

Hermitian operator
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((Definition))

Hermite conjugate (definition): or Hermitian adjoint

$$\langle \psi | \hat{A} | \varphi \rangle^* = \langle \varphi | \hat{A}^+ | \psi \rangle.$$

1. Complex number

What is the Hermitian adjoint of the complex number?

$$\langle \psi | c | \varphi \rangle^* = \langle \varphi | c^+ | \psi \rangle,$$

or

$$\begin{aligned} \langle \psi | c | \varphi \rangle^* &= (c \langle \psi | \varphi \rangle)^* = c^* \langle \psi | \varphi \rangle^* \\ &= c^* \langle \varphi | \psi \rangle = \langle \varphi | c^* | \psi \rangle \end{aligned}$$

where c is the complex number. Then we have

$$\langle \varphi | c^+ | \psi \rangle = \langle \varphi | c^* | \psi \rangle,$$

or

$$c^+ = c^*.$$

When $c = i$ (pure imaginary),

$$i^+ = i^* = -i.$$

In the quantum mechanics, the expectation value is real, i.e.,

$$\langle \psi | \hat{A} | \psi \rangle^* = \langle \psi | \hat{A} | \psi \rangle. \quad (1)$$

In this case \hat{A} is called a Hermitian operator. From the definition, we have

$$\langle \psi | \hat{A} | \psi \rangle^* = \langle \psi | \hat{A}^+ | \psi \rangle. \quad (2)$$

Hermitian operator satisfies the following condition:

$$\hat{A}^+ = \hat{A}.$$

2. Dirac notation (bra vector)

$$|\alpha\rangle = \hat{A}|\varphi\rangle \rightarrow \langle\alpha| = \langle\varphi|\hat{A}^+.$$

((Proof))

$$\langle\psi|\hat{A}|\varphi\rangle^* = \langle\varphi|\hat{A}^+|\psi\rangle.$$

Then we have

$$\langle\psi|\hat{A}|\varphi\rangle^* = \langle\psi|\alpha\rangle^* = \langle\alpha|\psi\rangle = \langle\varphi|\hat{A}^+|\psi\rangle.$$

Then

$$\langle\alpha|\psi\rangle = \langle\varphi|\hat{A}^+|\psi\rangle,$$

or

$$\langle\alpha| = \langle\varphi|\hat{A}^+.$$

3. The momentum operator is a Hermitian operator

$$\hat{p}^+ = \hat{p}.$$

((Proof))

$$\langle\psi|\hat{p}|\varphi\rangle^* = \langle\varphi|\hat{p}^+|\psi\rangle,$$

$$\begin{aligned} \langle\psi|\hat{p}|\varphi\rangle &= \int \langle\psi|x\rangle \langle x|\hat{p}|\varphi\rangle dx = \int \langle\psi|x\rangle \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|\varphi\rangle dx \\ &= \int \psi^*(x) \frac{\hbar}{i} \frac{\partial}{\partial x} \varphi(x) dx = - \int \frac{\hbar}{i} \frac{\partial}{\partial x} \psi^*(x) \varphi(x) dx \end{aligned}$$

Here we use the formula without proof

$$\langle x|\hat{p}|\varphi\rangle = \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x|\varphi\rangle,$$

or

$$\begin{aligned}\langle \psi | \hat{p} | \varphi \rangle^* &= \int \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x) \varphi^*(x) dx = \int \varphi^*(x) \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x) dx \\ &= \int \langle \varphi | x \rangle \langle x | \hat{p} | \psi \rangle dx = \langle \varphi | \hat{p} | \psi \rangle\end{aligned}$$

Thus we have

$$\langle \varphi | \hat{p}^+ | \psi \rangle = \langle \varphi | \hat{p} | \psi \rangle,$$

or

$$\hat{p}^+ = \hat{p}.$$

4. Formula

$$(\hat{A}\hat{B})^+ = \hat{B}^+ \hat{A}^+.$$

((Proof))

$$\langle \psi | (\hat{A}\hat{B})^+ | \varphi \rangle = \langle \varphi | (\hat{A}\hat{B}) | \psi \rangle^* = \langle \alpha | \beta \rangle^* = \langle \beta | \alpha \rangle = \langle \psi | \hat{B}^+ \hat{A}^+ | \varphi \rangle,$$

where

$$\langle \alpha | = \langle \varphi | (\hat{A} \quad \text{and} \quad | \beta \rangle = \hat{B} | \psi \rangle,$$

or

$$| \alpha \rangle = \hat{A}^+ | \varphi \rangle \quad \text{and} \quad \langle \beta | = \langle \psi | \hat{B}^+.$$

