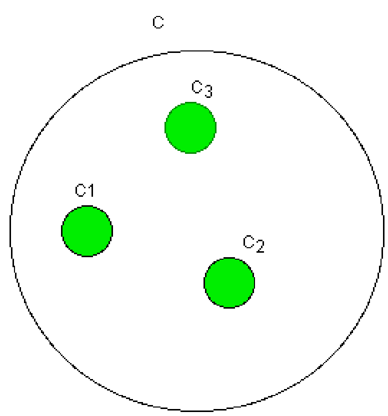
$$\frac{1}{z}-\frac{1}{2}+\frac{z}{12}-\frac{z^3}{720}+\frac{z^5}{30240}-\frac{z^7}{1209600}+\frac{z^9}{47900160}+O[z]^{11}$$

```
Series[e^(1/z), {z, ∞, 10}]
```

$$1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\frac{1}{24z^4}+\frac{1}{120z^5}+\frac{1}{720z^6}+\frac{1}{5040z^7}+\frac{1}{40320z^8}+\frac{1}{362880z^9}+\frac{1}{3628800z^{10}}+O\left[\frac{1}{z}\right]^{11}$$

9.12 Calculation of residue

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \ldots\ldots$$
$$= 2\pi i[\mathrm{Res}(z=z_1) + \mathrm{Res}(z=z_2) + \mathrm{Res}(z=z_3) + \ldots.$$



$$f(z) = a_0 + a_1(z-z_0) + \ldots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots$$

Using Cauchy's theorem,

$$\oint_C f(z)dz = \oint_C [a_0 + a_1(z - z_0) + ...]dz + \oint[\frac{b_{-1}}{z - z_0} + \frac{b_{-2}}{(z - z_0)^2} + ..]dz$$

or

$$\oint_C f(z)dz = \oint[\frac{b_{-1}}{z - z_0} + \frac{b_{-2}}{(z - z_0)^2} + ..]dz$$

We now calculate

$$I = \oint (z - z_0)^{-n} dz$$

$$z - z_0 = re^{i\theta}$$

$$dz = r(id\theta)e^{i\theta}$$

$$I = \int_0^{2\pi} (re^{i\theta})^{-n} r(id\theta)e^{i\theta} = ir^{-n+1}\int_0^{2\pi} e^{-i(n-1)\theta} d\theta = 2\pi i r^{-n+1}\delta_{n,1}$$

Then we have

$$\oint_C f(z)dz = \oint(\frac{b_1}{z - z_0})dz = 2\pi i b_1 = 2\pi i \operatorname{Re} s(z = z_0)$$

The problem is reduced to finding the value of b_1.

(i) ***f***(z) has a simple pole.

$$f(z) = a_0 + a_1(z - z_0) + ... + \frac{b_1}{z - z_0}$$

$$(z - z_0) f(z) = a_0(z - z_0) + a_1(z - z_0)^2 + ... + b_1$$

$$\lim_{z \to z_0}(z - z_0) f(z) = b_1 = \operatorname{Re} s(z = z_0)$$

(ii) A pole of order n

$$f(z) = a_0 + a_1(z - z_0) + ... + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ... + \frac{b_n}{(z - z_0)^n}$$

$$(z - z_0)^n f(z) = (z - z_0)^n [a_0 + a_1(z - z_0) + ...] + b_1(z - z_0)^{n-1} + b_2(z - z_0)^{n-2} + ... + b_n$$

$$\frac{d^{n-1}(z - z_0)^n f(z)}{dz^{n-1}} = b_1(n-1)!$$

$$b_1 = \mathrm{Re}\,s[z = z_0] = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}[(z - z_0)^n f(z)]}{dz^{n-1}}$$

((Note**))** **Mathematica**

Residue[expr, {*x*, *x*$_0$}];

to find the residue of expression when x equals x_0.

The residue is defined as the coefficient of `1/(z-z0)` in the Laurent expansion of *expr*.

NResidue[expr, {x, x0}];

to find numerically the residue of expression when x equals x_0.

Series[*f*, {*x*, *x*$_0$, *n*}];

to generate a power series expansion for f about the point $x=x_0$ to order $(x - x_0)^n$.

9.13 Mathematica

9.13.1 Example-1

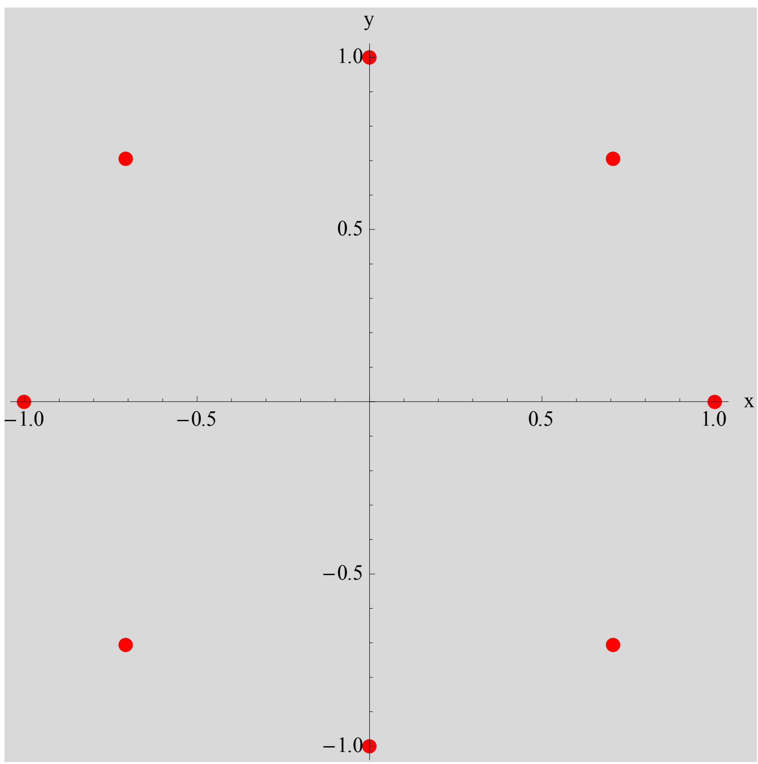
(a) Find the poles of

$$f(z) = \frac{1}{z^8 - 1}$$

```
Clear["Global`*"];

h[z_] = 1/(-1 + z^8);

eq1 = NSolve[Denominator[h[z]] == 0, z];

list1 = Table[{Re[z] /. eq1[[i]], Im[z] /. eq1[[i]]},
  {i, Length[eq1]}]
```

```
{{-1., 0}, {-0.707107, -0.707107},
 {-0.707107, 0.707107}, {0., -1.}, {0., 1.},
 {0.707107, -0.707107}, {0.707107, 0.707107}, {1., 0}}
```

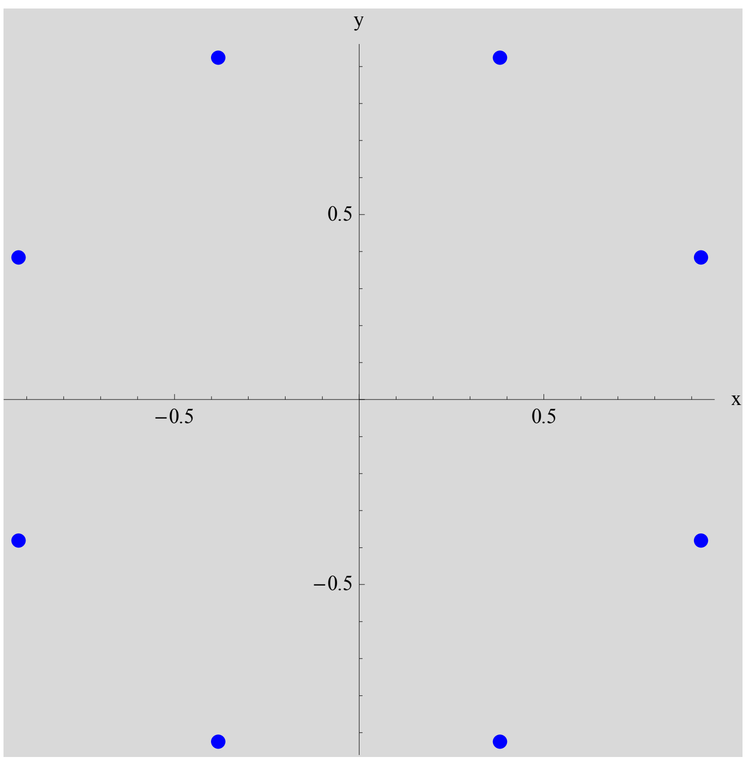
```
ListPlot[list1, PlotStyle → {Red, Thick, PointSize[0.02]},
 AspectRatio → 1, Background → LightGray,
 AxesLabel → {"x", "y"}]
```



(b) Find the poles of

$$f(z) = \frac{1}{z^8 + 1}$$

```
Clear["Global`*"];

h[z_] = 1/(1 + z^8);

eq1 = NSolve[Denominator[h[z]] == 0, z];

list1 = Table[{Re[z] /. eq1[[i]], Im[z] /. eq1[[i]]},
  {i, Length[eq1]}]

{{-0.92388, -0.382683}, {-0.92388, 0.382683},
 {-0.382683, -0.92388}, {-0.382683, 0.92388},
 {0.382683, -0.92388}, {0.382683, 0.92388},
 {0.92388, -0.382683}, {0.92388, 0.382683}}

ListPlot[list1, PlotStyle → {Blue, Thick, PointSize[0.02]},
 AspectRatio → 1, Background → LightGray,
 AxesLabel → {"x", "y"}]
```



9.13.2 Example-2

((**Mathematica**))

The residue is the coefficient of the term $(z-z_0)^{-1}$ in the Laurent expansion of a function about the point z_0. (For a very detailed survey of applications of residues, In *Mathematica*, we get the residue of a function at a given point via Residue.

Find the residue of the function

$$f(z) = \frac{1}{z^n} \frac{1}{e^z - 1}$$

at $z = 0$. (a) $n = 4$. (b) $n = 0, 1, \ldots, 10$.

`Residue[expr, {x, x0}]` the residue of *expr* when x equals x_0

```
Clear["Global`*"];
```

```
F[z_, n_] := 1/z^n 1/(E^z - 1);
```

```
Series[F[z, 4], {z, 0, 10}] // Normal
```

$$\frac{1}{z^5} - \frac{1}{2 z^4} + \frac{1}{12 z^3} - \frac{1}{720 z} + \frac{z}{30\,240} - \frac{z^3}{1\,209\,600} + \frac{z^5}{47\,900\,160} - \frac{691 z^7}{1\,307\,674\,368\,000} + \frac{z^9}{74\,724\,249\,600}$$

```
Residue[F[z, 4], {z, 0}]
```

$$-\frac{1}{720}$$

```
Table[{n, Residue[F[z, n], {z, 0}]}, {n, 0, 10}] // TableForm
```

0	1
1	$-\frac{1}{2}$
2	$\frac{1}{12}$
3	0
4	$-\frac{1}{720}$
5	0
6	$\frac{1}{30\,240}$
7	0
8	$-\frac{1}{1\,209\,600}$
9	0
10	$\frac{1}{47\,900\,160}$

9.13.3 Example-3

Find the residues of the function given by

$$f(z) = \frac{z^2 e^z}{1 + e^{2z}}$$

```
Clear["Global`*"];
```

$$f1 = \frac{z^2 e^z}{1 + e^{2z}};$$

```
eq1 = Solve[Denominator[f1] == 0, z];

z1 = z /. eq1[[1]]
```

$$-\frac{i \pi}{2}$$

```
z2 = z /. eq1[[2]]
```

$$\frac{i \pi}{2}$$

```
Series[f1, {z, z1, 3}] // Normal
```

$$\frac{\pi}{2} - \frac{i \pi^2}{8 \left(\frac{i \pi}{2} + z\right)} + \frac{1}{48} i \left(24 + \pi^2\right) \left(\frac{i \pi}{2} + z\right) - \frac{1}{12} \pi \left(\frac{i \pi}{2} + z\right)^2 - \frac{i \left(240 + 7 \pi^2\right) \left(\frac{i \pi}{2} + z\right)^3}{2880}$$

```
Series[f1, {z, z2, 3}] // Normal
```

$$\frac{\pi}{2} + \frac{i \pi^2}{8 \left(-\frac{i \pi}{2} + z\right)} - \frac{1}{48} i \left(24 + \pi^2\right) \left(-\frac{i \pi}{2} + z\right) - \frac{1}{12} \pi \left(-\frac{i \pi}{2} + z\right)^2 + \frac{i \left(240 + 7 \pi^2\right) \left(-\frac{i \pi}{2} + z\right)^3}{2880}$$

```
Residue[f1, {z, z1}]
```

$$-\frac{i \pi^2}{8}$$

```
Residue[f1, {z, z2}]
```

$$\frac{i \pi^2}{8}$$

9.13.4 Examples 4

Find the residue of $f[z] = \frac{\pi \, \mathrm{Cot}[\pi z]}{z^2}$ at $z_0 = 0$.

```
Clear["Global`*"];
```

$$f[z\_] = \frac{\pi \, \mathrm{Cot}[\pi z]}{z^2};$$

```
Series[f[z], {z, 0, 10}]
```

$$\frac{1}{z^3} - \frac{\pi^2}{3 z} - \frac{\pi^4 z}{45} - \frac{2 \pi^6 z^3}{945} - \frac{\pi^8 z^5}{4725} - \frac{2 \pi^{10} z^7}{93\,555} - \frac{1382 \pi^{12} z^9}{638\,512\,875} + O[z]^{11}$$

```
Residue[f[z], {z, 0}]
```

$$-\frac{\pi^2}{3}$$

9.13.5 Example 5

Show that $\int_C \frac{2z}{z^2+2}\,dz = 4\pi i$, where C is the circle $C: |z| = 2$ taken with positive orientation.

```
f[z_] = 2 z / (2 + z^2);
```

