Chapter 10S Johnson-Nyquist noise

Masatsugu Sei Suzuki
Department of Physics, SUNY at Binghamton
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Johnson noise
Johnson-Nyquist theorem
Boltzmann constant
Parseval relation
Correlation function
Spectral density
Wiener-Khinchin (or Khintchine)
Flicker noise
Shot noise
Poisson distribution
Brownian motion
Fluctuation-dissipation theorem
Langevin function

John Bertrand "Bert" Johnson (October 2, 1887–November 27, 1970) was a Swedish-born American electrical engineer and physicist. He first explained in detail a fundamental source of random interference with information traveling on wires. In 1928, while at Bell Telephone Laboratories he published the journal paper "*Thermal Agitation of Electricity in Conductors*". In telecommunication or other systems, *thermal noise* (or *Johnson noise*) is the noise generated by thermal agitation of electrons in a conductor. Johnson's papers showed a statistical fluctuation of electric charge occur in all electrical conductors, producing random variation of potential between the conductor ends (such as in vacuum tube amplifiers and thermocouples). Thermal noise power, per hertz, is equal throughout the frequency spectrum. Johnson deduced that thermal noise is intrinsic to all resistors and is not a sign of poor design or manufacture, although resistors may also have excess noise.

http://en.wikipedia.org/wiki/John_B._Johnson

Harry Nyquist (February 7, 1889 – April 4, 1976) was an important contributor to information theory.

http://en.wikipedia.org/wiki/Harry_Nyquist

10S.1 Histrory

In 1926, experimental physicist John Johnson working in the physics division at Bell Labs was researching noise in electronic circuits. He discovered random fluctuations in the voltages across electrical resistors, whose power was proportional to temperature. Harry Nyquist, a theorist in that division, got interested in the phenomenon and developed an elegant explanation based on fundamental physics. hence this type of noise is called Johnson noise, Nyquist noise, or thermal noise.

10S.2 Spectral density

Suppose we measure a physical quantity $x(t)$ as a function of time t. We define the Fourier transform of $x(t)$ as

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} x(t)$$

Note that

$$X^*(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} x(t) = X(-\omega).$$

since $x(t)$ is real. The inverse Fourier transform is given by

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} X(\omega).$$

(a) Parseval relation.

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)x^*(t)dt &= \int_{-\infty}^{\infty} dt \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} X(\omega) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' e^{i\omega' t} X^*(\omega') \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega X(\omega) \int_{-\infty}^{\infty} d\omega' X^*(\omega') \int_{-\infty}^{\infty} dt e^{i(\omega'-\omega)t} \\
&= \frac{1}{2\pi} 2\pi \int_{-\infty}^{\infty} d\omega X(\omega) \int_{-\infty}^{\infty} d\omega' X^*(\omega') \delta(\omega-\omega') \\
&= \int_{-\infty}^{\infty} d\omega |X(\omega)|^2 \\
&= 2\int_{0}^{\infty} d\omega |X(\omega)|^2
\end{aligned}$$

since

$$|X(\omega)|^2 = X(\omega)X^*(\omega) = X(\omega)X(-\omega) = |X(-\omega)|^2.$$

(b) We use a truncated signal.

It is sometimes necessary, in order to avoid convergence problem, to approach the definition of the integral as a limiting process. Using the Parseval relation, we get

$$\langle x^2(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{T/2}^{T/2} x^2(t) dt = \lim_{T\to\infty} \frac{2}{T} \int_{0}^{\infty} d\omega |X(\omega)|^2.$$

We define the spectral density as

$$\lim_{T\to\infty}\frac{2}{T}\overline{\left|X(\omega)\right|^2} = G(\omega)\,, \tag{1}$$

which is the ensemble average of $\left|X(\omega)\right|^2$. Then we have

$$\overline{\left\langle x^2(t)\right\rangle} = \int_0^\infty d\omega G(\omega)\,. \tag{2}$$

((**Note**))
We assume that the average of process parameter over time and the average over the statistical ensemble are the same (ergodicity).

10S.3 Correlation function

The limiting process required to calculate $G(\omega)$ is sometimes awkward to use in practice. Here we show that correlation function can be closely related to the spectral density. The correlation function associated with a stationary random function of time t, $x(t)$, is defined by

$$\left\langle x(t)x(t+\tau)\right\rangle\,. \tag{3}$$

which means that Eq.(3) is independent of t. The Fourier transform of $x(t+\tau)$ is given by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dte^{i\omega t}x(t+\tau) = \frac{1}{\sqrt{2\pi}}e^{-i\omega\tau}\int_{-\infty}^{\infty} dte^{i\omega t'}x(t') = e^{-i\omega\tau}X(\omega)\,.$$

Correspondingly, the inverse Fourier transform is given by

$$x(t+\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t+\tau)}X(\omega)\,.$$

We need to calculate the product given by

$$x(t)x(t+\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\int_{-\infty}^{\infty} d\omega' e^{-i\omega(t+\tau)}e^{-i\omega'\tau}X(\omega)X(\omega')\,.$$

It is sometimes necessary, in order to avoid convergence problem, to approach the definition of Eq.(3) as a limiting process. The integral of this product is

$$
\begin{aligned}
\int_{-T/2}^{T/2} x(t)x(t+\tau)dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \int_{-T/2}^{T/2} dt e^{-i(\omega'+\omega)t} e^{-i\omega'\tau} X(\omega)X(\omega') \\
&= \frac{1}{2\pi} 2\pi \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' e^{-i\omega'\tau} \delta(\omega+\omega') X(\omega)X(\omega') \\
&= \int_{-\infty}^{\infty} d\omega e^{i\omega\tau} X(\omega)X(-\omega) \\
&= \int_{-\infty}^{\infty} d\omega e^{i\omega\tau} \left|X(\omega)\right|^2 \\
&= 2\int_{0}^{\infty} d\omega \cos(\omega\tau) \left|X(\omega)\right|^2
\end{aligned}
$$

Note that

$$
\int_{-T/2}^{T/2} dt e^{-i(\omega'+\omega)t} = 2\pi\delta(\omega'+\omega) .
$$

Then we have

$$
\begin{aligned}
\left\langle x(t)x(t+\tau)\right\rangle &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt \\
&= \lim_{T\to\infty} \frac{2}{T}\int_{0}^{\infty} d\omega \cos(\omega\tau)\left|X(\omega)\right|^2
\end{aligned}
$$

