

Addition of angular momentum
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Here we discuss the addition of angular momentum, such as the orbital angular momentum $\mathbf{L}$ and the spin angular momentum $\mathbf{S}$. The addition of the angular momentum is encountered in all area of the modern physics, especially in quantum mechanics. The total angular momentum $\hat{\mathbf{J}}$ is expressed by

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$$

in terms of the angular momenta $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$. We show that the simultaneous eigenket $|j, m\rangle$ of $\hat{\mathbf{J}}$ and $\hat{J}_z$ can be expressed in terms of the linear combination of $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ with the so-called Clebsch-Gordan co-efficient, where $|j_1, m_1\rangle$ is the simultaneous eigenket of $\hat{\mathbf{J}}_1$ and $\hat{J}_{1z}$, and the simultaneous eigenket of $\hat{\mathbf{J}}_2$ and $\hat{J}_{2z}$. The Clebsch-Gordan co-efficient $\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle$ is discussed.

1. Clebsch-Gordan (CG) Coefficient

Let $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ be the two angular momenta, and let

$$[\hat{\mathbf{J}}_1, \hat{\mathbf{J}}_2] = 0,$$

or

$$[\hat{J}_{1i}, \hat{J}_{2j}] = 0. \quad (i, j = x, y, z)$$

Define the total angular momentum

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2,$$

where

$$\hat{\mathbf{J}}_1^2 |j_1, m_1\rangle = \hbar^2 j_1(j_1 + 1) |j_1, m_1\rangle,$$

$$\hat{J}_{1z}|j_1, m_1\rangle = \hbar m_1|j_1, m_1\rangle,$$

$$\hat{J}_{1\pm}|j_1, m_1\rangle = \hbar\sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)}|j_1, m_1 \pm 1\rangle,$$

Similarly,

$$\hat{J}_2^2|j_2, m_2\rangle = \hbar^2 j_2(j_2+1)|j_2, m_2\rangle,$$

$$\hat{J}_{2z}|j_2, m_2\rangle = \hbar m_2|j_2, m_2\rangle,$$

$$\hat{J}_{2\pm}|j_2, m_2\rangle = \hbar\sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)}|j_2, m_2 \pm 1\rangle.$$

We now consider a new ket defined by

$$|j_1, j_2; m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle.$$

The total angular momentum: commutation relation

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$$

$$[\hat{J}_z, \hat{J}_1^2] = [\hat{J}_z, \hat{J}_2^2] = 0$$

$$[\hat{J}^2, \hat{J}_1^2] = [\hat{J}^2, \hat{J}_2^2] = 0$$

((Note))

$$[\hat{J}_z, \hat{J}_1^2] = [\hat{J}_{1z} + \hat{J}_{2z}, \hat{J}_1^2] = [\hat{J}_{1z}, \hat{J}_1^2] = 0$$

$$[\hat{J}_z^2, \hat{J}_1^2] = -[\hat{J}_1^2, \hat{J}_z^2] = -([\hat{J}_1^2, \hat{J}_z]\hat{J}_z + \hat{J}_z[\hat{J}_1^2, \hat{J}_z]) = 0$$

$$[\hat{J}^2, \hat{J}_1^2] = [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \hat{J}_1^2] = 0$$

Furthermore, $\hat{J}_{1z}$ and $\hat{J}_{2z}$ commute with $\hat{J}_z$.

$$[\hat{J}_{1z}, \hat{J}_z] = [\hat{J}_{2z}, \hat{J}_z] = 0,$$

but not with $\hat{J}^2$.

$$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{\mathbf{J}}_1 \cdot \hat{\mathbf{J}}_2 = \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}$$

$$\begin{aligned}
[\hat{J}_{1z}, \hat{J}^2] &= [\hat{J}_{1z}, \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}] \\
&= [\hat{J}_{1z}, \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}] \\
&= [\hat{J}_{1z}, \hat{J}_{1+}]\hat{J}_{2-} + [\hat{J}_{1z}, \hat{J}_{1-}]\hat{J}_{2+} \\
&= \hbar\hat{J}_{1+}\hat{J}_{2-} - \hbar\hat{J}_{1-}\hat{J}_{2+}
\end{aligned}$$

$$\begin{aligned}
[\hat{J}_{2z}, \hat{J}^2] &= [\hat{J}_{2z}, \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}] \\
&= \hat{J}_{1+}[\hat{J}_{2z}, \hat{J}_{2-}] + \hat{J}_{1-}[\hat{J}_{2z}, \hat{J}_{2+}] \\
&= -\hbar\hat{J}_{1+}\hat{J}_{2-} + \hbar\hat{J}_{1-}\hat{J}_{2+}
\end{aligned}$$