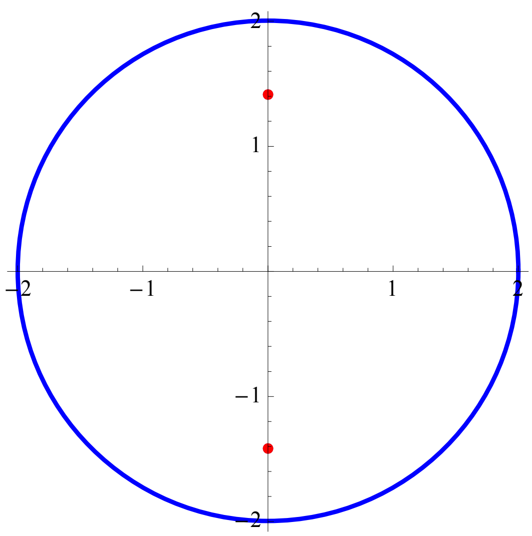
```
eq1 = Solve[Denominator[f[z]] == 0, z]
```

$\{\{z \to -i\sqrt{2}\}, \{z \to i\sqrt{2}\}\}$

```
data1 = Table[{Re[z], Im[z]} /. eq1, {i, 1, Length[eq1]}]
```

$\{\{\{0, -\sqrt{2}\}, \{0, \sqrt{2}\}\}, \{\{0, -\sqrt{2}\}, \{0, \sqrt{2}\}\}\}$

```
f1 = ListPlot[data1, PlotStyle → {PointSize[0.02], Red},
   AspectRatio → Automatic];
g1 = Graphics[{Blue, Thick, Circle[{0, 0}, 2]}];
Show[f1, g1]
```




```
r1 = Residue[f[z], {z, √2 i}]
```

1

```
r2 = Residue[f[z], {z, -√2 i}]
```

1

```
sol = 2 π i r1 + 2 π i r2
```

$4 i \pi$

9.13.6 Example 6

Use residues to integrate $\int_C \frac{1}{z^4+z^3-2z^2}\, dz$ around the circle $C: |z| = 3$.

```
f[z_] = 1/(-2 z^2 + z^3 + z^4);

eq1 = Solve[Denominator[f[z]] == 0, z]
```

{{z → -2}, {z → 0}, {z → 0}, {z → 1}}

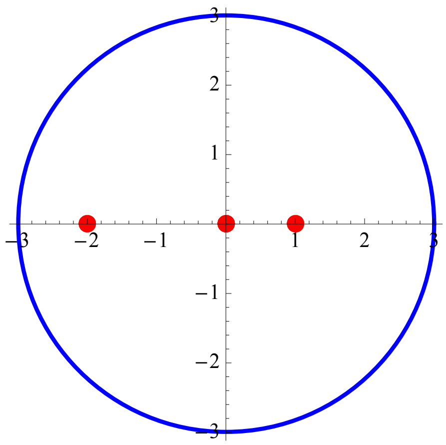
```
Length[eq1]
```

4

```
data1 = Table[{Re[z], Im[z]} /. eq1, {i, 1, Length[eq1]}]
```

{{{-2, 0}, {0, 0}, {0, 0}, {1, 0}}, {{-2, 0}, {0, 0}, {0, 0}, {1, 0}},
{{-2, 0}, {0, 0}, {0, 0}, {1, 0}}, {{-2, 0}, {0, 0}, {0, 0}, {1, 0}}}

```
f1 = ListPlot[data1, PlotStyle → { PointSize[0.04], Red},
  AspectRatio → Automatic]; g1 = Graphics[{Blue, Thick, Circle[{0, 0}, 3]}];
Show[f1, g1]
```




```
Table[{z /. eq1[[i]], Residue[f[Z], {Z, z /. eq1[[i]]}]}, {i, 1, Length[eq1]}]
```

$\left\{\left\{-2, -\frac{1}{12}\right\}, \left\{0, -\frac{1}{4}\right\}, \left\{0, -\frac{1}{4}\right\}, \left\{1, \frac{1}{3}\right\}\right\}$

```
Sum[2 π ⅈ Residue[f[Z], {Z, z /. eq2[[i]]}], {i, 1, Length[eq2]}]
```

0

9.14 Calculus of residue

9.4.1 Jordan's lemma

Jordan's lemma is a result frequently used in conjunction with the residue theorem to evaluate contour integrals and improper integrals. The theorem is named after the French mathematician Camille Jordan.

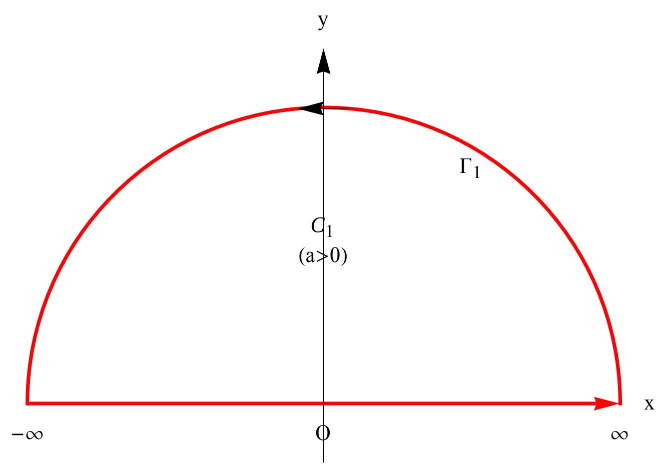
We want to calculate the integral given by

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx,$$

where a>0 or a<0, and

$$\lim_{|z|\to\infty} f(z) = 0.$$

In order to do that, we consider the path integral around the contour C_1 for a>0 (the upper half plane) and the contour C_2 for a<0 (the lower half plane) in the complex plane.



(i) a>0

$$\oint_{C_1} f(z)e^{iaz}dz = \int_{-\infty}^{\infty} f(x)e^{iax}dx + \lim_{R\to\infty} I_R$$

If the integral reduces to zero in the upper semi-circle (Γ_1, $R\to\infty$)

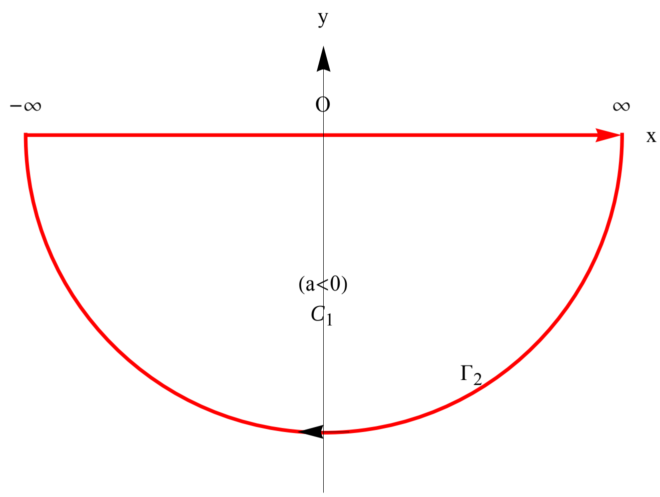
$$\lim_{R\to\infty} I_R = \lim_{R\to\infty}\int_{\Gamma_1} e^{iaz} f(z)dz = 0.$$

Note: $iaz = ia(\alpha + i\beta) = -a\beta + ia\alpha$. When β>0, the integral on the large semi-circle becomes zero. Then we have

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = \oint_{C_1} f(z)e^{iaz}dz = 2\pi i \sum_{\substack{upper \\ half \\ plane}} \mathrm{Re}\, sidues$$

where $f(z)$ has poles inside the contour C_1.

(ii) $a<0$



$$\int_{-\infty}^{\infty} f(x)e^{iax}dx + \lim_{R\to\infty} I_R = \oint_{C_2} f(z)e^{iaz}dz = 2\pi i \sum_{\substack{lolwer \\ half \\ plane}} \mathrm{Re}\, sidues$$

when

$$\lim_{R\to\infty} I_R = \lim_{R\to\infty} \int_{\Gamma_2} e^{iaz} f(z)dz = 0$$

((Note))

$iaz = ia(\alpha + i\beta) = -a\beta + ia\alpha$. When $\beta<0$, the integral on the large semi-circle becomes zero. Then we have

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = \oint_{C_2} f(z)e^{iaz}dz = 2\pi i \sum_{\substack{lolwer \\ half \\ plane}} \mathrm{Re}\, sidues$$

In summary,

For $a>0$,

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = \oint_{C_1} f(z)e^{iaz}dz = 2\pi i \sum_{\substack{upper\\half\\plane}} \mathrm{Re}\,sidues\,.$$

For a<0,

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = \oint_{C_2} f(z)e^{iaz}dz = 2\pi i \sum_{\substack{lower\\half\\plane}} \mathrm{Re}\,sidues\,.$$

(2) Integral along the contour of half-circle

Suppose that $f(z)$ has a simple pole on the real axis. We now consider the contour integral along the half circle centered at the pole ($z = z_0$),

$$\int_{C1} f(z)dz$$

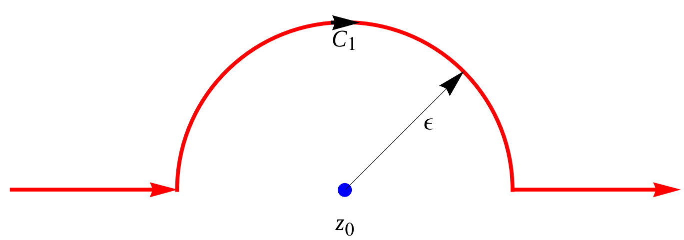
For simplicity we suppose that $f(z)$ has a single pole at $z = z_0$.

$$f(z) = \frac{a_{-1}}{z - z_0} + g(z) = \frac{b_1}{z - z_0} + g(z),$$

where $a_{-1} = b_1$, $g(z)$ is analytic.

Now we have the integral around the contour C_1 (upper-half circle with the radius ε located at the point $z = z_0$)

$$I = \oint_{C_1} f(z)dz = \oint_{C_1}[\frac{b_1}{z - z_0} + g(z)]dz$$
$$= \oint_{C_1} \frac{b_1}{z - z_0}dz = b_1\int_{\pi}^{0} \frac{1}{\varepsilon e^{i\theta}}\varepsilon e^{i\theta} id\theta = -b_1\int_{0}^{\pi} id\theta = -\pi i b_1$$



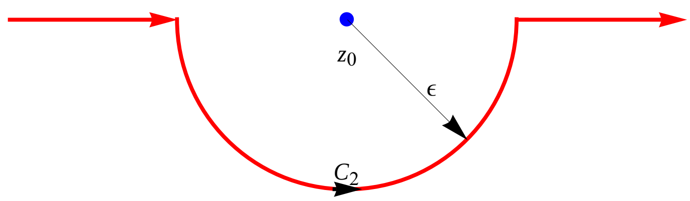
or

$$I = \oint_{C_1} f(z)dz = (-\pi i)\,\mathrm{Re}\,s(z = z_0)$$ [clock-wise (CW) rotation]

where $a_{-1} = \mathrm{Re}\, s(z = z_0)$.

Next we have the integral around the contour C_2 (lower-half circle with the radius ε located at the point $z = z_0$)

$$I = \oint_{C_2} f(z)dz = \oint_{C_2} [\frac{b_1}{z - z_0} + g(z)]dz$$
$$= \oint_{C_2} \frac{b_1}{z - z_0} dz = b_1 \int_{\pi}^{2\pi} \frac{1}{\varepsilon e^{i\theta}} \varepsilon e^{i\theta} i d\theta = b_1 \int_{\pi}^{2\pi} i d\theta = \pi i b_1 .$$



or

$$I = \oint_{C_2} f(z)dz = \pi i \,\mathrm{Re}\, s(z = z_0) \qquad \text{[counter clock-wise(CCW) rotation]}$$

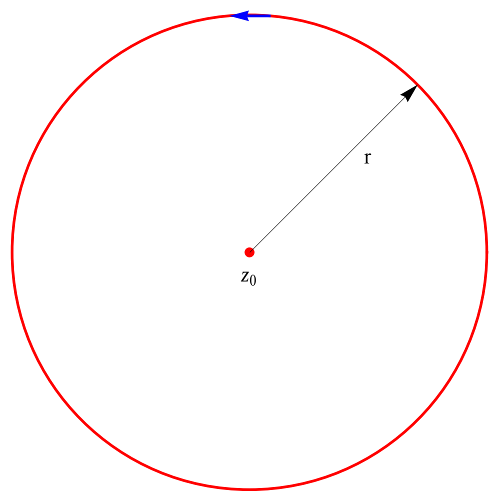
9.15 Calculation of residue

Residue theorem

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n ,$$

$$I = a_n \oint_C (z - z_0)^n dz ,$$

$$z - z_0 = re^{i\theta} ,$$

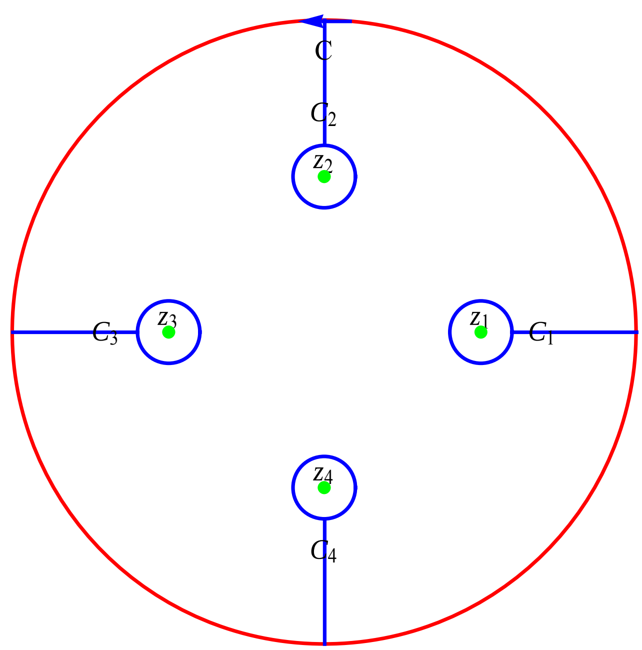


$$I = a_n \int_0^{2\kappa} (re^{i\theta})^n re^{i\theta} i d\theta = i a_n r^{n+1} \int_0^{2\kappa} e^{i(n+1)\theta} d\theta = 2\pi i a_n r^{n+1} \delta_{n+1,0}.$$

Then we have

$$\frac{1}{2\pi i} \oint f(z) dz = a_{-1} = \mathrm{Re}\, s[z = z_0].$$

A set of isolated singularities;



$$\oint_C f(z) dz + \oint_{C_1} f(z) dz + \oint_{C_2} f(z) dz + \oint_{C_3} f(z) dz + \ldots = 0.$$

Here C is a path with CCW and C_1, C_2, … are paths with CW.

$$\oint_C f(z)dz = -\oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz - \oint_{C_3} f(z)dz$$
$$= 2\pi i[\mathrm{Res}(z_1) + \mathrm{Res}(z_2) + \mathrm{Res}(z_3) + ...]$$

((Note))

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \frac{b_3}{(z - z_0)^3} + ... + \frac{b_n}{(z - z_0)^n}$$

(a) As long as one of the *b*'s are non zero, z_0 is a singular point.