The correlation function $C(\tau)$ is obtained as the ensemble average of $\left\langle x(t)x(t+\tau)\right\rangle$,

$$
\begin{aligned}
C(\tau) &= \overline{\left\langle x(t)x(t+\tau)\right\rangle} \\
&= \lim_{T\to\infty} \frac{2}{T}\int_{0}^{\infty} d\omega \cos(\omega\tau)\overline{\left|X(\omega)\right|^2} \\
&= \int_{0}^{\infty} d\omega \cos(\omega\tau) G(\omega) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} d\omega e^{-i\omega\tau} G(\omega)
\end{aligned}
, \qquad (4)
$$

where we use the relation(Eq.(1)). The inverse Fourier transform of $G(\omega)$ is given by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{-i\omega\tau} G(\omega) = \frac{2C(\tau)}{\sqrt{2\pi}} .$$

The corresponding Fourier transform is

$$\begin{aligned} G(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \frac{2C(\tau)}{\sqrt{2\pi}} \\ &= \frac{2}{2\pi}\int_{-\infty}^{\infty} d\tau e^{i\omega\tau} C(\tau) \quad . \qquad (5) \\ &= \frac{4}{2\pi}\int_{0}^{\infty} d\tau \cos(\omega\tau) C(\tau) \end{aligned}$$

Equation (5) is the Wiener-Khinchin (WK) theorem. $G(\omega)$ and $C(\tau)$ are a Fourier transform pair;

$G(\omega)$:	spectral density
$C(\tau)$:	correlation function

10S.4 Properties of the correlation function

The correlation function $C(\tau) = \overline{\langle x(t)x(t+\tau\rangle}$ is an even function of τ.

$$C(-\tau) = \overline{\langle x(t)x(t-\tau\rangle} = \overline{\langle x(t-\tau)x(t)\rangle} = \overline{\langle x(t)x(t+\tau)\rangle} = C(\tau) .$$

We also have

$$\overline{\langle [x(t) \pm x(t+\tau)]^2 \rangle} \geq 0 ,$$

or

$$\overline{\langle x^2(t)\rangle} + \overline{\langle x^2(t+\tau)\rangle} \pm 2\overline{\langle x(t)x(t+\tau)\rangle} \geq 0 ,$$

or

$$C(\tau = 0) = \overline{\langle x^2(t)\rangle} \geq \left| \overline{\langle x(t)x(t+\tau)\rangle} = C(\tau) \right| .$$

10S.5 Johnson-Nyquist theorem

At any non-zero temperature we can think of the moving charges as a sort of *electron gas* trapped inside the resistor box. The electrons move about in a randomised way — similar to Brownian motion — bouncing and scattering off one another and the atoms. At any particular instant there may be more electrons near one end of the box than the other.

We consider a resistance R which has a length l, a cross section A. The relaxation time of the carriers (electrons) is τ_e. There are n electrons per unit volume (n is called the electron density). The total number of electrons is N;

$$N = nAl .$$

From the Ohm's law, the open circuit voltage is

$$V = IR = RJA = RA(ne\bar{u}),$$

where V is a voltage, I is a current, J is the current density, A is the conductor cross section area, e is the absolute value of the charge of electron (e>0), and $\bar{u}$ is the average velocity of electrons. Since

$$\bar{u} = \frac{1}{N}\sum_{i=1}^{N} u_i ,$$

we have

$$V = RA(\frac{N}{Al}e)\frac{1}{N}\sum_{i=1}^{N} u_i = \frac{Re}{l}\sum_{i=1}^{N} u_i = \sum_{i+1}^{N} V_i ,$$

where

$$V_i = \frac{Re}{l}u_i .$$

Suppose that the correlation function is expressed by

$$C(\tau) = \overline{\langle V_i(t)V_i(t+\tau\rangle} = \overline{V_i^2}e^{-\frac{\tau}{\tau_0}} = \left(\frac{Re}{l}\right)^2 \overline{u_i^2}e^{-\frac{\tau}{\tau_0}} .$$

The spectral density is given by

$$
\begin{aligned}
G(\omega) &= \frac{4}{2\pi}\int_0^\infty d\tau\cos(\omega\tau)C(\tau) \\
&= \frac{4}{2\pi}\left(\frac{Re}{l}\right)^2 \overline{u_i^{\,2}}\int_0^\infty d\tau\cos(\omega\tau)e^{-\frac{\tau}{\tau_0}} \\
&= \frac{4}{2\pi}\left(\frac{Re}{l}\right)^2 \overline{u_i^{\,2}}\frac{\tau_0}{1+(\omega\tau_0)^2}
\end{aligned}
$$

(Lorentz-type spectral density)

In general, the spectral density $G(\omega)$ has a Lorentz-type when the correlation function $C(\tau)$ is described by a exponential decay; exp($-\tau/t_0$). For the voltage signals, the spectral density $G(\omega)$ is customary to use units of [V^2 s] = [V^2 Hz^{-1}].

We note that

$$\frac{1}{2}m\overline{u_i^{\,2}} = \frac{1}{2}k_BT\ , \qquad \text{(equipartition theorem)}$$

where m is the mass of electron and k_B is the Boltzmann constant. In metal at room temperature, τ_e is very small; τ_e<10^{-12} s. For the DC and microwave regions, $2\pi f\tau_c << 1$. Then we get

$$G(\omega) \approx \frac{4}{2\pi}\left(\frac{\mathrm{Re}}{l}\right)^2 \frac{k_BT}{m}\tau_0\ , \qquad \text{(white noise)}$$

which is independent of ω. We get

$$
\begin{aligned}
\overline{\langle V_i^2\rangle} &= \int_0^\infty d\omega G(\omega) \approx G(\omega)\Delta\omega \\
&= 2\pi(\Delta f)G(\omega) \qquad , \\
&= 4(\Delta f)\left(\frac{Re}{l}\right)^2\frac{k_BT}{m}\tau_0
\end{aligned}
$$

since $\omega = 2\pi f$. We have

$$\overline{\langle V^2\rangle} = N\overline{\langle V_i^2\rangle} = 4k_BTR(\Delta f)(\frac{Re^2\tau_0 NA}{ml^2}) = 4k_BTR(\Delta f).$$

Here we note that the resistance is define by

$$R = \frac{1}{\sigma}\frac{l}{A} = \frac{l}{\frac{N}{Al}\frac{e^2\tau_0}{m}A} = \frac{ml^2}{NAe^2\tau_0},$$

and

$$\frac{Re^2\tau_0 NA}{ml^2} = \frac{ml^2}{NAe^2\tau_0}\frac{e^2\tau_0 NA}{ml^2} = 1.$$

Experimentally it is predicted that the plot of $\overline{\langle V^2 \rangle}$ vs R may exhibits a straight line. The slope will give some estimation for the value of the Boltzmann constant k_B.

10S.6 Evaluation of $\overline{\langle V^2 \rangle}$

The Boltzmann constant is given by

$k_B = 1.380650410 \text{ x } 10^{-23}$ J/K.

$$\sqrt{\overline{\langle V^2 \rangle}} = \sqrt{4k_B TR(\Delta f)},$$

where T is the temperature (K), R is the resistance, and Δf is the frequency range.