Thus we have

$$[\hat{J}_z, \hat{J}^2] = 0$$

Since

$$[\hat{J}^2, \hat{J}_z] = [\hat{J}^2, \hat{J}_1^2] = [\hat{J}^2, \hat{J}_2^2] = [\hat{J}_1^2, \hat{J}_2^2] = [\hat{J}_1^2, \hat{J}_z] = [\hat{J}_2^2, \hat{J}_z] = 0,$$

We have simultaneous eigenkets of $\hat{J}^2, \hat{J}_1^2, \hat{J}_2^2, \hat{J}_z$. We use the eigenket,

$$|j_1, j_2; j, m\rangle$$

to denote the basis.

$$\hat{J}_1^2 |j_1, j_2; j, m\rangle = \hbar^2 j_1(j_1 + 1) |j_1, j_2; j, m\rangle$$

$$\hat{J}_2^2 |j_1, j_2; j, m\rangle = \hbar^2 j_2(j_2 + 1) |j_1, j_2; j, m\rangle$$

$$\hat{J}^2 |j_1, j_2; j, m\rangle = \hbar^2 j(j + 1) |j_1, j_2; j, m\rangle$$

$$\hat{J}_z |j_1, j_2; j, m\rangle = \hbar m |j_1, j_2; j, m\rangle$$

Here we have the relation

$$|j_1, j_2; j, m\rangle = \sum_{m_1} \sum_{m_2} |j_1, j_2; m_1, m_2\rangle \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle$$

where we use the closure relation.

$$\sum_{m_1} \sum_{m_2} |j_1, j_2; m_1, m_2\rangle \langle j_1, j_2; m_1, m_2| = 1 :$$

$\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle$: Clebsch Gordan co-efficient

For simplicity we use

$$|j_1, j_2; m_1, m_2\rangle = |j, m\rangle,$$

$$|j_1, j_2; m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle = |j_1, m_1\rangle |j_2, m_2\rangle$$

2. Properties of CG co-efficient

(1) The co-efficient vanishes unless $m = m_1 + m_2$.

((Proof))

Note that

$$(\hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z}) |j_1, j_2; j, m\rangle = 0$$

$$\langle j_1, j_2; m_1, m_2 | \hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z} | j_1, j_2; j, m \rangle = 0$$

or

$$(m - m_1 - m_2) \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle = 0$$

For $m - m_1 - m_2 \neq 0$,

$$\langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle = 0$$

(2) The CG co-efficient vanishes unless

$$|j_1 - j_2| \leq j \leq j_1 + j_2$$

triangle inequality, which tells us the possible range of values of j when we add two angular momenta j_1 and j_2 whose values are fixed.

((Proof))

For a proof of the triangle rule, let us consider the possible values of m . Since $m = m_1 + m_2$, its maximum value is $j_1 + j_2$. The value is realized for the single state $|j_1, j_2; j_1, j_2\rangle$. The total J of this state is $j = j_1 + j_2$. The next highest value of m , $j_1 + j_2 - 1$ is realized for the linear combinations of two states in the uncoupled representations; $|j_1, j_2; j_1 - 1, j_2\rangle$

and $|j_1, j_2; j_1, j_2 - 1\rangle$. One of the linear combinations belongs to $j = j_1 + j_2$ and the second one to $j = j_1 + j_2 - 1$.

For $m = j_1 + j_2 - 2$, there are three linear combinations of the three uncoupled states; $|j_1, j_2; j_1 - 2, j_2\rangle$, $|j_1, j_2; j_1 - 1, j_2 - 1\rangle$, and $|j_1, j_2; j_1, j_2 - 2\rangle$, corresponding to three of j , $j = j_1 + j_2$, $j = j_1 + j_2 - 1$, and $j = j_1 + j_2 - 2$.