(b) If b_n does not vanish, but all b_j with $j>n$ does vanish, then *f*(*z*) has a pole of order *n*.

(c) If $b_1 \neq 0$, and all other b_j's = 0, then the pole is a simple pole.

(d) If *f*(z) has a pole of order n, 1/*f*(z) has a zero of order n.

(e) A function that is analytic in a region except for isolated singular points is

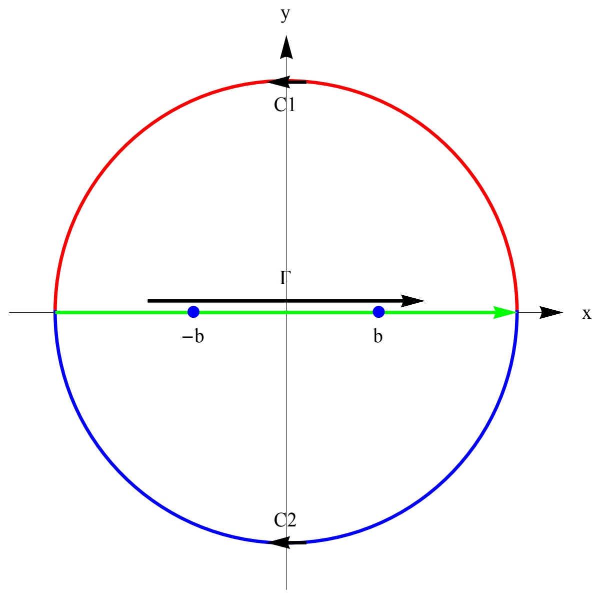
(f) Isolated essential singularity (all the b_j's) normally does not occur in physics.

9.16 Mathematica

9.16.1 Application I

$$I = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 - b^2} dx \qquad (b>0).$$

$$f(z) = \frac{1}{z^2 - b^2}$$



Note that

$$| f(z)| < \frac{1}{R^2} \to 0 \qquad\qquad (R \to \infty).$$

For a>0,

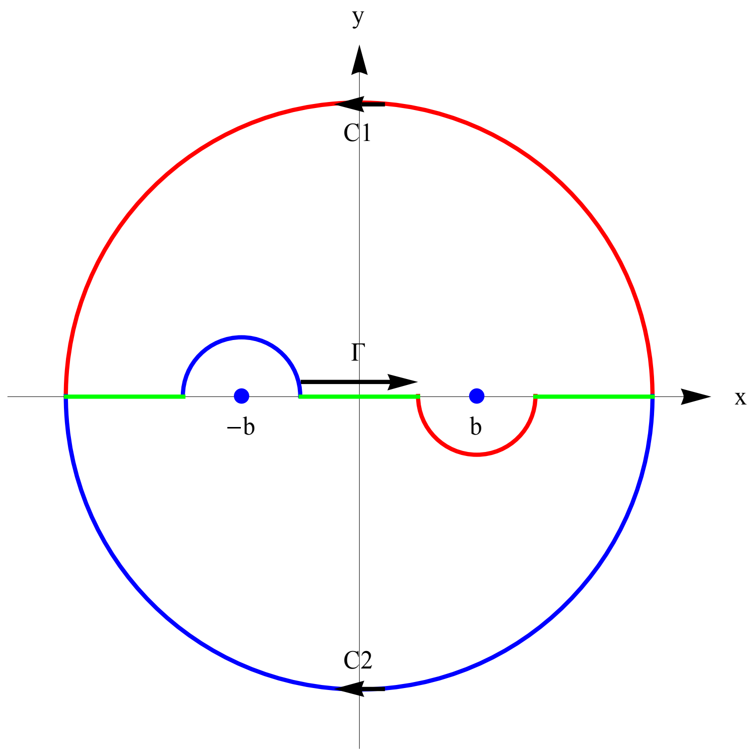
$$I = \int_{C1+\Gamma} dz e^{iaz} f(z)$$

For a<0,

$$I = \int_{C2+\Gamma} dz e^{iaz} f(z)$$

since iaz = ia (α + i β) = i$a\alpha$ - $a\beta$). We need to choose the path (upper half plane) for a>0 and the path (lower half plane) for a<0 (Jordan's lemma). There are poles on the real axis at $z = \pm b$. We must specify how to go around. We consider the four cases (Cases I - IV).

((**Case I**))



For a>0 (upper half-plane),

$$\oint_{C1+\Gamma} \frac{e^{iaz}}{z^2-b^2}dz = P\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2-b^2}dx - \pi i \,\mathrm{Res}(z=-b) + \pi i\,\mathrm{Res}(z=b)$$
$$= 2\pi i\,\mathrm{Res}(z=b)$$

The second term of the right-hand side clock-wise
The third term of the right hand side counter clock-wise

Then we have

$$P\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2-b^2}dx = \pi i[\mathrm{Res}(z=-b)+\mathrm{Res}(z=b)] = -\frac{\pi}{b}\sin(ab)$$

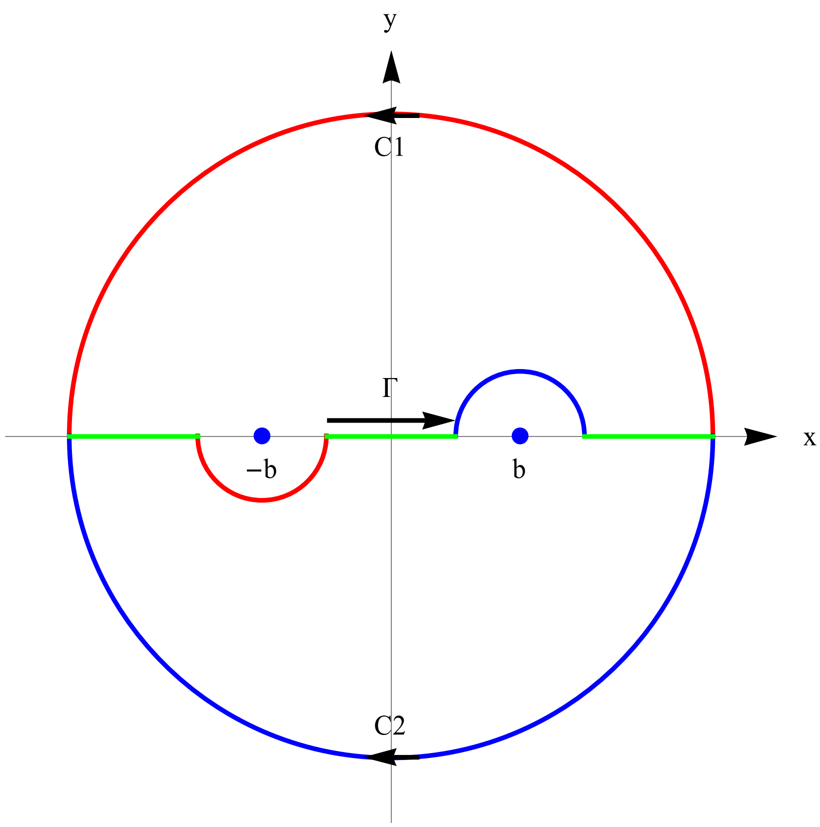
For a<0 (lower half-plane),

$$\oint_{C2+\Gamma} \frac{e^{iaz}}{z^2-b^2}dz = P\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2-b^2}dx - \pi i \,\mathrm{Res}(z=-b) + \pi i\,\mathrm{Res}(z=b)$$
$$= -2\pi i\,\mathrm{Res}(z=-b)$$

or

$$P\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2-b^2}dx=-\pi i[\mathrm{Res}(z=-b)+\mathrm{Res}(z=b)]=\frac{\pi}{b}\sin(ab)$$

((**Case II**))



For a>0 (upper half-plane)

$$\oint_{C1+\Gamma}\frac{e^{iaz}}{z^2-b^2}dz=P\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2-b^2}dx+\pi i\,\mathrm{Res}(z=-b)-\pi i\,\mathrm{Res}(z=b)$$
$$=2\pi i\,\mathrm{Res}(z=-b)$$

The second term of the right-hand side counter clock-wise
The third term of the right hand side clock-wise

Then we have

$$P\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2-b^2}dx=\pi i[\mathrm{Res}(z=-b)+\mathrm{Res}(z=b)]=-\frac{\pi}{b}\sin(ab)$$

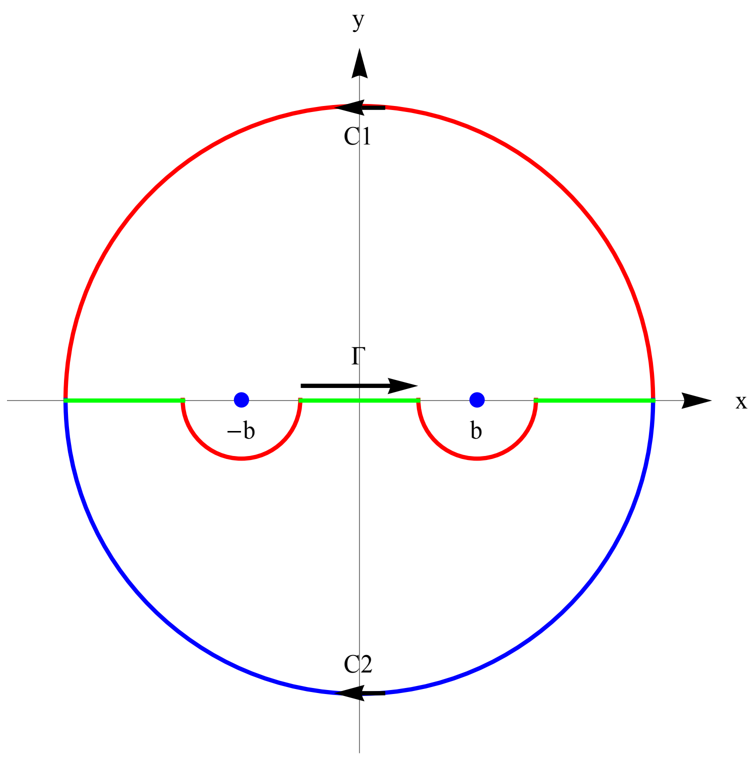
For a<0 (lower half-plane)

$$\oint_{C2+\Gamma} \frac{e^{iaz}}{z^2-b^2}dz = P\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2-b^2}dx + \pi i \operatorname{Res}(z=-b) - \pi i \operatorname{Res}(z=b)$$
$$= -2\pi i \operatorname{Res}(z=b)$$

or

$$P\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2-b^2}dx = -\pi i[\operatorname{Res}(z=-b) + \operatorname{Res}(z=b)] = \frac{\pi}{b}\sin(ab)$$

((**Case-III**))



(a) For a>0 (upper half-plane), we have

$$\oint_{C1+\Gamma} \frac{e^{iaz}}{z^2-b^2}dz = P\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2-b^2}dx + \pi i \operatorname{Res}(z=-b) + \pi i \operatorname{Res}(z=b)$$
$$= 2\pi i \operatorname{Res}(z=b) + 2\pi i \operatorname{Res}(z=-b)$$

since there are two poles inside the contour. Then we have

$$P\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2-b^2}dx = \pi i[\mathrm{Re}\,s(z=-b)+\mathrm{Re}\,s(z=b)]$$

(for a>0).

$$= -\frac{\pi}{b}\sin(ab)$$

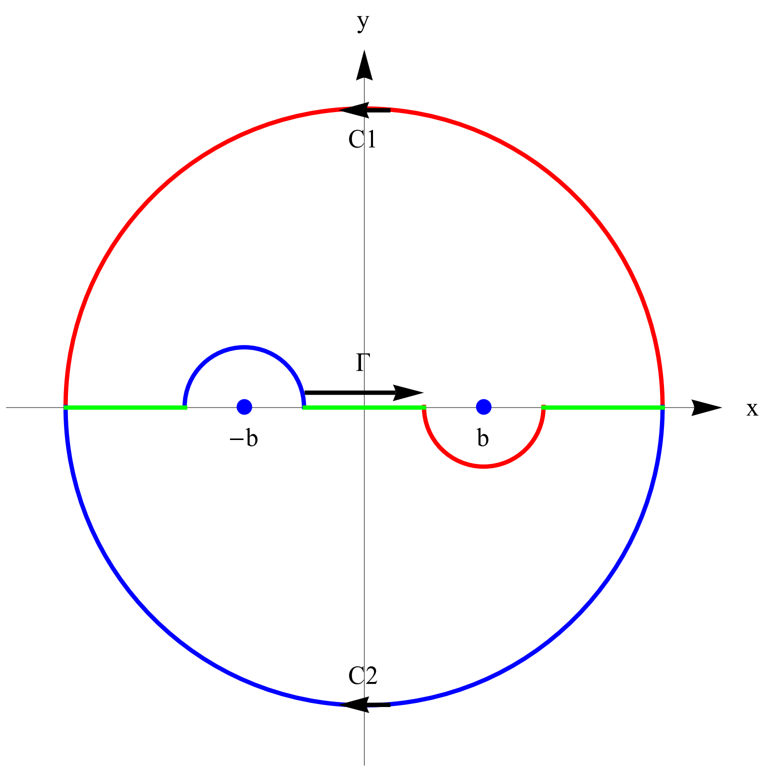
(b) For a<0 (lower half-plane), we have

$$\oint_{C2+\Gamma}\frac{e^{iaz}}{z^2-b^2}dz = P\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2-b^2}dx + \pi i\,\mathrm{Re}\,s(z=-b)+\pi i\,\mathrm{Re}\,s(z=b)$$
$$=0$$

since there is no pole in the contour.