(i) $T = 293.15$ K, $\Delta f = 10^6$ Hz, $R = 1$ MΩ

$$\sqrt{\overline{\langle V^2 \rangle}} = 127.238\mu V.$$

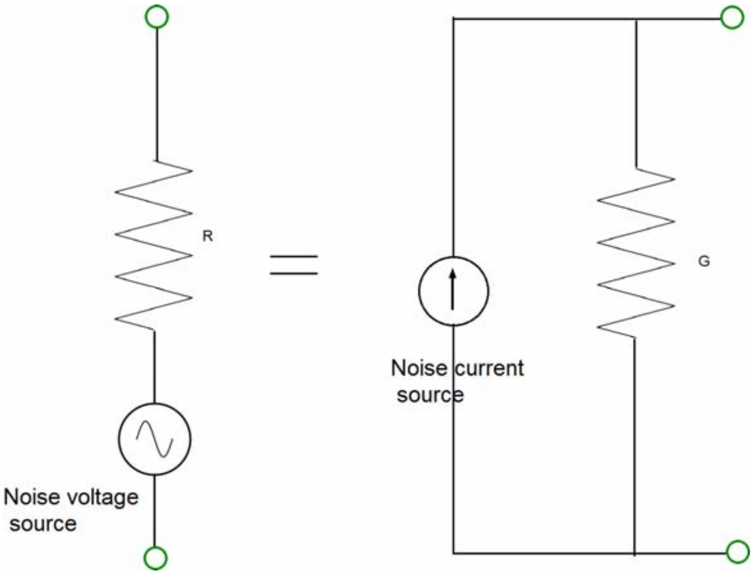
(ii) $T = 293.15$ K, $\Delta f = 1.9980 \text{ x } 10^4$ Hz, $R = 200\Omega$

$$\sqrt{\overline{\langle V^2 \rangle}} = 254.349 nV = 0.254\mu V.$$

(iii) $T = 293.15$ K, $\Delta f = 1.0 \text{ x } 10^4$ Hz, $R = 1000\Omega$

$$\sqrt{\overline{\langle V^2 \rangle}} = 402.362 nV = 0.402\mu V.$$

10S.7 Equivalent circuit (Thevinin theorem and Norton theorem)



Noise volatge source (the left of the above figure) and noise current source (the right of the above figure);

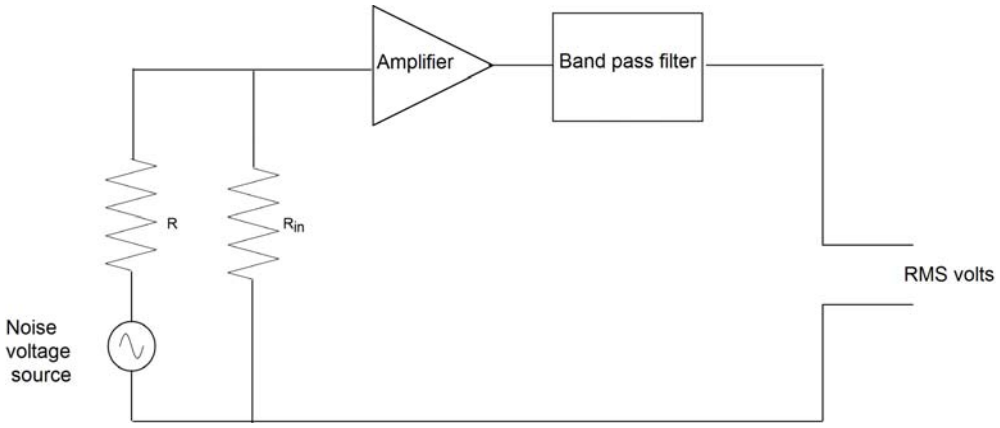
$$G = \frac{1}{R}$$

$$\overline{\langle V^2 \rangle} = 4k_B TR(\Delta f)$$

$$\overline{\langle I^2 \rangle} = \frac{\overline{\langle V^2 \rangle}}{R^2} = \frac{4k_B TR(\Delta f)}{R^2} = 4k_B TG(\Delta f)\,.$$

In Fig. we show the equivalent circuits for the noise voltage source and noise current source. These circuits are equivalent according to the Thevinin's theorem and Norton's theorem.

10S.8 Measurement of Johnson noise



The above figure shows an equivalent circuit for the Johnson noise produced by a resistor. ε_n is an invisible random voltage generator, connected in series with an ideal (noise free) resistor *R*. The voltage fluctuations are amplified and passed through a band-pass filter to a voltmeter. The filter only allows through frequencies in some range (the bandwidth),

$$\Delta f = f_{\max} - f_{\min}, \qquad \text{(Hz)}$$

R_{in} is the input resistance of the amplifier. The voltage due to the Johnson noise is

$$e_n = \sqrt{\langle V^2 \rangle} = \sqrt{4k_B TR(\Delta f)}.$$

From the Ohm's law, the current flowing in the input resistance of the amplifier is

$$i = \frac{e_n}{R + R_{in}}.$$

The voltage seen at the amplifier's input across R_{in} is

$$v = Ri = \frac{Re_n}{R + R_{in}}.$$

The noise power entering in the amplifier is

$$P = iv = \frac{Re_n^{\ 2}}{(R + R_{in})^2}.$$

The maximum noise power is obtained as

$$P_{\max} = \frac{e_n^{\ 2}}{4R} = \frac{\overline{\langle V^2 \rangle}}{4R} = k_B T(\Delta f),$$

when $R_{\mathrm{in}} = R$ (impedance matching). The unit of $P_{\max}$ is Watt (J/s)

10S.9 Shot noise

Shot noise is distinct from current fluctuations in thermal equilibrium, which happen without any applied voltage and without any average current flowing. These thermal equilibrium current fluctuations are known as Johnson-Nyquist noise or thermal noise. Shot noise is a Poisson process and the charge carriers which make up the current will follow a Poisson distribution.
We define the Fourier transform of the current,

$$Y(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i\omega t} I(t) .$$

Then we have

$$\langle I^2 \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{T/2}^{T/2} I^2(t) dt = \lim_{T\to\infty} \frac{2}{T} \int_0^{\infty} d\omega |Y(\omega)|^2 .$$

We define the spectral density as

$$\lim_{T\to\infty} \frac{2}{T} \overline{|Y(\omega)|^2} = S(\omega), \tag{1}$$

which is the ensemble average of $|Y(\omega)|^2$. Then we have

$$\overline{\langle I^2 \rangle} = \int_0^{\infty} d\omega S(\omega) .$$

The correlation function of current is defined by

$$\begin{aligned} C(\tau) &= \overline{\langle I(t)I(t+\tau)\rangle} \\ &= \lim_{T\to\infty}\frac{2}{T}\int_0^\infty d\omega\cos(\omega\tau)\overline{|Y(\omega)|^2} \\ &= \int_0^\infty d\omega\cos(\omega\tau)S(\omega) \\ &= \frac{1}{2}\int_{-\infty}^\infty d\omega e^{-i\omega\tau}S(\omega) \end{aligned}$$

or

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty d\omega e^{-i\omega\tau}S(\omega) = \frac{2C(\tau)}{\sqrt{2\pi}}.$$