((Comments))

1. $(\hat{A}^+)^+ = \hat{A}.$
2. $(c\hat{A})^+ = c^* \hat{A}^+.$
3. $(\hat{A} + \hat{B})^+ = \hat{A}^+ + \hat{B}^+.$
4. If $\hat{A}^+ = \hat{A}$ and $\hat{B}^+ = \hat{B}$,

$$(\hat{A}\hat{B})^+ = \hat{B}^+ \hat{A}^+ = \hat{B}\hat{A}.$$

So $\hat{A}\hat{B}$ is not Hermitian.

5.

$$(\hat{A}\hat{B} + \hat{B}\hat{A})^+ = \hat{B}\hat{A} + \hat{A}\hat{B},$$

is Hermitian.

6.

$$(\hat{A}^2)^+ = (\hat{A}\hat{A})^+ = \hat{A}^+\hat{A}^+ = \hat{A}\hat{A} = \hat{A}^2,$$

is Hermitian.

7. \hat{p} is Hermitian.

$$\hat{p} = -i\hbar\hat{D} \quad \text{with } \hat{D}^+ = -\hat{D}.$$

$$\hat{p}^+ = (-i\hbar\hat{D})^+ = -i^+\hbar\hat{D}^+ = -i\hbar\hat{D} = \hat{p}.$$

$$|\alpha\rangle \Rightarrow -\hat{A}|\alpha\rangle.$$

Hermitian conjugate

$$\langle\alpha| \Rightarrow \langle\alpha|\hat{A}^+.$$

Outer product of $|\beta\rangle$ and $\langle\alpha|$ is an operator

$$\hat{A} = |\beta\rangle\langle\alpha|.$$

$$\hat{A}|\chi\rangle = (|\beta\rangle\langle\alpha|)|\chi\rangle = \langle\alpha|\chi\rangle|\beta\rangle.$$

$$\hat{A}^+ = |\alpha\rangle\langle\beta|.$$

((Proof))

$$\begin{aligned}
\langle \gamma | \hat{A}^+ | \delta \rangle &= \langle \delta | \hat{A} | \gamma \rangle^* \\
&= (\langle \delta | \beta \rangle \langle \alpha | \gamma \rangle)^* \\
&= \langle \delta | \beta \rangle^* \langle \alpha | \gamma \rangle^* \\
&= \langle \beta | \delta \rangle \langle \gamma | \alpha \rangle \\
&= \langle \gamma | \alpha \rangle \langle \beta | \delta \rangle
\end{aligned}$$

Thus we have

$$\hat{A}^+ = |\alpha\rangle\langle\beta|.$$

5. Properties of Hermitian

Eigenvalue problem:

$$\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle,$$

where

$$\begin{aligned}
\hat{A} \text{ is the Hermitian: } & \hat{A}^+ = \hat{A} \\
|\varphi_n\rangle \text{ is the eigenket} & \\
a_n \text{ is the eigenvalue} &
\end{aligned}$$

Since \hat{A} is the Hermitian,

$$\langle \varphi_m | \hat{A} | \varphi_n \rangle^* = \langle \varphi_n | \hat{A}^+ | \varphi_m \rangle = \langle \varphi_n | \hat{A} | \varphi_m \rangle. \quad (1)$$

The matrix element

$$A_{nm} = \langle \varphi_n | \hat{A} | \varphi_m \rangle,$$

$$A_{mn}^* = A_{nm}.$$

The matrix element of \hat{A}^+ is the complex conjugate of the matrix element for the transpose of the matrix \hat{A} .

$n \times n$ matrix elements

$$\begin{array}{cccccccc}
A_{11} & A_{12} & A_{13} & \cdot & \cdot & \cdot & \cdot & A_{1n} \\
A_{12}^* & A_{22} & A_{23} & \cdot & \cdot & \cdot & \cdot & A_{2n} \\
A_{13}^* & A_{23}^* & A_{33} & \cdot & \cdot & \cdot & \cdot & A_{3n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
A_{1n}^* & A_{2n}^* & A_{3n}^* & \cdot & \cdot & \cdot & \cdot & A_{nn}
\end{array}$$

The diagonal elements are real.