$$P\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2-b^2}dx = -\pi i[\mathrm{Re}\,s(z=-b)+\mathrm{Re}\,s(z=b)]$$
$$=\frac{\pi}{b}\sin(ab)$$

((**Case-IV**))



For a>0 (upper half-plane)

$$\oint_{C1+\Gamma} \frac{e^{iaz}}{z^2-b^2}dz = P\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2-b^2}dx - \pi i \operatorname{Res}(z=-b) + \pi i \operatorname{Res}(z=b)$$
$$= 2\pi i \operatorname{Res}(z=b)$$

or

$$P\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2-b^2}dx = \pi i[\operatorname{Res}(z=-b) + \operatorname{Res}(z=b)] \qquad (a>0)$$
$$= -\frac{\pi}{b}\sin(ab)$$

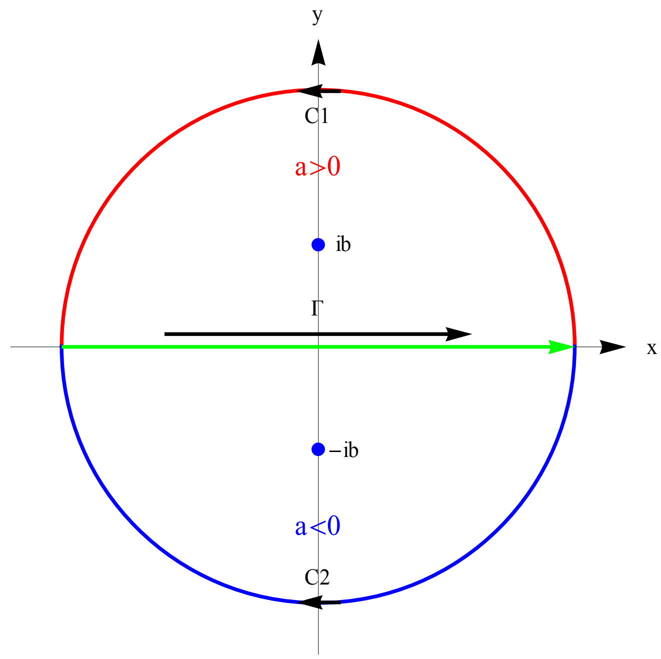
For $a<0$ (lower half plane)

$$\oint_{C1+C3} \frac{e^{iaz}}{z^2-b^2}dz = P\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2-b^2}dx - \pi i \operatorname{Res}(z=-b) + \pi i \operatorname{Res}(z=b)$$
$$= -2\pi i \operatorname{Res}(z=-b)$$

or

$$P\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2-b^2}dx = -\pi i[\operatorname{Res}(z=-b) + \operatorname{Res}(z=b)] \qquad (a<0)$$
$$= \frac{\pi}{b}\sin(ab)$$

9.16.2 Application II



$$I = \int_0^\infty \frac{\cos(ax)}{x^2+b^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}dx \qquad (b>0).$$

The last equality holds because of

$$\int_{-\infty}^{\infty} \frac{\sin(ax)}{x^2+b^2}dx = 0 \qquad \text{(the integrand is an odd function of } x\text{)}$$

So we calculate $I = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}dx$.

For a>0 (upper-half plane),

$$\int_{C1+\Gamma} \frac{e^{iaz}}{z^2+b^2}dz = \int_{C1} \frac{e^{iaz}}{z^2+b^2}dz + \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}dx = 2\pi i \operatorname{Re}s(z=ib) = \frac{\pi e^{-ab}}{b}$$

since there is one single pole at $z = ib$ and the contour integral around the path C1 is zero (Jordan's lemma).

For a<0 (lower half-plane),

$$\int_{C2+\Gamma} \frac{e^{iaz}}{z^2+b^2}dz = \int_{C2} \frac{e^{iaz}}{z^2+b^2}dz + \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}dx = -2\pi i \operatorname{Re}s(z=-ib) = \frac{\pi e^{ab}}{b}$$

since there is one single pole at $z = -ib$ and the contour integral around the path C2 is zero (Jordan's lemma).

9.16.3 Application III

We consider the derivation of familiar integral

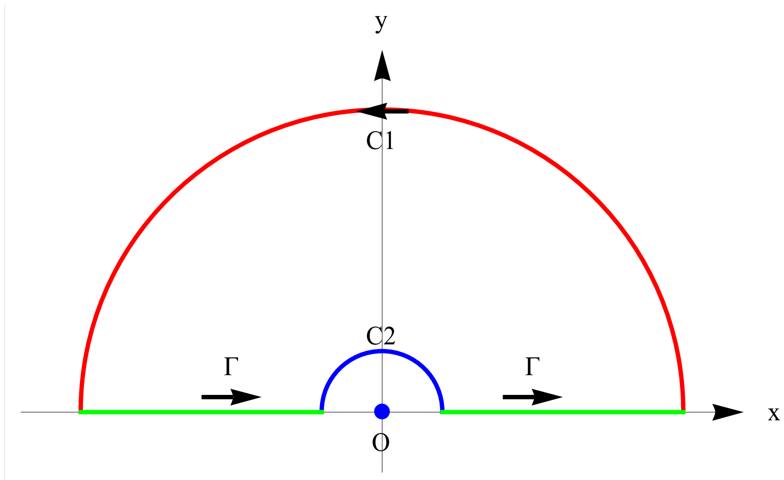
$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

((**Case-I**))

It can be found by integrating

$$I = \oint_{C1+\Gamma+C2} \frac{e^{iz}}{z} dz$$

around the contour of Fig. The integrand has a simple pole at $z = 0$. This pole is avoided by placing a semicircular path C2 (the radius r →0) around it. There are no poles inside the contour. So $I = 0$.



$$I = \oint_{C1+\Gamma+C2} \frac{e^{iz}}{z} dz = \int_{C1} \frac{e^{iz}}{z} dz + \int_{\Gamma} \frac{e^{iz}}{z} dz + \int_{C2} \frac{e^{iz}}{z} dz = 0$$

The first term is equal to zero (the radius of C1 $R\rightarrow\infty$, Jordan's lemma). Then we have

$$P\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx - \pi i \operatorname{Res}(z = 0) = 0$$

or

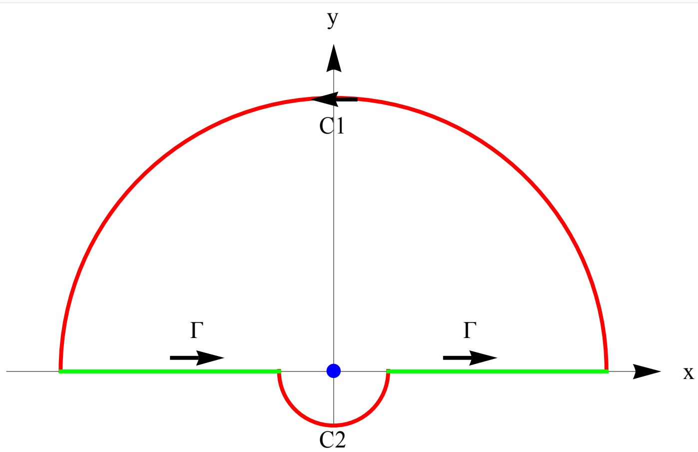
$$P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx = 2i\int_{0}^{\infty}\frac{\sin x}{x}dx = i\pi$$

or

$$\int_{0}^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$$

((**Case-II**))

For the calculation of the integral, we can choose the different contour shown in Fig.



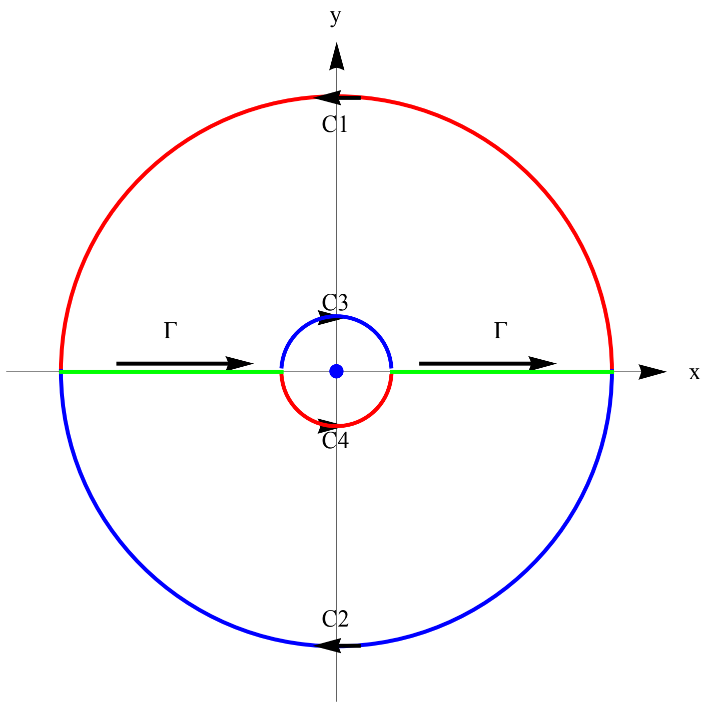
$$I = \oint_{C1+\Gamma+C2}\frac{e^{iz}}{z}dz = \int_{C1}\frac{e^{iz}}{z}dz + \int_{\Gamma}\frac{e^{iz}}{z}dz + \int_{C2}\frac{e^{iz}}{z}dz = 2\pi i \,\mathrm{Re}\, s(z=0)$$

The integrand has a simple pole at $z = 0$ in this contour. Using the Jordan's lemma for the contour integral around the contour C1, we have

$$P\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx = \pi i \,\mathrm{Re}\, s(z=0) = \pi i\ .$$

9.16.4 Application IV Unit step function

$$I(s) = \int_{-\infty}^{\infty}\frac{e^{isx}}{x}dx$$



For s>0,

$$\oint_{C1+\Gamma+C3} \frac{e^{isz}}{z} dz = \int_{C1} \frac{e^{isz}}{z} dz + P\int_{-\infty}^{\infty} \frac{e^{isx}}{x} dx + \int_{C3} \frac{e^{isz}}{z} dz = 0$$

since there is no pole inside the contour. From the Jordan's lemma, the first term (the contour integral around C1) is equal to zero. Then we have

$$P\int_{-\infty}^{\infty} \frac{e^{isx}}{x} dx = -\int_{C3} \frac{e^{isz}}{z} dz = -(-\pi i)\operatorname{Re}s(z=0) = \pi i$$

For s<0

$$\oint_{C2+\Gamma+C4} \frac{e^{isz}}{z} dz = \int_{C2} \frac{e^{isz}}{z} dz + P\int_{-\infty}^{\infty} \frac{e^{isx}}{x} dx + \int_{C4} \frac{e^{isz}}{z} dz = 0$$

since there is no pole inside the contour. From the Jordan's lemma, the first term (the contour integral around C2) is equal to zero. Then we have

$$P\int_{-\infty}^{\infty} \frac{e^{isx}}{x} dx = -\int_{C4} \frac{e^{isz}}{z} dz = -(\pi i)\operatorname{Re}s(z=0) = -\pi i$$

In summary, we obtain

$$I(s)=\begin{cases} i\pi \;\; (s>0) \\ -i\pi\,(s<0)\end{cases}$$

For $s = 0$,

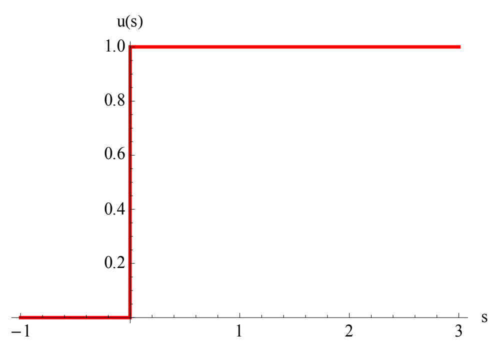
$$\int_{-\infty}^{\infty}\frac{1}{x}dx=0$$

We consider

$$u(s)=\frac{1}{2}+\frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{e^{isx}}{x}dx=\begin{cases}1,(s>0)\\0,(s<0)\end{cases}$$

and

$u(s) = 1/2$ at $s = 0$.