The Wiener-Khinchin (or Khintchine) states that the noise spectrum is the Fourier transform of the correlation function (the spectral density)

$$\begin{aligned} S(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty d\tau e^{i\omega\tau}\frac{2C(\tau)}{\sqrt{2\pi}} \\ &= \frac{2}{2\pi}\int_{-\infty}^\infty d\tau e^{i\omega\tau}C(\tau) \\ &= \frac{4}{2\pi}\int_0^\infty d\tau\cos(\omega\tau)C(\tau) \end{aligned}$$

((Note))

$S(\omega)$:	spectral density
$C(\tau)$:	correlation function

The delta-function current pulses are given by

$$I(t) = e\sum_k \delta(t-t_k), \qquad I(t+\tau) = e\sum_k \delta(t-t_k+\tau).$$

Then the correlation function is obtained as

$$
\begin{aligned}
C(\tau) &= \lim_{T\to\infty}\frac{e^2}{T}\sum_k\sum_{k'}\int_{-T/2}^{T/2}\delta(t-t_k)\delta(t-t_{k'}+\tau)dt \\
&= \lim_{T\to\infty}\frac{e^2}{T}\sum_k\sum_{k'}\delta(t_k-t_{k'}+\tau) \\
&= e\bar{I}\delta(\tau)
\end{aligned}
$$

where

$$
\frac{N}{T}=\frac{\bar{I}}{e},
$$

and $\bar{I}$ is the DC current. When the summation indices are $k = k'$, it means that the arrival times are equal $t_k = t_k'$. Then we just have $\delta(\tau)$. If there are N values of t_k such that $-T/2<t_k<T/2$, these terms will contribute $N\,\delta(\tau)$ to the correlation function. For $t_k \neq t_{k'}$ the delta functions will occur at randomly distributed, nonzero values of τ. The contributions from these delta functions to the $C(\tau)$ will vanish.

Taking the Fourier transform, we find

$$
S(\omega)=\frac{2}{2\pi}e\bar{I}\int_{-\infty}^{\infty}d\tau e^{i\omega\tau}\delta(\tau)=\frac{1}{2\pi}2e\bar{I}
$$

The spectrum is uniform and extends to all frequencies. Such a spectrum is called white. Then we have

$$
\overline{\langle I^2\rangle}=\int_0^{\infty}d\omega S(\omega)=\int_0^{\infty}d\omega\frac{1}{2\pi}2e\bar{I}=\frac{2\pi\Delta f}{2\pi}2e\bar{I}=2e\bar{I}\Delta f\ .
$$

The current fluctuations have a standard deviation of

$$
\sqrt{\overline{\langle I^2\rangle}}=\sqrt{2e\bar{I}\Delta f}\ ,
$$

where e (>0) is the absolute value of the charge of electron, Δf is the bandwidth in hertz over which the noise is measured, and $\bar{I}$ is the average current through the device. For a current of 100 mA this gives a value of

$$\sqrt{\overline{\langle I^2 \rangle}} = 0.179nA$$

if the noise current is filtered with a filter having a bandwidth of Δf = 1 Hz. The plot of $\overline{\langle I^2 \rangle}$ vs $\bar{I}$ may exhibits a straight line. This slope will give some estimation for the value of charge e.

e = 1.602176487 x 10^{-19} C (from NIST Physics constant)

((**Note**))
Derivation of the formula by van der Ziel (Poisson distribution)

We define N as the number of carriers passing a point in a time T at a rate $n(t)$.

$$N = \int_0^T n(t)dt \,, \qquad \text{and} \qquad \overline{N} = \overline{n}T \,.$$

where $\overline{N}$ and $\overline{n}$ are ensemble averages and this result follows from that fact that time average equals ensemble average (the ergodicity). We assume that n follows the Poisson statistics. The average current is

$$\bar{I} = e\frac{\overline{N}}{T} = e\overline{n}$$

The spectral density S is given by

$$S = 2e^2\overline{n^2} = 2e^2\overline{n} = 2e^2\frac{\bar{I}}{e} = 2e\bar{I}$$

where $\overline{n^2} = \overline{n}$ for the Poisson distribution. Since S is independent of the frequency f (white noise), we have

$$\overline{\langle I^2 \rangle} = S\Delta f = 2e\bar{I}\Delta f \,.$$

10S.10` Flicker noise (1/*f* noise, pink noise): DC current related noise

Flicker noise, also known as **1/*f* noise**, is a signal or process with a frequency spectrum that falls off steadily into the higher frequencies, with a pink spectrum. It occurs in almost all electronic devices, and results from a variety of effects, though always related to a DC current.

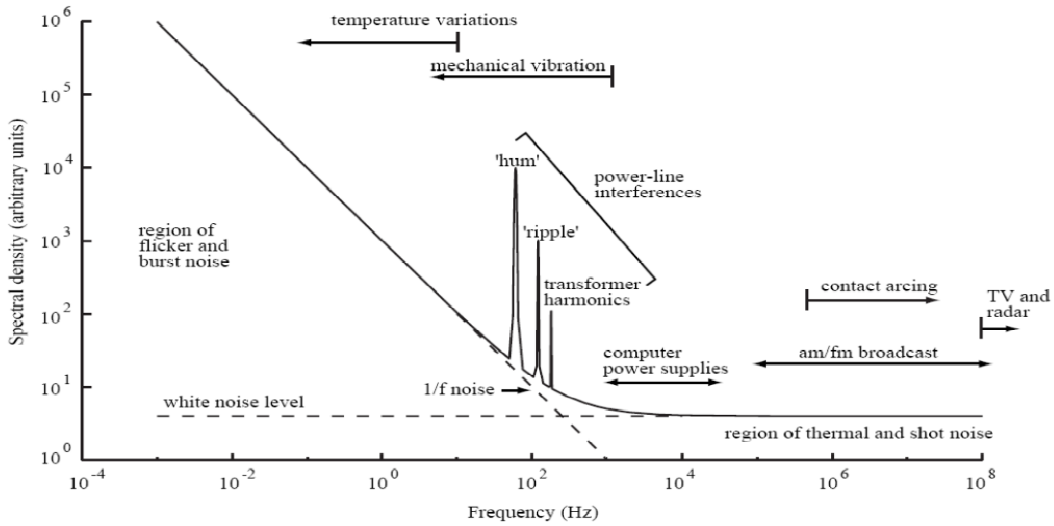


Fig. Spectral density vs frequency. $1/f$ noise in the low frequency range (the flicker noise). White noise in the higher frequency range.