If we continue this process, we can see that each time we lower m by 1, a new value of j appears. The argument continues until we reach a stage where we can no longer go one more step down in one of them to make new states. We assume that $j_1 > j_2$.

$$j = j_1 + j_2; m = j_1 + j_2, j_1 + j_2 - 1, \dots, -(j_1 + j_2)$$

$$j = j_1 + j_2 - 1, m = j_1 + j_2 - 1, j_1 + j_2 - 2, \dots, -(j_1 + j_2 - 1),$$

.....

$$j = j_1 - j_2, m = j_1 - j_2, j_1 - j_2 - 1, \dots, -(j_1 - j_2).$$

Here we show that if the triangle rule is valid, then the dimension of the space spanned by $\{|j_1, j_2; j, m\rangle\}$ is the same as that of the space spanned by $\{|j_1, j_2; m_1, m_2\rangle\}$.

For the (m_1, m_2) way of counting, we obtain $N = (2j_1 + 1)(2j_2 + 1)$.

As for the $\{j, m\}$ way of counting, note that for each j , there are $(2j + 1)$ states. According to the inequality $(|j_1 - j_2| \leq j \leq j_1 + j_2)$, j runs (we assume $j_1 > j_2$) from $j_1 - j_2$ to $j_1 + j_2$.

$$\begin{aligned} N &= \sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = \sum_{j=0}^{j_1+j_2} (2j+1) - \sum_{j=0}^{j_1-j_2} (2j+1) \\ &= (j_1 + j_2 + 1)^2 - (j_1 - j_2)^2 \\ &= (2j_1 + 1)(2j_2 + 1) \end{aligned}$$

((Note))

$$D_{j_1} \times D_{j_2} = D_{j_1+j_2} + D_{j_1+j_2-1} + \dots + D_{|j_1-j_2|}$$

where

$$D_{j_1+j_2}; j = j_1 + j_2; m = j_1 + j_2, j_1 + j_2 - 1, \dots, -(j_1 + j_2),$$

$$D_{j_1+j_2-1}; j = j_1 + j_2 - 1; m = j_1 + j_2 - 1, j_1 + j_2 - 2, \dots, -(j_1 + j_2 - 1),$$

.....

$$D_{|j_1-j_2|}; j = |j_1 - j_2|; m = |j_1 - j_2|, |j_1 - j_2| - 1, \dots, -|j_1 - j_2|,$$

The CG coefficients form an unitary matrix. Furthermore, the matrix elements are taken to be real by convention. A real unitary matrix is orthogonal. We have the orthogonal condition.

$$\langle j_1, j_2, j, m | j_1, j_2, m_1, m_2 \rangle = \langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m \rangle^* = \langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m \rangle$$

Closure relation

$$\begin{aligned} \langle j_1, j_2, m_1, m_1' | j_1, j_2, m_1', m_2' \rangle &= \sum_{j, m} \langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m \rangle \langle j_1, j_2, j, m | j_1, j_2, m_1', m_2' \rangle \\ &= \sum_{j, m} \langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m \rangle \langle j_1, j_2, m_1', m_2' | j_1, j_2, j, m \rangle \\ &= \delta_{m_1, m_1'} \delta_{m_2, m_2'} \end{aligned}$$

or

$$\sum_{j, m} \langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m \rangle \langle j_1, j_2, m_1', m_2' | j_1, j_2, j, m \rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'}$$

Similarly,

$$\sum_{m_1, m_2} \langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m \rangle \langle j_1, j_2, m_1, m_2 | j_1, j_2, j', m' \rangle = \delta_{j, j'} \delta_{m, m'}$$

As a special case of this, we may set $j = j', m' = m = m_1 + m_2$.

$$\sum_{\substack{m_1, m_2 \\ m = m_1 + m_2}} \left| \langle j_1, j_2, m_1, m_2 | j_1, j_2, j, m \rangle \right|^2 = 1$$

which is just the normalization condition of $|j_1, j_2, j, m\rangle$.

3 Recursion relation-1 for the CG co-efficients

((First step))

$$\hat{J}_{\pm} |j_1, j_2; j, m\rangle = (\hat{J}_{1\pm} + \hat{J}_{2\pm}) \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} |j_1, j_2; m_1', m_2'\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle$$

$$\sqrt{(j \mp m)(j \pm m + 1)} |j_1, j_2; j, m \pm 1\rangle$$

$$= \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} [\sqrt{(j_1 \mp m_1')(j_1 \pm m_1' + 1)} |j_1, j_2; m_1' \pm 1, m_2'\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle$$

$$+ \sqrt{(j_2 \mp m_2')(j_2 \pm m_2' + 1)} |j_1, j_2; m_1', m_2' \pm 1\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle]$$

((Second step))