9.16.5 The $i\varepsilon$ prescription

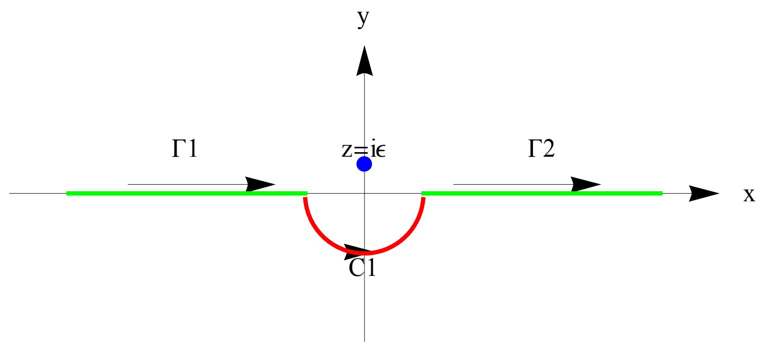
We derive the formula

$$\frac{1}{x\mp i\varepsilon}=P\frac{1}{x}\pm i\pi\delta(x)$$

where $\varepsilon\,(\to 0)$ is a positive infinitesimally small quantity.

(1) Case-I

$$I_1=\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{f(x)}{x-i\varepsilon}dx$$



Since the only singularity near the real axis is $z = i\varepsilon$, we make the following deformation of the contour without changing the value of I. The contour runs along the real axis (the path $\Gamma 1$) and goes around counterclockwise, below the origin in a semicircle (C1), and resumes along the real axis (the path $\Gamma 2$).

$$I_1 = \int_{\Gamma 1} \frac{f(x)}{x} dx + \int_{C1} \frac{f(z)}{z} dz + \int_{\Gamma 2} \frac{f(x)}{x} dx$$

$$= P \int_{-\infty}^{\infty} \frac{f(x)}{x} dx + \pi i \operatorname{Re} s(z = 0)$$

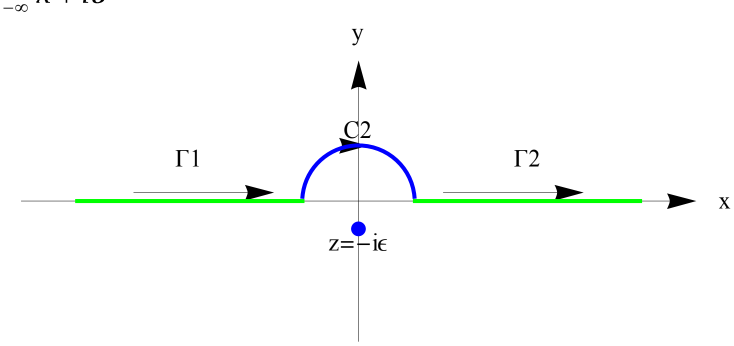
$$= P \int_{-\infty}^{\infty} \frac{f(x)}{x} dx + \pi i f(0)$$

or

$$\frac{1}{x - i\varepsilon} = P\frac{1}{x} + i\pi\delta(x).$$

(2) Case-II

$$I_2 = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x + i\varepsilon} dx$$



Since the only singularity near the real axis is z = -iε, we make the following deformation of the contour without changing the value of I_2 The contour runs along the real axis (the path $\Gamma 1$) and goes around clockwise, above the origin in a semicircle (C2), and resumes along the real axis (the path $\Gamma 2$).

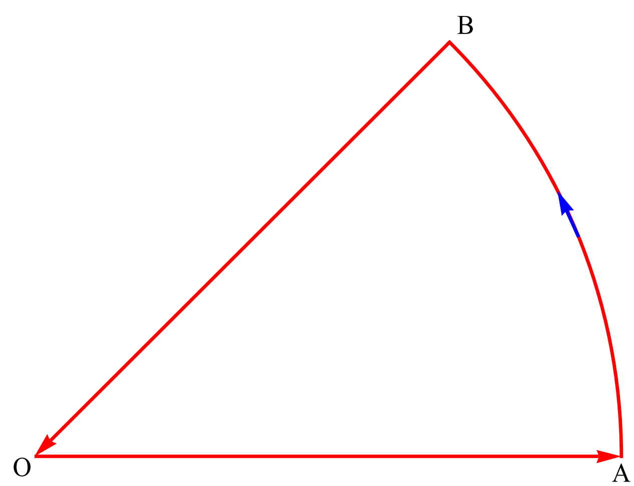
$$I_2 = \int_{\Gamma 1} \frac{f(x)}{x} dx + \int_{C2} \frac{f(z)}{z} dz + \int_{\Gamma 2} \frac{f(x)}{x} dx$$
$$= P \int_{-\infty}^{\infty} \frac{f(x)}{x} dx - \pi i \operatorname{Res}(z = 0)$$
$$= P \int_{-\infty}^{\infty} \frac{f(x)}{x} dx - \pi i f(0)$$

or

$$\frac{1}{x + i\varepsilon} = P\frac{1}{x} - i\pi\delta(x)$$

9.16.7 Fresnel integral

$$\int_0^{\infty} \cos(x^2) dx = \int_0^{\infty} \sin(x^2) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$



$$\int_{OA} e^{-z^2} dz + \int_{AB} e^{-z^2} dz + \int_{BO} e^{-z^2} dz = 0$$

<u>OA</u>

$$I_1 = \int_{OA} e^{-z^2} dz = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

<u>BO</u>

$$I_2 = \int_{BO} e^{-z^2} dz = -\int_{OB} e^{-z^2} dz$$

$$z = re^{i\pi/4}, \qquad dz = e^{i\pi/4} dr, \quad z^2 = r^2 e^{i\pi/2} = ir^2 .$$

$$I_2 = -e^{i\pi/4} \int_0^{\infty} e^{-ir^2} dr = -\frac{(1+i)}{\sqrt{2}} [\int_0^{\infty} \cos(r^2) - i\sin(r^2)] dr$$

<u>AB</u>

$$z = \mathrm{R}e^{i\theta}, \qquad dz = \mathrm{R}e^{i\theta}\, id\theta, \qquad z^2 = R^2 e^{i2\theta}.$$

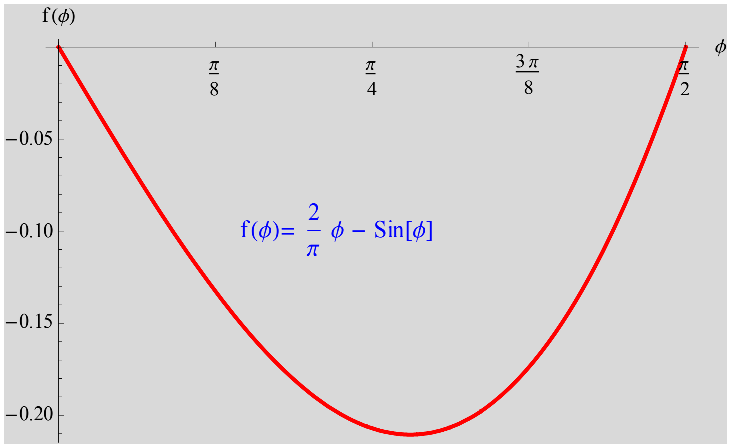
$$I_3 = \int_{AB} e^{-z^2} dz = \int_0^{\pi/4} \exp(-R^2 e^{2i\theta}) \mathrm{R}e^{i\theta}\, id\theta$$

$$|I_3| \le \int_0^{\pi/4} \exp[-R^2 \cos(2\theta)] R d\theta = \frac{R}{2} \int_0^{\pi/2} \exp[-R^2 \cos(\varphi)] d\varphi = \frac{R}{2} \int_0^{\pi/2} \exp[-R^2 \sin(\varphi)] d\varphi$$

or

$$|I_3| \le \frac{R}{2} \int_0^{\pi/2} \exp[-R^2 \sin(\varphi)] d\varphi \le \frac{\pi(1-e^{-R^2})}{4R} \to 0$$

as $R \to \infty$. Note that we use the inequality $\sin\varphi \le \frac{2}{\pi}\varphi$ for $0 \le \varphi \le \pi/2$.



Since $I_1 + I_2 = 0$, we have

$$\frac{\sqrt{\pi}}{2} = \frac{1+i}{\sqrt{2}}\int_0^\infty [\cos(r^2) - i\sin(r^2)]dr$$

or

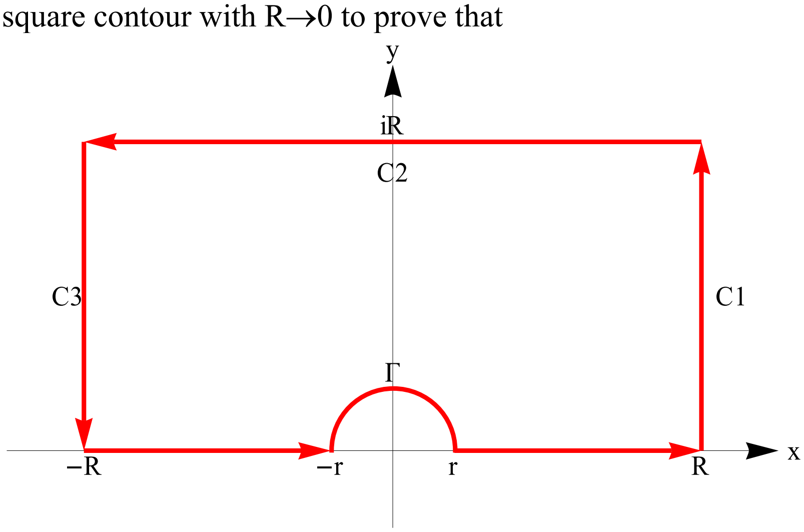
$$\frac{1}{2}\sqrt{\frac{\pi}{2}}(1-i) = \int_0^\infty [\cos(r^2) - i\sin(r^2)]dr$$

or

$$\int_0^\infty \cos(r^2)dr = \int_0^\infty \sin(r^2)dr = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

9.16.8 Arfken 7-1-15

Use the large square contour with R→0 to prove that



$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

((**Solution**))

$$0 = \oint \frac{e^{iz}}{z} dz = \int_{-R}^{-r} \frac{e^{ix}}{x} dx + \int_{r}^{R} \frac{e^{ix}}{x} dx + \int_{C1} \frac{e^{iz}}{z} dz + \int_{C2} \frac{e^{iz}}{z} dz + \int_{C3} \frac{e^{iz}}{z} dz + \int_{\Gamma} \frac{e^{iz}}{z} dz$$

$$\left| \int_{C1} \frac{e^{iz}}{z} dz \right| = \left| \int_{0}^{R} idy \frac{e^{i(R+iy)}}{R+iy} \right| \le \int_{0}^{R} dy \frac{e^{-y}}{R} = \frac{1}{R}(1-e^{-R}) \to 0 .$$

$$\left| \int_{C2} \frac{e^{iz}}{z} dz \right| = \left| \int_{-R}^{R} dx \frac{e^{i(x+iR)}}{x+iR} \right| \le \frac{1}{R} \int_{R}^{R} dx e^{-R} = 2e^{-R} \to 0$$

$$\left| \int_{C3} \frac{e^{iz}}{z} dz \right| = \left| \int_{R}^{0} idy \frac{e^{i(-R+iy)}}{-R+iy} \right| \le \frac{1}{R} \int_{0}^{R} dy e^{-y} = \frac{1}{R}(1-e^{-R}) \to 0$$

Then

$$\int_{C1} \frac{e^{iz}}{z} dz + \int_{C2} \frac{e^{iz}}{z} dz + \int_{C3} \frac{e^{iz}}{z} dz = 0 .$$

Since

$$\int_{\Gamma} \frac{e^{iz}}{z} dz = -\pi i \operatorname{Re} s(z=0) = -\pi i$$

we have

$$P \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i$$

or

$$P \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

9.17 Cut line for the multivalued function

Single valued function: a function which is mostly analytical with some singularities

We can generalize discussion to multi-valued functions using a geometrical construction known as Riemann surfaces.

We consider logarithmic functions

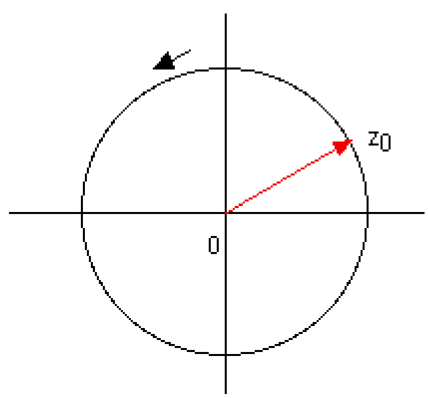
$$\ln z = \ln|z| + i \arg z$$

Argz is uniquely defined only up to multiple of 2π.

$$Argz = \theta + 2\pi n$$

(i) To see what this means, we consider a closed path C enclosing $z = 0$. Start at $z = z_0$ and follow the value of lnz as it changes continuously as we go along the contour C. Whatever we had for argz_0 initially we find that argz_0 has increased by 2π when we return to z_0.

$$(\ln z_0)_{final} = (\ln z_0)_{initial} + 2\pi i \text{ .}$$



((**Branch point**))

A point in complex plane that has this property, i.e.,

$$f(z_0)_{final} \neq f(z_0))_{initial}$$

for any loop around it, is called a "branch point," of the function.

lnz has a branch point at $z = 0$.

(ii) Let $z = 1/z'$, then we have

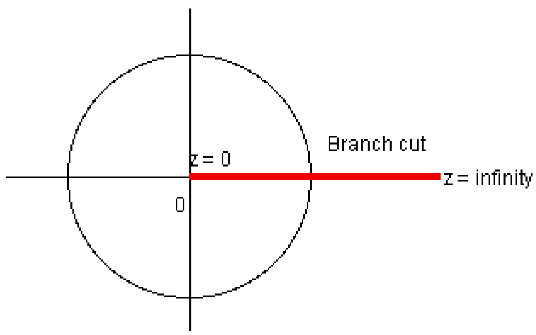
$$\ln z = \ln z' = \ln\left|\frac{1}{z'}\right| - i\arg z'$$

It is easy to see that lnz' has a branch point at z' = 0 and z = ∞.

In summary, lnz has two branch points at |z| = 0 and |z| = ∞.

(iii) If we draw line or curve joining the two branch points, this line is referred to as a branch cut. So the complex plane is cut from branch point to another.