10S.11 Brownian motion

Suppose that a force $\eta(t)$ is applied to a particle with a mass m along the x direction. According to the Newton's second law, The velocity $v(t) = \dot{x}(t)$ satisfies the following differential equation,

$$m\frac{dv(t)}{dt} + \varsigma v(t) = \eta(t)\,. \tag{1}$$

In a situation such that the particle moves randomly in the fluid at a constant temperature T, the parameters ζ and $\eta(t)$ no longer be regarded as independent ones. It is required that ζ and $\eta(t)$ are closely related to each other. This is the key point of the Brownian motion (Einstein's relation). Note that $\eta(t)$ is a force applied to the particle as a result of the collision of molecules of fluid with the particle. This force is considered to be random force (fluctuating force). We assume that the time average of $\eta(t)$ is zero and exhibits a white random force (Gaussian);

$$\langle \eta(t) \rangle = 0\,,$$ and

$$\langle \eta(t)\eta(t') \rangle = 2\varepsilon\delta(t-t')\,, \tag{2}$$

where ε is the magnitude of the random force. The first term of Eq.(1) is the effect of friction, where ζ is the coefficient of friction. We define the relaxation time τ as

$$\tau = \frac{m}{\varsigma}.$$

The solution of Eq.(1) is obtained as

$$v(t) = e^{-(t-t_0)/\tau} v(t_0) + \frac{1}{m}\int_{t_0}^{t} ds e^{-(t-s)/\tau} \eta(s).$$

When $v(t_0)$ is fixed,

$$\langle v(t) \rangle = e^{-(t-t_0)/\tau} v(t_0) + \frac{1}{m}\int_{t_0}^{t} ds e^{-(t-s)/\tau} \langle \eta(s) \rangle = e^{-(t-t_0)/\tau} v(t_0),$$

indicating that $\langle v(t) \rangle$ decays with time (relaxation time τ). For simplicity, we assume that $v(t_0)$ is finite. In the limit of $t_0 \to -\infty$, we have

$$v(t) = \frac{1}{m}\int_{-\infty}^{t} ds e^{-(t-s)/\tau} \eta(s).$$

Then we get

$$\langle v(t) \rangle = +\frac{1}{m}\int_{-\infty}^{t} ds e^{-(t-s)/\tau} \langle \eta(s) \rangle = 0,$$

since $\langle \eta(s) \rangle = 0$. The correlation function is given by

$$\begin{aligned}
\langle v(t_1) v(t_2) \rangle &= \frac{1}{m^2} < \int_{-\infty}^{t_1} e^{-\frac{t_1 - s_1}{\tau}} \eta(s_1) ds_1 \int_{-\infty}^{t_2} e^{-\frac{t_2 - s_2}{\tau}} \eta(s_2) ds_2 > \\
&= \frac{1}{m^2} \int_{-\infty}^{t_1} e^{-\frac{t_1 - s_1}{\tau}} ds_1 \int_{-\infty}^{t_2} e^{-\frac{t_2 - s_2}{\tau}} ds_2 \langle \eta(s_1) \eta(s_2) \rangle \\
&= \frac{2\varepsilon}{m^2} \int_{-\infty}^{t_1} e^{-\frac{t_1 - s_1}{\tau}} ds_1 \int_{-\infty}^{t_2} e^{-\frac{t_2 - s_2}{\tau}} ds_2 \delta(s_1 - s_2) \\
&= \frac{2\varepsilon}{m^2} \int_{0}^{\infty} e^{-\frac{u_1}{\tau}} du_1 \int_{0}^{\infty} e^{-\frac{u_2}{\tau}} du_2 \delta(t_1 - t_2 - u_1 + u_2)
\end{aligned}$$

where t_1 - s_1 = u_1 and t_2 - s_2 = u_2. Using the Mathematica, we get

$$\langle v(t_1)v(t_2)\rangle = \frac{\varepsilon\tau}{m^2} e^{-\frac{|t_1-t_2|}{\tau}} . \tag{3}$$

((**Mathematica**))

```
eq1 =
  Integrate[
    Integrate[Exp[(-u1 - u2)/τ]
      DiracDelta[t1 - t2 - u1 + u2],
     {u1, 0, ∞}], {u2, 0, ∞}] //
   Simplify[#, t1 - t2 ∈ Reals] & ;

Simplify[eq1, t1 > t2]
```

$\frac{1}{2} e^{\frac{-t1+t2}{\tau}} \tau$

```
Simplify[eq1, t1 < t2]
```

$\frac{1}{2} e^{\frac{t1-t2}{\tau}} \tau$

When $t = t'$, we have

$$\langle v^2(t)\rangle = \frac{\varepsilon\tau}{m^2} . \tag{4}$$

Using the equipartition law in the classical limit,

$$\frac{1}{2} m\langle v^2(t)\rangle = \frac{1}{2} k_B T , \tag{5}$$

we get the relation

$$\langle v^2(t)\rangle = \frac{k_B T}{m} = \frac{\varepsilon\tau}{m^2} , \tag{6}$$

or

$$\varepsilon = \frac{mk_B T}{\tau}. \qquad (7)$$

This is the relation of dissipation-fluctuation. The parameter ε is the magnitude of the fluctuating force and $\varsigma = m/\tau$ is the co-efficient of friction (dissipation). This relation is first derived by Einstein (1905).

10S.12 Diffusion constant *D*

The velocity correlation function is rewritten as

$$\langle v(t_1)v(t_2)\rangle = \frac{k_B T}{m} e^{-\frac{|t_1-t_2|}{\tau}}.$$

The displacement correlation function can be obtained as follows.

$$\begin{aligned}
\langle [x(t)-x(0)]^2\rangle &= \int_0^t dt_1 \int_0^t dt_2 \langle v(t_1)v(t_2)\rangle \\
&= \frac{k_B T}{m}\int_0^t dt_1 \int_0^t dt_2 e^{-\frac{|t_1-t_2|}{\tau}} \\
&= \frac{2k_B T}{m}\{\tau t - \tau^2(1-e^{-t/\tau})\}
\end{aligned}$$

When $t>>\tau$, we have

$$\langle (\Delta x)^2\rangle = \langle [x(t)-x(0)]^2\rangle = \frac{2k_B T}{m}\tau t = \frac{2k_B T}{m}\frac{m}{\varsigma}t = \frac{2k_B T}{\varsigma}t. \qquad (8)$$

We define the Diffusion constant *D* as

$$D = \lim_{t\to\infty}\frac{\langle [x(t)-x(0)]^2\rangle}{2t} = \frac{k_B T}{\varsigma}. \qquad (9)$$