Next step is to multiply by $\langle j_1, j_2; m_1, m_2 |$ on the left and to use orthonormality.

$$\sqrt{(j \mp m)(j \pm m + 1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \pm 1\rangle$$

$$= \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} [\sqrt{(j_1 \mp m_1')(j_1 \pm m_1' + 1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; m_1' \pm 1, m_2'\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle$$

$$+ \sqrt{(j_2 \mp m_2')(j_2 \pm m_2' + 1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; m_1', m_2' \pm 1\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle]$$

or

$$\sqrt{(j \mp m)(j \pm m + 1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \pm 1\rangle$$

$$= \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} [\sqrt{(j_1 \mp m_1')(j_1 \pm m_1' + 1)} \delta_{m_1, m_1' \pm 1} \delta_{m_2, m_2'} \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle$$

$$+ \sqrt{(j_2 \mp m_2')(j_2 \pm m_2' + 1)} \delta_{m_1, m_1'} \delta_{m_2, m_2' \pm 1} \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle]$$

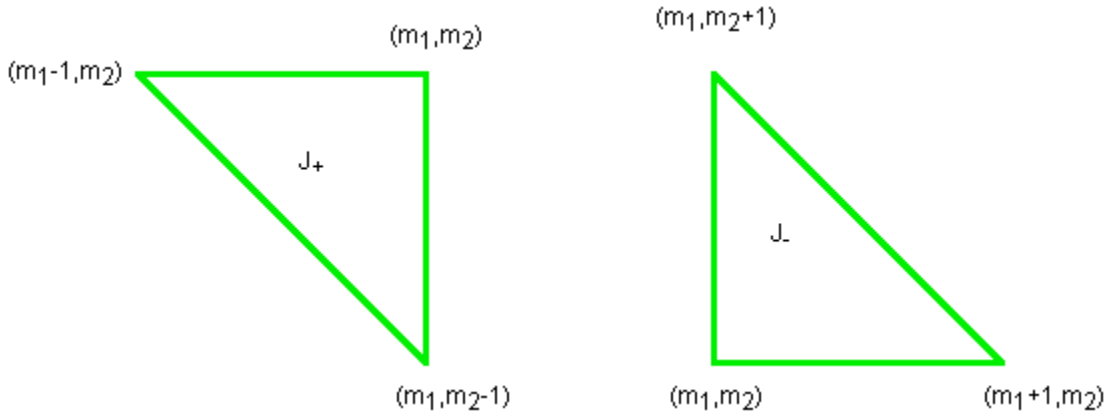
or

$$\begin{aligned}
& \sqrt{(j \mp m)(j \pm m + 1)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \pm 1 \rangle \\
&= \sqrt{[j_1 \mp (m_1 \mp 1)](j_1 \pm (m_1 \mp 1) + 1)} \langle j_1, j_2; m_1 \mp 1, m_2 | j_1, j_2; j, m \rangle \\
&+ \sqrt{[j_2 \mp (m_2 \mp 1)](j_2 \pm (m_2 \mp 1) + 1)} \langle j_1, j_2; m_1, m_2 \mp 1 | j_1, j_2; j, m \rangle
\end{aligned}$$

((Note)) We use a more convenient expression

$$\begin{aligned}
& \sqrt{(j \mp m + 1)(j \pm m)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\
&= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)} \langle j_1, j_2; m_1 \mp 1, m_2 | j_1, j_2; j, m \mp 1 \rangle \\
&+ \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)} \langle j_1, j_2; m_1, m_2 \mp 1 | j_1, j_2; j, m \mp 1 \rangle
\end{aligned}$$

This recursion relations are related to the CG coefficients at (m_1, m_2) , $(m_1 \mp 1, m_2)$, and $(m_1, m_2 \mp 1)$, where $m_1 + m_2 = m$.