((**Example**))



$$f(z) = \ln z$$

We cut the plane along the positive half of the real axis.

$$f_0(z) = \ln r + i\theta$$

where

$$z = re^{i\theta}$$

where r>0 and $0 \le \theta \le 2\pi$.
This is really a single valued function. Similarly we can define

$$f_1(z) = \ln r + i(\theta + 2\pi)$$

where r>0 and $0 \le \theta \le 2\pi$.

and

$$f_{-1}(z) = \ln r + i(\theta - 2\pi)$$

where r>0 and $0 \le \theta \le 2\pi$.

and a whole inifinite sequence for functions

$$f_0, f_{\pm 1}, f_{\pm 2}, f_{\pm 2}, \ldots\ldots\ldots\ldots\ldots\ldots.., f_{\pm n},$$

which are single value and can be used to replace the multivalued function lnz.

(iv) Each $f_n(z)$ suffers a discontinuity across the cut – i.e., 2πi- but the values of $f_n(z)$ above cut is the same as $f_{n+1}(z)$ below cut.

- Thus it is suggestive to say that we have an infinite series of cut planes on top of one another.
- The plane that $f_{n+1}(z)$ is defined on, lied right above the $f_n(z)$.
- The adjacent planes are connected across the cut.
- The lower lip pf the n+1 plane is connected to the upper lip of the n-th plane.

So when we cross a cut, we are going from one-cut-plane to another cut-plane.

Each plane is called a Rieman sheet. Supposition of all planes in the helix array is called Rieman surface.

(v) What has all of this achieved?

- Started with lnz. We have one single valued function defined on Rieman surface instead of multivalued function.
- Branch points: when we go around them, we go to the different sheet.
- We can define the order of branch points. We have an n-th order branch point if we have to go around it n+1 times to some back to a function original value [minimum n].

So $z = 0$ is a branch point of infinite order for lnz.

9.19 Cut-line

The cut line runs along some curve from the $z = 0$ out to infinity.
We consider

(a) $\sqrt{z}$

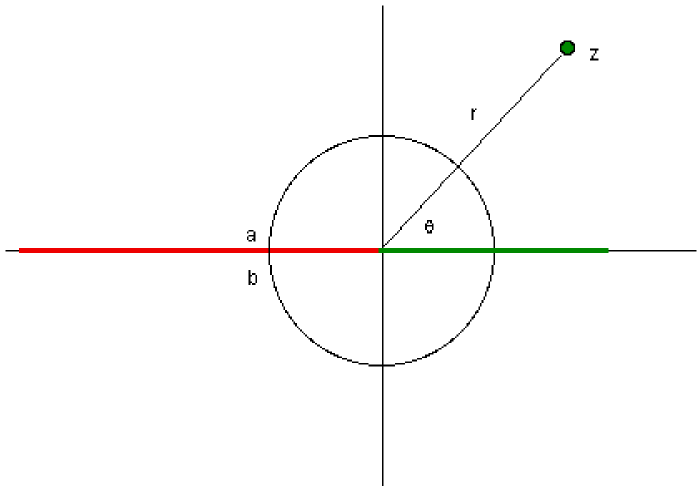
Cut along the negative real axis

$$z = re^{i\theta} \ \ (-\pi \le \theta < \pi)$$

The angle θ is restricted so as not to cross the cut line.

(i) Cut along the negative real axis as the cut line

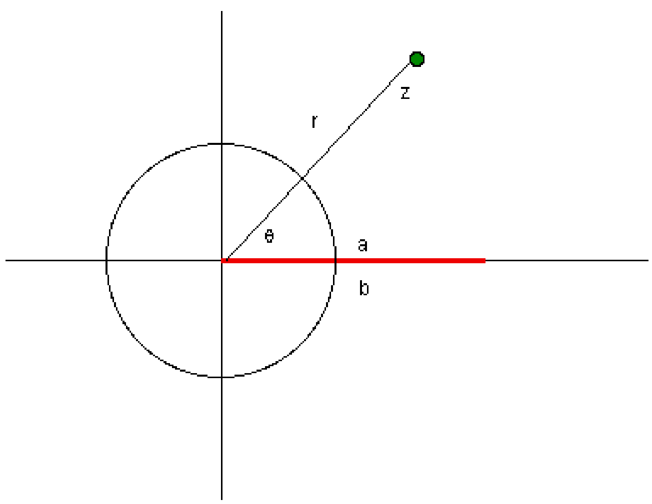
$$\sqrt{z} = \sqrt{r}e^{i\theta/2} \qquad \text{for } -\pi \le \theta < \pi$$



point a ($z_a = re^{i\pi}$) $\quad \to \sqrt{r}e^{i\pi/2} = i\sqrt{r}$
point b ($z_b = re^{-i\pi}$) $\quad \to \sqrt{r}e^{-i\pi/2} = -i\sqrt{r}$

There is thus a discontinuity across the cut. The function is not analytic on the cut

(ii) Cut along the positive real axis as the cut line



$$\sqrt{z} = \sqrt{r}e^{i\theta/2} \qquad \text{for } 0 \le \theta < 2\pi$$

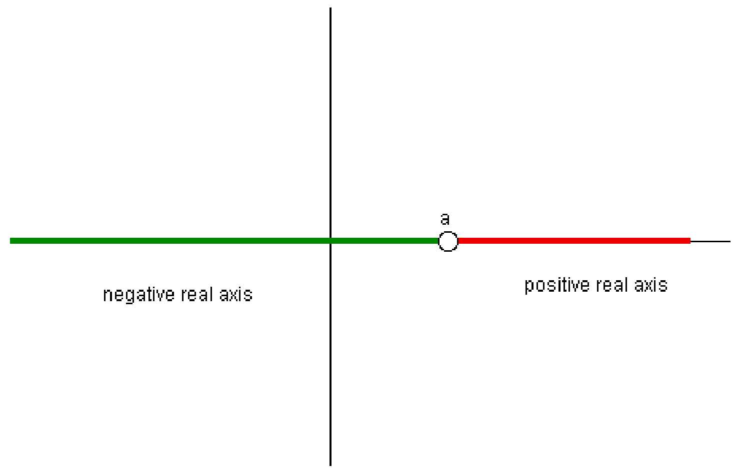
point a ($z_a = re^{i0}$)$\to$ $\quad \sqrt{r}e^{i0} = \sqrt{r}$
point b ($z_b = re^{i2\pi}$) $\quad \to \quad \sqrt{r}e^{i\pi} = -\sqrt{r}$

There is thus a discontinuity across the cut. The function is not analytic on the cut line

(b) $\sqrt{z-a} = \sqrt{z'}$

$$z' = z - a$$

z'=0 (or $z = a$) and infinity ($z = \infty$) are branch points.



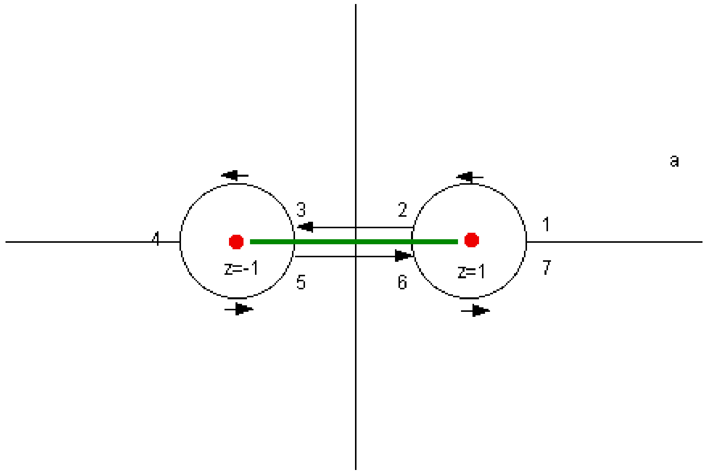
(c) $\sqrt{(z+1)(z-1)}$

Check on the possibility of taking the line segment joining z+1 and z-1 as a cut line

$$z+1 = re^{i\theta}$$
$$z-1 = \rho e^{i\phi}$$

where $0 \le \theta < 2\pi$ and $0 \le \phi < 2\pi$

$$f(z) = \sqrt{r\rho}e^{i(\theta+\phi)/2}$$



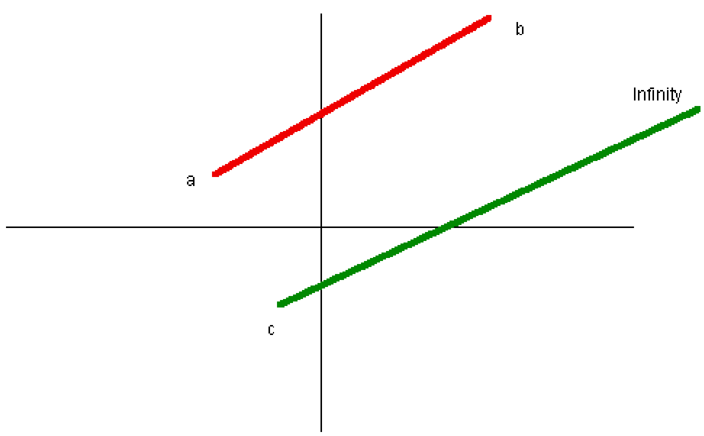
	θ	ϕ	$(\theta+\phi)/2$
1	0	0	0
2	0	π	$\pi/2$
3	0	π	$\pi/2$
4	π	π	π
5	2π	π	$3\pi/2$
6	2π	π	$3\pi/2$
7	2π	2π	2π

(i) The phase at points 5 and 6 is not the same as the points 2 and 3. This behavior can be expected at a branch point cut line.

(ii) The phase at point 7 exceeds that at 1 by 2π. So the function $f(z)$ is therefore single valued for the contour shown encircling both branch points.

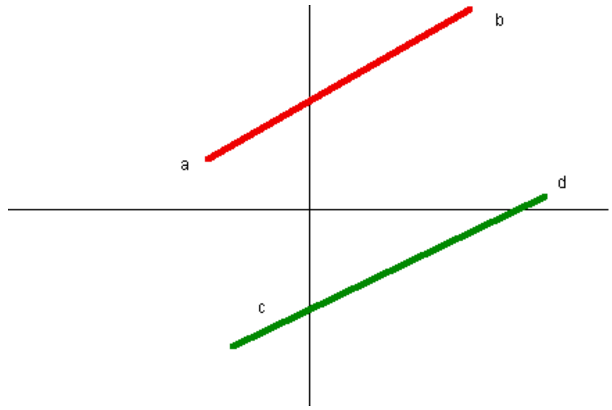
(d) $\sqrt{(z-a)(z-b)(z-c)}$

There are three branch points. So there are two cut lines.



(d) $\sqrt{(z-a)(z-b)(z-c)(z-d)}$

There are four branch points. So there are two cut lines.



((Note))

(i) A function with a branch point and a required cut line will not be a continuous across the cut line.

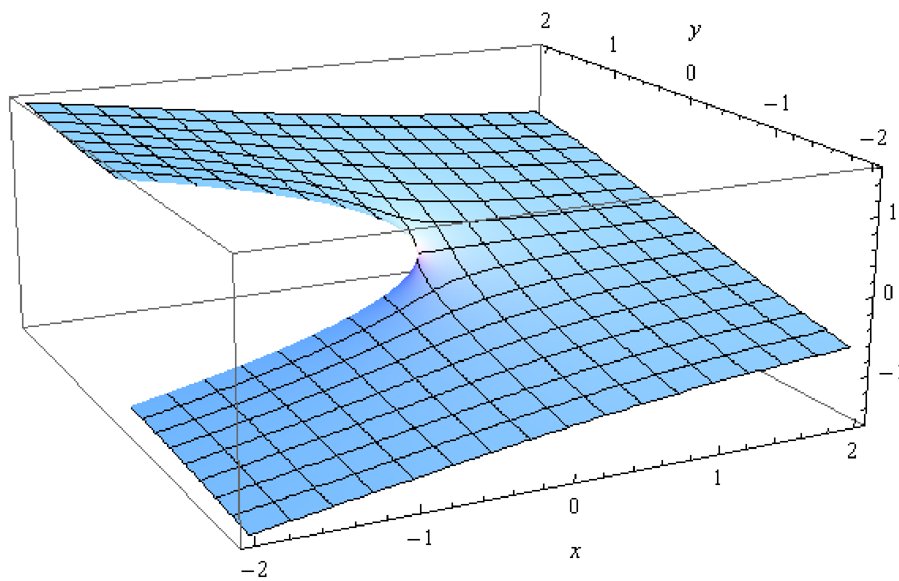
(ii) In general, there will be a phase difference on opposite sides of this cut line.

(iii) *Thus the line integrals on opposite sides of the cut line will not generally cancel each other.*

9.20 Examples of cut line (Mathematica)

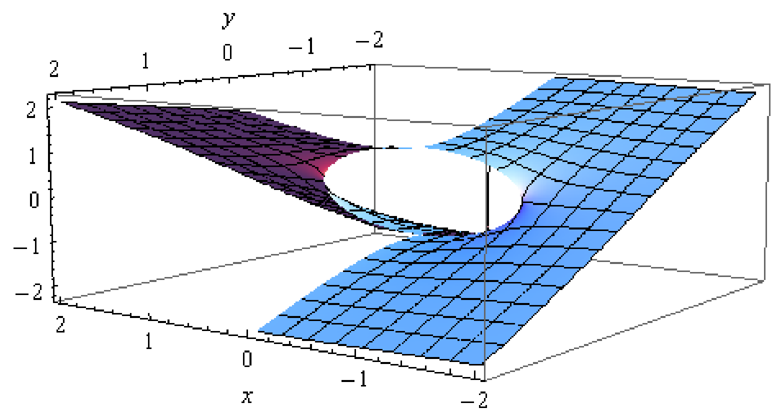
9.20.1 $\sqrt{z}$

Plot3D of Im[$\sqrt{z}$]→there is a cut line on the negative x axis (x<0).