((M athematica))

```
Clear["Global`*"];

f1 = Exp[-(t1 - t2)/τ] UnitStep[t1 - t2] +
   Exp[(t1 - t2)/τ] UnitStep[t2 - t1];

eq1 =
 Integrate[Integrate[f1, {t1, 0, t}],
    {t2, 0, t}] //
   Simplify[#, {t > 0, τ > 0}] &

2 τ (t + (-1 + e^(-t/τ)) τ)
```

10S.13 Langevin function

We consider the case that the finite force F is further applied to the particle. We set up the Lagrange equation

$$m\frac{dv(t)}{dt}+\varsigma v(t)=\eta(t)+F\,. \tag{10}$$

Using the assumaption that $\langle\eta(t)\rangle=0$, we get

$$m\frac{d\langle v(t)\rangle}{dt}+\varsigma\langle v(t)\rangle=F\,. \tag{11}$$

The solution of this equation is

$$\begin{aligned}\langle v(t)\rangle&=\frac{F}{\varsigma}[1-e^{-\frac{\varsigma}{m}(t-t_0)}]+\langle v(t_0)\rangle e^{-\frac{\varsigma}{m}(t-t_0)}\\&=\frac{F\tau}{m}(1-e^{-\frac{t-t_0}{\tau}})+\langle v(0)\rangle e^{-\frac{t-t_0}{\tau}}\end{aligned}$$

In the limit of $t\to\infty$ (in thermal equilibrium), $\langle v(t)\rangle$ becomes constant (steady state),

$$\langle v(t)\rangle=\frac{F\tau}{m}=\mu F\,, \tag{12}$$

where μ is the mobility and is defined by

$$\mu = \frac{\tau}{m} . \tag{13}$$

There is a relation between D and μ as

$$D = \frac{k_B T}{m}\tau = \mu k_B T \, , \qquad \text{(Einstein's relation)}, \tag{14}$$

10S.14 Fluctuation-dissipation (FD) theorem

(i) First-type FD theorem

The velocity correlation function is given by

$$\langle v(t)v(0)\rangle = \frac{k_B T}{m} e^{-\frac{|t|}{\tau}} .$$

Taking the integral over time, we have

$$\int_0^\infty \langle v(t)v(0)\rangle dt = \frac{k_B T}{m}\int_0^\infty e^{-\frac{t|}{\tau}} dt = \frac{k_B T\tau}{m} = \mu k_B T \, . \tag{15}$$

Then the mobility μ can be rewritten as

$$\mu = \frac{1}{k_B T}\int_0^\infty \langle v(t)v(0)\rangle dt . \qquad \text{(Type-1 FD theorem)} \tag{16}$$

The transport co-efficient (mobility, conductivity) can be described by the time correlation of the velocity (current density).

(ii) Second-type FD theorem

From the relation

$$\langle \eta(t)\eta(0)\rangle = 2\varepsilon\delta(t) ,$$

We have

$$\int_{-\infty}^{\infty} \langle \eta(t)\eta(0)\rangle dt = \int_{-\infty}^{\infty} 2\varepsilon\delta(t) dt = 2\varepsilon = 2\varsigma k_B T \, ,$$

or

$$\int_{-\infty}^{\infty}\langle\eta(t)\eta(0)\rangle dt = 2\int_0^{\infty}\langle\eta(t)\eta(0)\rangle dt = 2\varepsilon = 2\varsigma k_B T\ ,$$

or

$$\varsigma = \frac{1}{k_B T}\int_{-\infty}^{\infty}\langle\eta(t)\eta(0)\rangle dt\ . \tag{17}$$

The co-efficient of the friction can be described by the time correlation of the fluctuating force.

10S.15 Langevin equation for electrical conductivity

We consider the motion of the i-th particle with mass m and charge e in the presence of fluctuating electric field $\varepsilon_i(t)$;

$$m[\frac{d}{dt}v_i(t) + \frac{1}{\tau}v_i(t)] = e\varepsilon_i(t) + eE$$

We define the current density as

$$J(t) = \sum_{i=1}^{n} ev_i(t)$$

From the above equation, we have

$$m[\frac{d}{dt}\sum_{i=1}^{n} v_i(t) + \frac{1}{\tau}\sum_{i=1}^{n} v_i(t)] = e\sum_{i=1}^{n}\varepsilon_i(t) + neE\ ,$$

or

$$m[\frac{d}{dt}J(t) + \frac{1}{\tau}J(t)] = ne^2\varepsilon(t) + ne^2E\ ,$$

where n is the particle density and the fluctuating electric field is given by

$$\varepsilon(t) = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i(t)$$

which is a average fluctuating electric field applied to the *n* charged particles.

$$\langle \varepsilon(t) \rangle = 0 .$$

and

$$q^2 \langle \varepsilon_i(t)\varepsilon_j(t') \rangle = \delta_{ij}\frac{2mk_BT}{\tau}\delta(t-t') .$$

Here we assume the independence of $\{\varepsilon_i(t)\}$. In the steady state, we have

$$\langle J(t) \rangle = \frac{ne^2\tau}{m}E = \sigma E .$$

The electrical resistance *R* is

$$R = \rho\frac{l}{A} = \frac{1}{\sigma}\frac{l}{A} = \frac{ml}{\tau nq^2 A}$$

The fluctuating electric field ε(t) can be described by

$$\langle \varepsilon(t)\varepsilon(t') \rangle = \frac{2mk_BT}{\tau nq^2}\delta(t-t') = 2mk_BT\frac{RA}{ml}\delta(t-t') = 2k_BT\frac{RA}{l}\delta(t-t')$$

Using the relatrion $V(t) = l\,\varepsilon(t)$,

$$\langle V(t)V(t') \rangle = 2k_BTRAl\delta(t-t') .$$

or

$$\langle V(t)V(0) \rangle = 2k_BTRAl\delta(t)$$

or

$$R = \frac{1}{lA}\frac{1}{2k_BT}\int_{-\infty}^{\infty}\langle V(t)V(0)\rangle dt = \frac{1}{lA}\frac{1}{k_BT}\int_0^{\infty}\langle V(t)V(0)\rangle dt$$

10S.16 Fluctuation-dissipation theorem for Conductivity

Suppose that $E = 0$. Then we have

$$\frac{d}{dt}J(t) + \frac{1}{\tau}J(t) = \frac{ne^2}{m}\varepsilon(t)$$

The solution for this equation is given by

$$J(t) = \frac{ne^2}{m}\int_{t_0}^{t} e^{-\frac{(t-s)}{\tau}}\varepsilon(s)ds + J(t_0)e^{-\frac{t-t_0}{\tau}}$$

For simplicity, we assume that $J(t_0)$ is finite. In the limit of $t_0 \rightarrow -\infty$, we have

$$J(t) = \frac{ne^2}{m}\int_{-\infty}^{t} ds e^{-\frac{(t-s)}{\tau}}\varepsilon(s) .$$