From Eq.(1), we have

$$\langle \varphi_m | a_n | \varphi_n \rangle^* = a_n^* \langle \varphi_n | \varphi_m \rangle = \langle \varphi_n | \hat{A} | \varphi_m \rangle = a_m \langle \varphi_n | \varphi_m \rangle. \quad (2)$$

(1) *Eigen value is real*

$m = n$ in Eq.(2),

$$(a_n^* - a_n) \langle \varphi_n | \varphi_n \rangle = 0,$$

$$a_n^* = a_n.$$

Thus a_n is real.

(2) *The eigenkets are orthogonal*

In Eq.(2),

$$(a_n - a_m) \langle \varphi_n | \varphi_m \rangle = 0.$$

If $a_n \neq a_m$ (non-degenerate case),

$$\langle \varphi_n | \varphi_m \rangle = 0. \quad (\text{orthogonal})$$

(3) *The eigenkets of \hat{A} form a complete set.*

For any $|\psi\rangle$,

$$|\psi\rangle = \sum_{n=1}^{\infty} c_n |\varphi_n\rangle,$$

with

$$c_n = \langle \varphi_n | \psi \rangle.$$

6. Closure relation:

$$|\psi\rangle = \sum_{n=1}^{\infty} c_n |\varphi_n\rangle = \sum_{n=1}^{\infty} |\varphi_n\rangle \langle \varphi_n | \psi \rangle.$$

Because $|\psi\rangle$ is an arbitrary ket, we must have

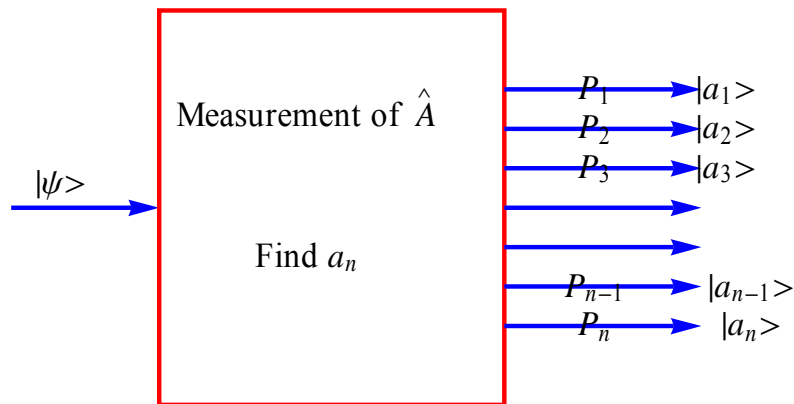
$$\sum_{n=1}^{\infty} |\varphi_n\rangle \langle \varphi_n | = \hat{1}.$$

For \hat{A} with $\hat{A}|\varphi_n\rangle = a_n|\varphi_n\rangle$,

$$\hat{A} = \hat{A} \sum_{n=1}^{\infty} |\varphi_n\rangle \langle \varphi_n | = \sum_{n=1}^{\infty} \hat{A} |\varphi_n\rangle \langle \varphi_n | = \sum_{n=1}^{\infty} a_n |\varphi_n\rangle \langle \varphi_n |.$$

7. Principle of measurement

For any $|\psi\rangle$,



$$|\psi\rangle = \left(\sum_{n=1}^{\infty} |\varphi_n\rangle \langle \varphi_n | \right) |\psi\rangle = \sum_{n=1}^{\infty} |\varphi_n\rangle \langle \varphi_n | \psi \rangle.$$

Probability of finding $|\varphi_n\rangle$

$$P(a_n) = |\langle \varphi_n | \psi \rangle|^2.$$

8. Commuting observables

The product of Hermitian operators \hat{A} and \hat{B}

$$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger = \hat{B}\hat{A}.$$

If

$$[\hat{A}, \hat{B}] = 0. \quad (\text{commutable})$$

$$(\hat{A}\hat{B})^\dagger = \hat{B}\hat{A} = \hat{A}\hat{B}.$$

Thus $\hat{A}\hat{B}$ is Hermitian.

9. Simultaneous eigenkets

We may use $|a, b\rangle$ to characterize the simultaneous eigenket.

$$\hat{A}|a, b\rangle = a|a, b\rangle,$$

$$\hat{B}|a, b\rangle = b|a, b\rangle.$$

Then

$$\hat{A}\hat{B}|a, b\rangle = b\hat{A}|a, b\rangle = ab|a, b\rangle,$$

$$\hat{B}\hat{A}|a, b\rangle = a\hat{B}|a, b\rangle = ab|a, b\rangle,$$

or

$$[\hat{A}, \hat{B}]|a, b\rangle = 0.$$