4 Recursion relation-2 for the CG co-efficients

$$\hat{\mathbf{J}}^2 |j_1, j_2; j, m\rangle = \hat{\mathbf{J}}^2 \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} |j_1, j_2; m_1', m_2'\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m \rangle$$

$$\hat{\mathbf{J}}^2 = \hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}$$

$$\hat{\mathbf{J}}^2 |j_1, j_2; j, m\rangle = \hbar^2 j(j+1) |j_1, j_2; j, m\rangle$$

$$\begin{aligned}
& \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} \hat{\mathbf{J}}^2 |j_1, j_2; m_1', m_2'\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle \\
&= \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} (\hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z} + \hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}) \\
&\quad \times |j_1, j_2; m_1', m_2'\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle
\end{aligned}$$

Here

$$\begin{aligned}
& \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} (\hat{\mathbf{J}}_1^2 + \hat{\mathbf{J}}_2^2 + 2\hat{J}_{1z}\hat{J}_{2z}) |j_1, j_2; m_1', m_2'\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle \\
&= \hbar^2 \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} \{j_1(j_1 + 1) + j_2(j_2 + 1) + 2m_1'm_2'\} \\
&\quad \times |j_1, j_2; m_1', m_2'\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle \\
& \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} (\hat{J}_{1+}\hat{J}_{2-} + \hat{J}_{1-}\hat{J}_{2+}) |j_1, j_2; m_1', m_2'\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle \\
&= \hbar^2 \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} \{\sqrt{(j_1 - m_1')(j_1 + m_1' + 1)}\sqrt{(j_2 + m_2')(j_2 - m_2' + 1)} \\
&\quad \times |j_1, j_2; m_1' + 1, m_2' - 1\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle \\
&+ \hbar^2 \sum_{\substack{m_1', m_2' \\ m_1' + m_2' = m}} \{\sqrt{(j_1 + m_1')(j_1 - m_1' + 1)}\sqrt{(j_2 - m_2')(j_2 + m_2' + 1)} \\
&\quad \times |j_1, j_2; m_1' - 1, m_2' + 1\rangle \langle j_1, j_2; m_1', m_2' | j_1, j_2; j, m\rangle
\end{aligned}$$

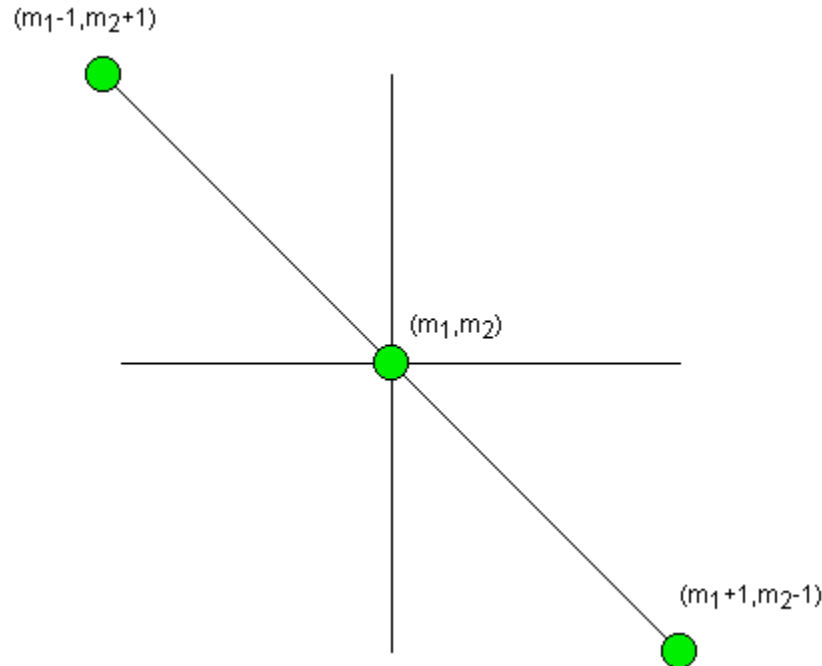
Note that

$$\langle j_1, j_2; m_1, m_2 | j_1, j_2; m_1', m_2' \rangle = \delta_{m_1, m_1'} \delta_{m_2, m_2'}$$

We multiply the above equation by bra $\langle j_1, j_2; m_1, m_2 |$

$$\begin{aligned}
& \{j(j + 1) - j_1(j_1 + 1) - j_2(j_2 + 1) - 2m_1m_2\} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m\rangle \\
&= \sqrt{(j_1 - m_1 + 1)(j_1 + m_1)}\sqrt{(j_2 + m_2 + 1)(j_2 - m_2)} \langle j_1, j_2; m_1 - 1, m_2 + 1 | j_1, j_2; j, m\rangle \\
&+ \sqrt{(j_1 + m_1 + 1)(j_1 - m_1)}\sqrt{(j_2 - m_2 + 1)(j_2 + m_2)} \langle j_1, j_2; m_1 + 1, m_2 - 1 | j_1, j_2; j, m\rangle
\end{aligned}$$

This recursion relations are related to the CG coefficients at (m_1, m_2) , (m_1-1, m_2+1) , and (m_1+1, m_2-1) , where $m_1+m_2 = m$.



$$D_{j_1} \times D_{j_2} = D_{j_1+j_2} + D_{j_1+j_2-1} + \dots + D_{|j_1-j_2|}$$

For fixed j , $m = j, j-1, \dots, -j$.