Then we get

$$\langle J(t)\rangle = \frac{ne^2}{m}\int_{-\infty}^{t} ds e^{-\frac{(t-s)}{\tau}}\langle\varepsilon(s)\rangle = 0 ,$$

since $\langle\varepsilon(s)\rangle = 0$. The correlation function is given by

$$\begin{aligned}
\langle J(t_1)J(t_2)\rangle &= \frac{n^2e^4}{m^2} < \int_{-\infty}^{t_1} e^{-\frac{t_1-s_1}{\tau}}\varepsilon(s_1)ds_1 \int_{-\infty}^{t_2} e^{-\frac{t_2-s_2}{\tau}}\varepsilon(s_2)ds_2 > \\
&= \frac{n^2e^4}{m^2}\int_{-\infty}^{t_1} e^{-\frac{t_1-s_1}{\tau}}ds_1 \int_{-\infty}^{t_2} e^{-\frac{t_2-s_2}{\tau}}ds_2\langle\varepsilon(s_1)\varepsilon(s_2)\rangle \\
&= \frac{n^2e^4}{m^2}\frac{2mk_BT}{\tau me^2}\int_{-\infty}^{t_1} e^{-\frac{t_1-s_1}{\tau}}ds_1 \int_{-\infty}^{t_2} e^{-\frac{t_2-s_2}{\tau}}ds_2\delta(s_1-s_2) \\
&= \frac{2ne^2k_BT}{\tau m}\int_0^{\infty} e^{-\frac{u_1}{\tau}}du_1\int_0^{\infty} e^{-\frac{u_2}{\tau}}du_2\delta(t_1-t_2-u_1+u_2)
\end{aligned}$$

where $t_1 - s_1 = u_1$ and $t_2 - s_2 = u_2$. Using the Mathematica, we get

$$\left\langle J(t_1)J(t_2)\right\rangle = \frac{2ne^2k_BT}{\tau m}\frac{\tau}{2}e^{-\frac{|t_1-t_2|}{\tau}} = \frac{ne^2k_BT}{m}e^{-\frac{|t_1-t_2|}{\tau}}.$$

Then we get

$$\int_0^\infty \left\langle J(t)J(0)\right\rangle dt = \frac{ne^2k_BT}{m}\int_0^\infty e^{-\frac{t}{\tau}}dt = \frac{ne^2\tau k_BT}{m} = k_BT\sigma .$$

The conductivity is expressed by the time correlation of the current dnsity;

$$\sigma = \frac{1}{k_BT}\int_0^\infty \left\langle J(t)J(0)\right\rangle dt .$$

REFERENCES

A.C. Melissinos and J. Napolitano, Experiments in Modern Physics, 2nd edition (Academic Press, Amsterdam, 2003).
J.B. Johnson, Phys. Rev. **32**, 97 (1928).
H. Nyquist, Phys. Rev. **32**, 110 (1928).
C. Kittel, Elementary Statistical Physics (John Wiley & Sons, New York, 1958).
C. Kittel and H. Kromer, 2nd edition (W.H. Freeman and Company, New York, 1980).
H.S. Robertson, Statistical Thermodynamics (PTR Prentice Hall, Englewood Cliffs, NJ, 1993).
Toshimitsu Musha, World of Fluctuation (Kozansha Blue Backs, Tokyo, Japan) [in Japanese].
Masuo Suzuki, Statistical Mechanics 2nd edition (Iwanami, Tokyo, 1996) [In Japanese].
Kazuo Kitahara, Nonequilibrium Statistical Physics, 2nd edition (Iwanami, Tokyo, 1998) [in japanese].

APPENDIX

A.1 Poisson distribution

The Poisson ditribution function is given by

$$P_n = \frac{\mu^n e^{-\mu}}{n!},$$

with the mean

$$\langle n \rangle = \sum_{n=0}^{\infty} n P_n = \mu ,$$

and the variance

$$\langle n^2 \rangle - \langle n \rangle^2 = \sum_{n=0}^{\infty} (n^2 - \mu^2) P_n = \mu ,$$

since

$$\langle n^2 \rangle = \sum_{n=0}^{\infty} n^2 P_n = \mu(\mu + 1) .$$

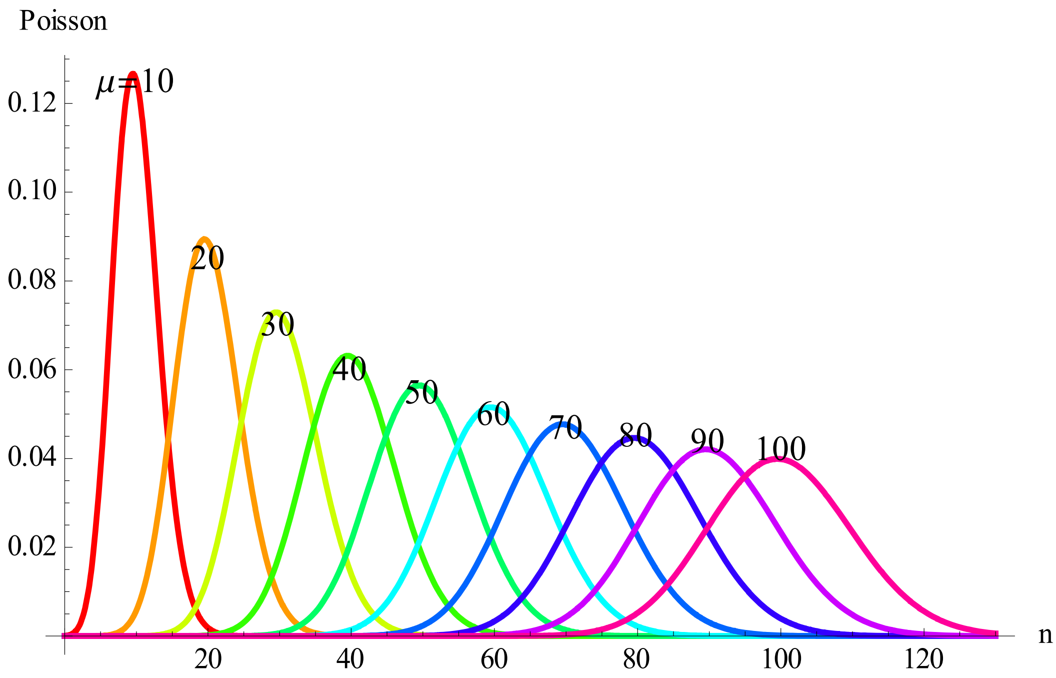


Fig. The Poisson distribution function with μ being changed as a parameter.