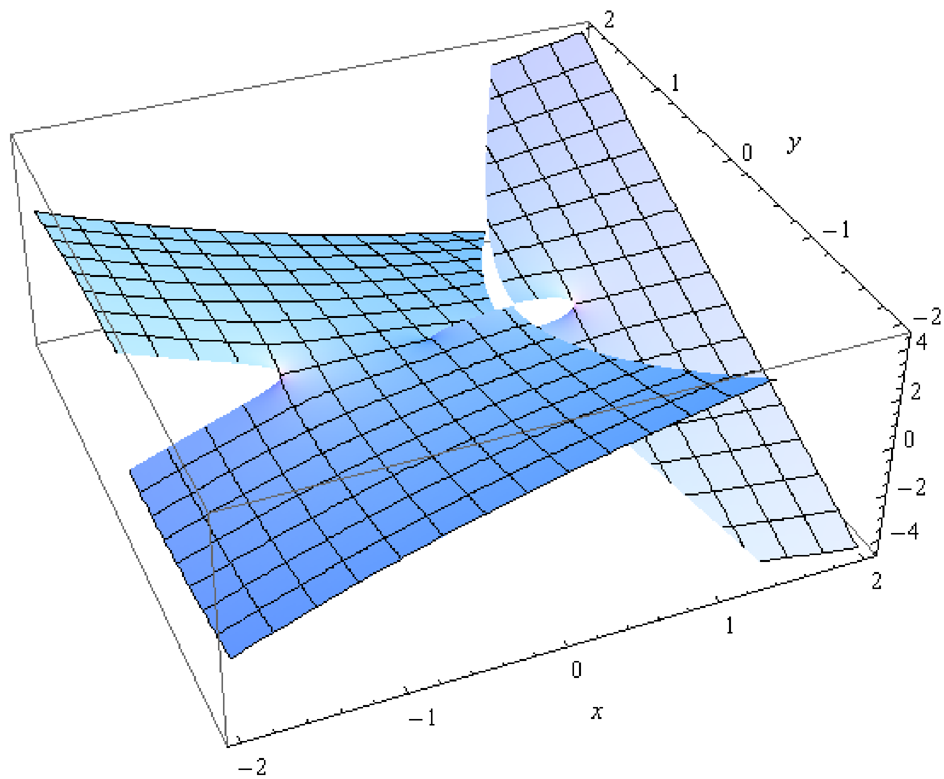
9.20.2 $\sqrt{(z+1)(z-1)}$

Plot3D of Im[$\sqrt{(z+1)(z-1)}$]→there is a cut line between z = 1 and z = -1.



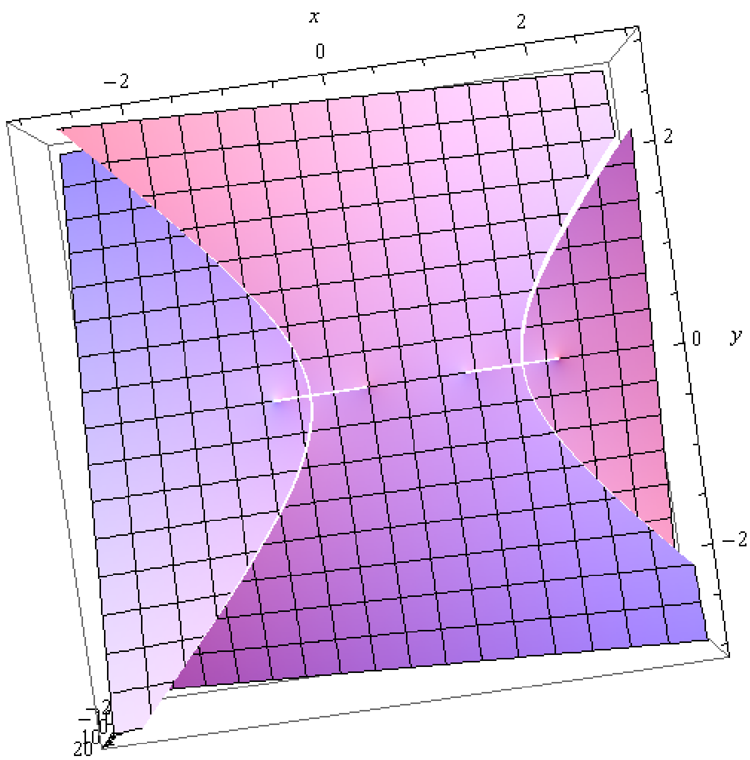
9.20.3 $\sqrt{(z+1)z(z-1)}$

Plot3D of Im[$\sqrt{(z+1)z(z-1)}$]→A cut line between z = 1 and z = 0, and a cut line between z = -1 and the negative x axis (x<-1).



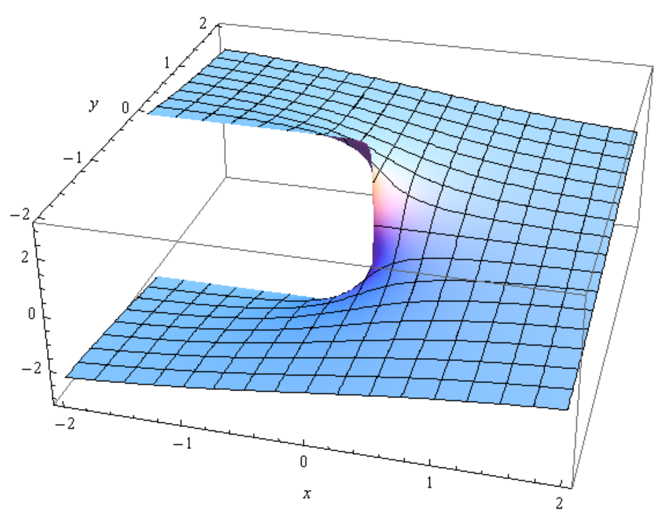
9.20.4 $\sqrt{(z+1)z(z-1)(z-2)}$

Plot3D of Im[$\sqrt{(z+1)z(z-1)(z-2)}$]→A cut line between z = -1 and z = 0, and a cut line between z = 1 and z = 2.



9.20.5 ln(z)

Plot3D of Im[ln(z)]→there is a cut line on the negative x axis.



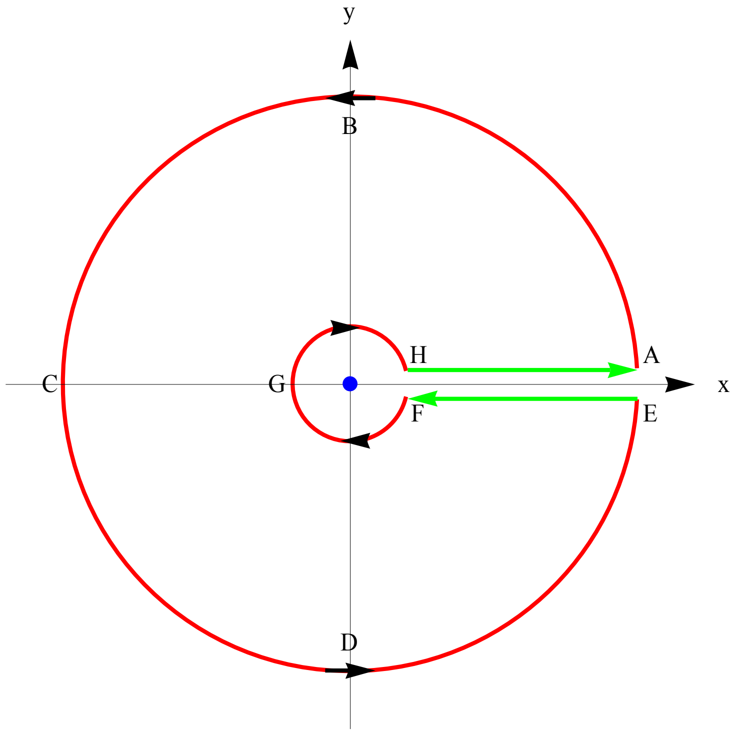
9.21 Applications

9.21.1 Arfken 7-1-18

Show that

$$\int_0^\infty \frac{x^a}{(x+1)^2}dx = \frac{\pi a}{\sin(\pi a)}$$

where $|a|<1$. We note that $z = 0$ is a branch point and the positive x axis is a cut line.



$$f(z) = \frac{z^a}{(z+1)^2}.$$

Since there is one pole at $z = e^{i\pi}$, we have

$$\int_{HA} f(z)dz + \int_{ABCDE} f(z)dz + \int_{EF} f(z)dz + \int_{HGF} f(z)dz = 2\pi i \operatorname{Res}(z = e^{i\pi}).$$

<u>HA</u>

$$z = xe^{i0}, \qquad z^a = x^a e^{ia0}$$

$$dz = dxe^{i0}$$

$$I_1 = \int_{.HA} f(z)dz = \int_0^\infty \frac{x^a}{(x+1)^2} e^{i0} dx = \int_0^\infty \frac{x^a}{(x+1)^2} dx$$

<u>EF</u>

$$z = xe^{2\pi i}, \qquad z^a = x^a e^{2\pi i a}$$

$$dz = dxe^{2\pi i}$$

$$I_2 = \int_{.EF} f(z)dz = \int_\infty^0 \frac{x^a}{(x+1)^2} e^{2\pi i a} dx = -e^{2\pi i a} \int_0^\infty \frac{x^a}{(x+1)^2} dx = -e^{2\pi i a} I_1$$

<u>ABCD (radius R)</u>

$$z = \mathrm{Re}^{i\theta}, \qquad z^a = R^a e^{ia\theta}$$

$$dz = \mathrm{Re}^{i\theta} id\theta$$

$$\left| \int_{ABCDE} f(z)dz \right| = \left| \int_0^{2\pi} \frac{R^a e^{ia\pi}}{(\mathrm{Re}^{i\theta}+1)^2} \mathrm{Re}^{i\theta} id\theta \right| \le \int_0^{2\pi} \frac{R^{a+1}}{R^2} d\theta = R^{a-1}(2\pi) \to 0 \qquad (\text{as } R \to \infty).$$

<u>HGF (radius ρ)</u>

$$z = \rho e^{i\theta}, \qquad z^a = \rho^a e^{ia\theta}$$

$$dz = \rho e^{i\theta} id\theta$$

$$\int_{HGF} f(z)dz = \int_0^{2\pi} \frac{\rho^a e^{ia\theta}}{(\rho e^{i\theta}+1)^2} \rho e^{i\theta} id\theta = i\rho^{a+1} \int_0^{2\pi} e^{i\theta(a+1)} d\theta \to 0 \qquad (\text{as } \rho \to 0).$$

Then we have

$$I_1 + I_2 = (1 - e^{i2\pi a}) I_1 = 2\pi i \,\mathrm{Re}\, s(z = e^{i\pi})$$

or

$$I_1 = \frac{2\pi i \,\mathrm{Re}\, s(z = e^{i\pi})}{1 - e^{i2\pi a}} = \frac{2\pi i(-a)e^{ia\pi}}{1 - e^{i2\pi a}} = \frac{\pi a}{\sin(a\pi)}$$

where

$$\mathrm{Res}(z=e^{i\pi})=\frac{1}{(2-1)!}\frac{d}{dz}[(z+1)^2\frac{z^a}{(z+1)^2}]_{z=e^{i\pi}}=az^{a-1}|_{z=e^{i\pi}}=ae^{i\pi(a-1)}=-ae^{ia\pi}$$

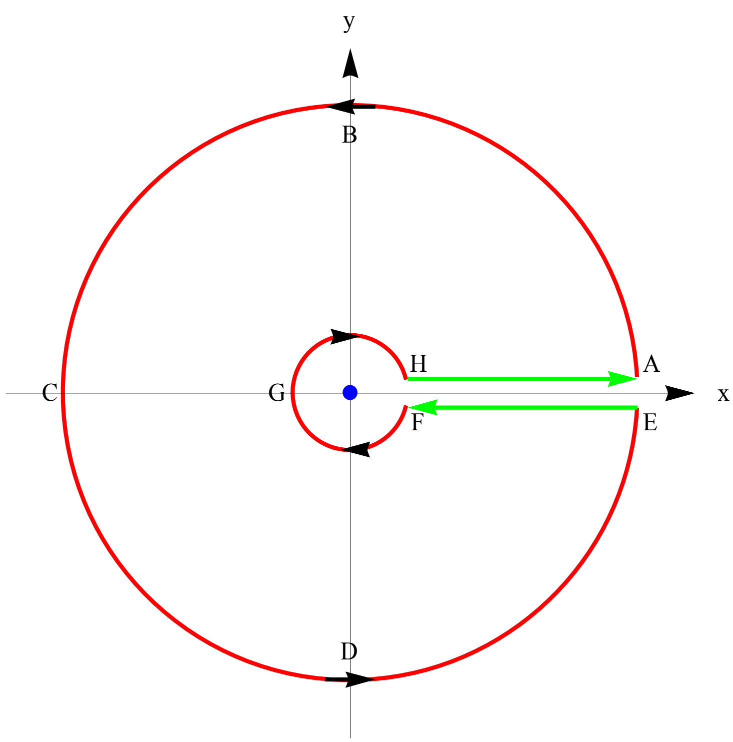
9.21.2 Application II

Evaluate

$$I=\int_0^\infty\frac{dx}{(x+a)^3\sqrt{x}} \qquad (a>0).$$

$$f(z)=\frac{1}{(z+a)^3\sqrt{z}}$$

$f(z)$ has a pole at $z = -a$. We note that $z = 0$ is a branch point and the positive x axis is a cut line.



$$\int_{HA}f(z)dz+\int_{ABCDE}f(z)dz+\int_{EF}f(z)dz+\int_{HGF}f(z)dz=2\pi i\,\mathrm{Res}(z=ae^{i\pi})$$

<u>HA</u>

$$z=xe^{i0}, \qquad z^{1/2}=x^{1/2}e^{ia0/2}=x^{1/2}$$

$$dz = dxe^{i0}$$

$$I_1 = \int_{.HA} f(z)dz = \int_0^\infty \frac{1}{(x+a)^3\sqrt{x}}dx$$

<u>EF</u>

$$z = xe^{2\pi i}, \qquad z^{1/2} = x^{1/2}e^{\pi i} = -\sqrt{x}$$

$$dz = dxe^{2\pi i} = dx$$

$$I_2 = \int_{.EF} f(z)dz = \int_\infty^0 \frac{-1}{(x+a)^3\sqrt{x}}dx = \int_0^\infty \frac{x^a}{(x+1)^2}dx = I_1$$

<u>ABCD (radius R)</u>

$$z = \mathrm{Re}^{i\theta}, \qquad z^{1/2} = R^{1/2}e^{i\theta/2}$$

$$dz = \mathrm{Re}^{i\theta}\, id\theta$$

$$\left|\int_{ABCDE} f(z)dz\right| = \left|\int_0^{2\pi} \frac{1}{(\mathrm{Re}^{i\theta}+a)^3 R^{1/2}e^{i\theta/2}}\mathrm{Re}^{i\theta}\, id\theta\right| \le \int_0^{2\pi} \frac{R^{1/2}}{R^3}d\theta = R^{-3+1/2}(2\pi) \to 0$$

as $R \to \infty$.