((**Mathematica**))

```
PDF[PoissonDistribution[μ], k]
```

$$\frac{e^{-\mu}\,\mu^{k}}{k\,!}$$

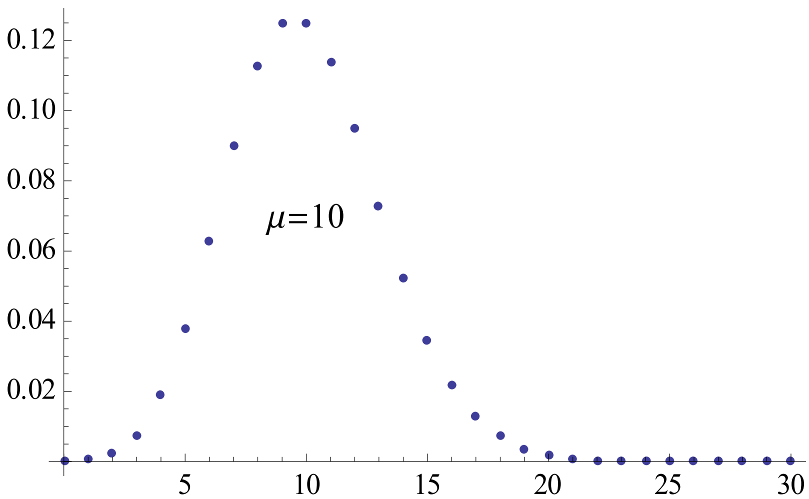
```
Mean[PoissonDistribution[μ]]
```

μ

```
Variance[PoissonDistribution[μ]]
```

μ

```
f1 =
 ListPlot[
  Table[
   {k, PDF[PoissonDistribution[10], k]},
   {k, 0, 30}]];
f2 =
 Graphics[
  {Text[Style["μ=10", Black, 12],
    {10, 0.07}]}];
Show[f1, f2]
```

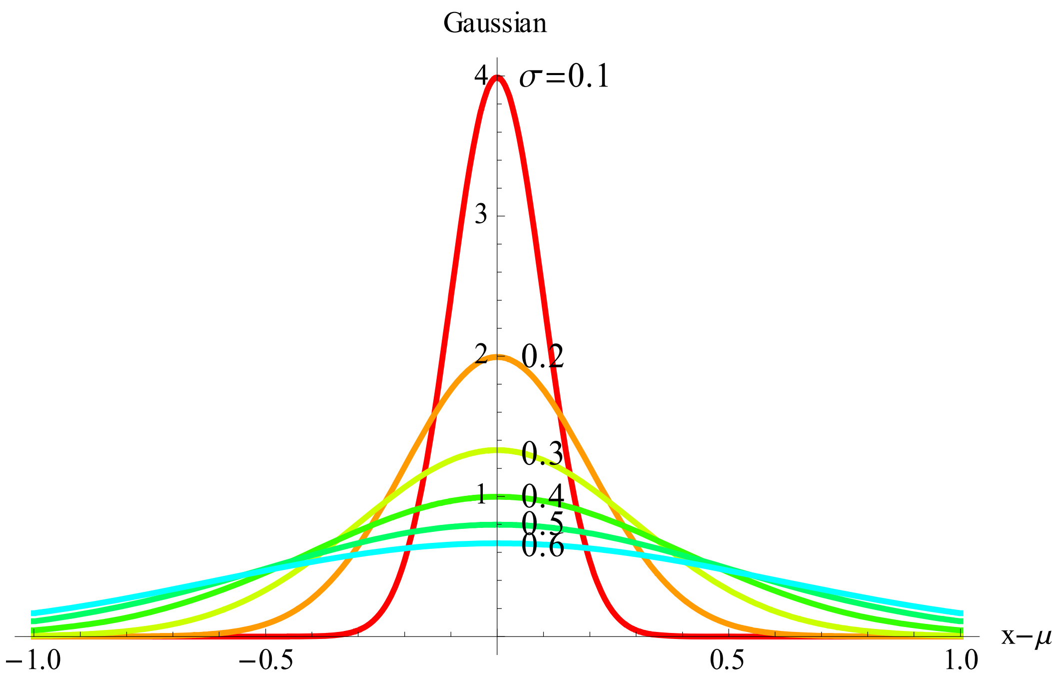


A.2 Gaussian distribution (normal distribution)

$$f(x;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}\exp[-\frac{(x-\mu)^2}{2\sigma^2}].$$

$$\langle x\rangle=\int_{-\infty}^{\infty}xf(x,\mu,\sigma)dx=\mu.$$

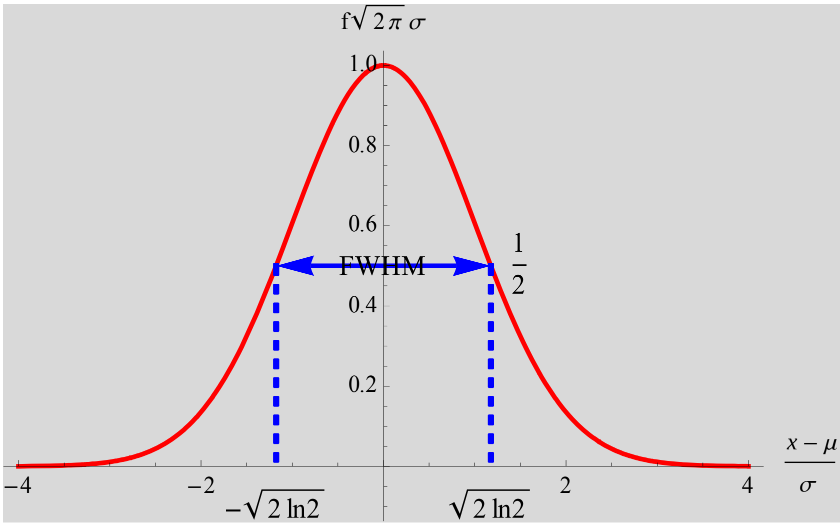
$$\langle x^2\rangle-\langle x\rangle^2=\int_{-\infty}^{\infty}(x^2-\mu^2)f(x,\mu,\sigma)dx=\sigma^2.$$



μ: mean

σ: standard deviation

$$\sqrt{2\pi}\sigma f(x;\mu,\sigma)=\exp[-\frac{(x-\mu)^2}{2\sigma^2}]=\exp[-\frac{1}{2}(\frac{x-\mu}{\sigma})^2].$$



The full width at half maximum (FWHM);

$$\Gamma = 2.35482\sigma$$

$$\sqrt{2\ln 2} = 1.17741$$

((**Mathematica**))

```
PDF[NormalDistribution[μ, σ], x]
```

$$\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

```
Mean[NormalDistribution[μ, σ]]
```

μ

```
Variance[NormalDistribution[μ, σ]]
```

σ^2

```
f1 = Plot[PDF[NormalDistribution[0, 1], x],
   {x, -4, 4}];
f2 =
 Graphics[
  {Text[Style["μ=0, σ=1", Black, 12],
    {1, 0.3}]}];
Show[f1, f2]
```