<u>HGF (radius ρ)</u>

$$z = \rho e^{i\theta}, \qquad z^{1/2} = \rho^{1/2}e^{i\theta/2}$$

$$dz = \rho e^{i\theta} id\theta$$

$$\int_{HGF} f(z)dz = \int_0^{2\pi} \frac{1}{(\rho e^{i\theta}+a)^3\rho^{1/2}e^{i\theta/2}}\rho e^{i\theta}id\theta = \frac{i\rho^{1/2}}{a^3}\int_0^{2\pi} e^{i\theta/2}d\theta \to 0$$

as $\rho \to 0$.

Then we have

$$I_1 + I_2 = 2I_1 = 2\pi i \operatorname{Re}s(z = ae^{i\pi}) = 2\pi i(-\frac{3i}{8a^{5/2}}) = \frac{3\pi}{4a^{5/2}}$$

or

$$I_1 = \frac{3\pi}{8a^{5/2}}$$

((**Mathematica**))

```
Residue[1/z^(1/2) 1/(z + a)^3, {z, a Exp[ⅈ π]}] // Simplify[#, a > 0] &
```

$$-\frac{3\,i}{8\,a^{5/2}}$$

```
Series[1/z^(1/2) 1/(z + a)^3, {z, a Exp[ⅈ π], 2}] // Simplify[#, a > 0] & // Normal
```

$$-\frac{5\,i}{16\,a^{7/2}} - \frac{i}{\sqrt{a}\,(a+z)^3} - \frac{i}{2\,a^{3/2}\,(a+z)^2} - \frac{3\,i}{8\,a^{5/2}\,(a+z)} - \frac{35\,i\,(a+z)}{128\,a^{9/2}} - \frac{63\,i\,(a+z)^2}{256\,a^{11/2}}$$

Note that the residue is equal to the coefficient of $1/(z+a)$.

9.22 Kronig-Kramers relation

We consider the motion of a particle (mass m and charge q) in the presence of electric field.

$$m(\ddot{x} + \rho_0 \dot{x} + \omega_0^{\ 2} x) = qE = F$$

$$x = \mathrm{Re}[X_\omega e^{-i\omega t}], \qquad F = \mathrm{Re}[F_\omega e^{-i\omega t}].$$

where ω is the angular frequency, x is the total displacement and F is the applied electric field.

$$m(-\omega^2 - \rho_0 i\omega + \omega_0^{\ 2})X_\omega = F_\omega = qE_\omega$$

$$qX_\omega = \frac{q^2 E_\omega}{m}\frac{1}{\omega_0^{\ 2} - \omega^2 - i\omega\rho_0} = \alpha(\omega)E_\omega$$

Here $\alpha(\omega)$ is called response function and is defined by

$$\alpha(\omega) = \frac{q^2}{m}\frac{1}{\omega_0^{\ 2} - \omega^2 - i\omega\rho_0}$$

We need not to assume this form of $\alpha(\omega)$, but we make use of three properties of the response function viewed as a function of the complex variables. Now we regard $\alpha(\omega)$ as a function of complex variable ω.

$$f_0 = q^2/m,$$

and

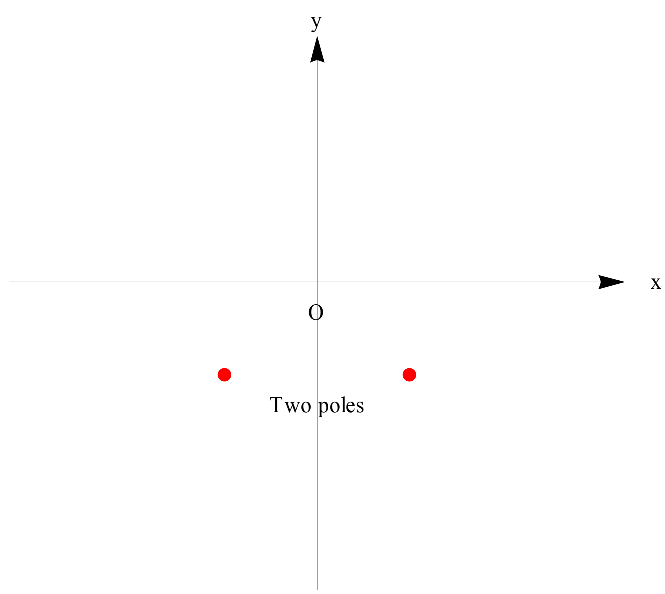
$$\alpha(\omega) = \frac{f_0}{\omega_0^{\ 2} - \omega^2 - i\omega\rho_0}$$

The pole of $\alpha(\omega)$:

$$\omega^2 + i\omega\rho_0 - \omega_0^{\ 2} = 0,$$

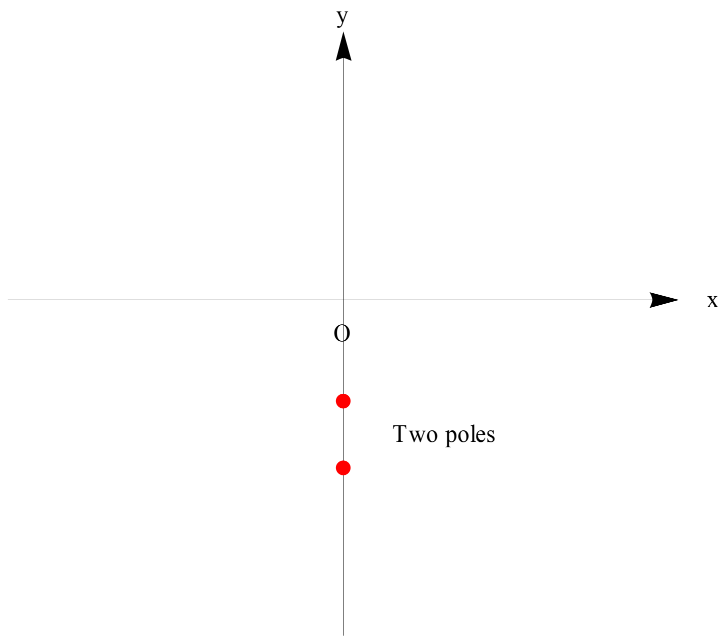
$$\omega = \frac{-i\rho_0 \pm \sqrt{4\omega_0^{\ 2} - \rho_0^{\ 2}}}{2}.$$

Case (1): $4\omega_0^{\ 2} - \rho_0^{\ 2} \geq 0$.



Case (2): $4\omega_0^{\ 2} - \rho_0^{\ 2} < 0$.

$$\omega = \frac{-i\rho_0 \pm i\sqrt{\rho_0^{\ 2} - 4\omega_0^{\ 2}}}{2}.$$



Thus we have the following features.

(1) The poles of $\alpha(\omega)$ are all below the real axis.
(2) The integral of $\alpha(\omega)/\omega$ vanishes when taken around an infinite semicircle in the upper half of the complex plane. It suffices that $\alpha(\omega)/\omega \to 0$ uniformly as $|\omega| \to \infty$.

$$\lim_{|\omega|\to\infty} \frac{\alpha(\omega)}{\omega} = 0.$$

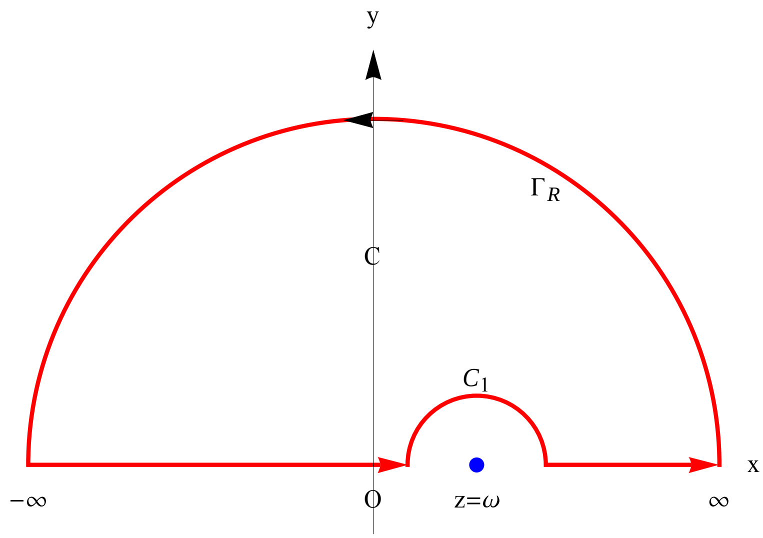
(3) $\alpha(\omega) = \alpha'(\omega) + i\alpha''(\omega)$,

$$\alpha(\omega) = \frac{f_0(\omega_0{}^2 - \omega^2 + i\omega\rho)}{(\omega_0{}^2 - \omega^2)^2 + \omega^2 \rho_0{}^2}$$

where $\alpha'(\omega)$ is even and $\alpha''(\omega)$ is odd with respect to real ω. Now we calculate

$$\oint_C \frac{\alpha(z)}{z-\omega} dz.$$

We note that Integrant has a simple pole at the real axis.



$$\oint_C \frac{\alpha(z)}{z-\omega} dz = \int_{\Gamma_R} \frac{\alpha(z)}{z-\omega} dz + P\int_{-\infty}^{\infty} \frac{\alpha(x)}{x-\omega} dx - i\pi \operatorname{Re} s(z=\omega) = 0$$

Since

$$\int_{\Gamma_R} \frac{\alpha(z)}{z-\omega} dz \text{,}$$

$$P\int_{-\infty}^{\infty} \frac{\alpha(x)}{x-\omega} dx = i\pi \operatorname{Re} s(z=\omega) = i\pi\alpha(\omega)\,,$$

or

$$\alpha(\omega) = \frac{1}{\pi i} P\int_{-\infty}^{\infty} \frac{\alpha(x)}{x-\omega} dx\,.$$

Now we have

$$\begin{aligned}\alpha(\omega) &= \alpha'(\omega) + i\alpha''(\omega) \\ &= \frac{1}{\pi i} P[\int_{-\infty}^{\infty} \frac{\alpha'(x) + i\alpha''(x)}{x-\omega} dx]\text{'}\end{aligned}$$

$$\alpha'(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\alpha''(x)}{x-\omega} dx$$
$$= \frac{1}{\pi} P[\int_{0}^{\infty} \frac{\alpha''(x)}{x-\omega} dx + \int_{-\infty}^{0} \frac{\alpha''(x)}{x-\omega} dx] .$$

Since , $\alpha''(-x) = -\alpha''(x)$ (odd function), we have

$$\alpha'(\omega) = \frac{1}{\pi} P[\int_{0}^{\infty} \frac{\alpha''(x)}{x-\omega} dx + \int_{0}^{\infty} \frac{\alpha''(x)}{x+\omega} dx]$$
$$= \frac{2}{\pi} P \int_{0}^{\infty} \frac{x\alpha''(x)}{x^2-\omega^2} dx .$$

Similarly

$$\alpha''(\omega) = \frac{1}{\pi} P[\int_{0}^{\infty} \frac{\alpha''(x)}{x-\omega} dx + \int_{0}^{\infty} \frac{\alpha''(x)}{x+\omega} dx]$$
$$= -\frac{2\omega}{\pi} P \int_{0}^{\infty} \frac{\alpha'(x)}{x^2-\omega^2} dx .$$

APPENDIX

A.1 Mathematica

Residue[expr, {*x*, *x*₀}];

to find the residue of expression when x equals x_0.

The residue is defined as the coefficient of `1/(z-z0)` in the Laurent expansion of *expr*.

NResidue[expr, {x, x0}];

to find numerically the residue of expression when x equals x_0.

Series[*f*, {*x*, *x*₀, *n*}];

to generate a power series expansion for f about the point $x=x_0$ to order $(x - x_0)^n$.

A.2 Principal part of an integral

We consider a real function f(x) that blows up at x = a. The principal part of the integral is defined as

$$P\int_{-\infty}^{\infty} f(x)dx = \lim_{\varepsilon\to 0}[\int_{-\infty}^{a-\varepsilon} f(x)dx + \int_{a+\varepsilon}^{\infty} f(x)dx]$$

((Example))

$$P\int_{-\infty}^{\infty}\frac{1}{x^3}dx = \lim_{\varepsilon\to 0}[\int_{-\infty}^{-\varepsilon}\frac{1}{x^3}dx + \int_{\varepsilon}^{\infty}\frac{1}{x^3}dx] = \lim_{\varepsilon\to 0}(-\frac{1}{2\varepsilon^2} + \frac{1}{2\varepsilon^2}) = 0$$