In the (m_1, m_2) plane, we plot the boundary of the allowed region determined by

$$|m_1| \leq j_1, \quad |m_2| \leq j_2$$

$$m_1 + m_2 = m$$

The J_+ recursion relation tells us that the co-efficient at (m_1, m_2) is related to those at (m_1-1, m_2) and (m_1, m_2-1) . Likewise, the J_- recursion relation (lower sign) relates the three co-efficients whose m_1, m_2 values are given in Fig.

(Recursion relations + Normalization conditions) $\rightarrow$ uniquely determine the CG co-efficients.

$$|l_1, l_2; j, m\rangle = |j, m\rangle = \sum_{\substack{m_1, m_2 \\ m_1 + m_2 = m}} C_{m_1, m_2} |l_1, m_1\rangle |l_2, m_2\rangle$$

5 CG values: $j_1 = 1/2, j_2 = 1/2$

$$j_1 = 1/2, j_2 = 1/2 \quad (|m_1| \leq 1/2, |m_2| \leq 1/2)$$

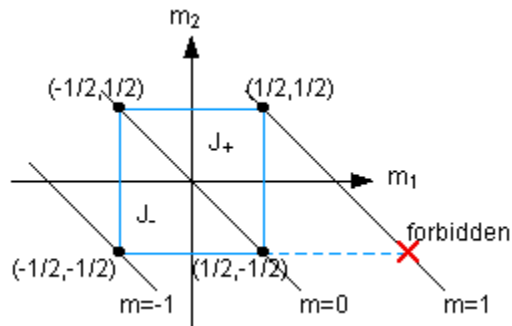
$$D_{1/2} \times D_{1/2} = D_1 + D_0 \rightarrow j = 1 \text{ and } j = 0$$

(i) $j = 1$

$$m = 1, 0, -1$$

where

$$m = m_1 + m_2$$



$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle \quad (m = 1)$$

$$\frac{\left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle}{\sqrt{2}} \quad (m = 0)$$

$$\left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \quad (m = -1)$$

(ii) $j = 0$ ($m = 0$)

$$\frac{\left| \frac{1}{2}, \frac{1}{2} \right\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle - \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle}{\sqrt{2}} \quad (m=0)$$

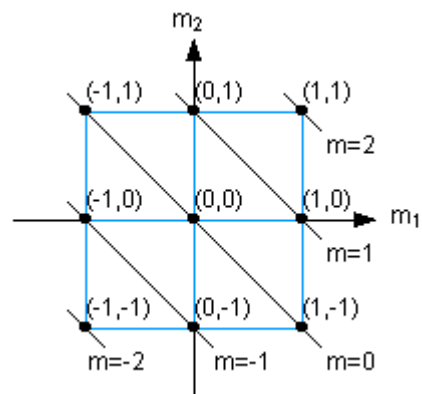
6 CG values: $j_1 = 1, j_2 = 1$

$$j_1 = 1, j_2 = 1 \quad (|m_1| \leq 1, |m_2| \leq 1)$$

$$D_1 \times D_1 = D_2 + D_1 + D_0$$

(i) $j = 2 \quad (|m| \leq 2)$

$$m = m_1 + m_2$$



$$|1,1\rangle|1,1\rangle \quad (m=2)$$

$$\frac{|1,1\rangle|1,0\rangle + |1,0\rangle|1,1\rangle}{\sqrt{2}} \quad (m=1)$$

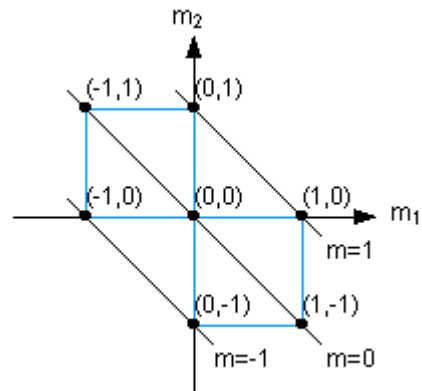
$$\frac{|1,1\rangle|1,-1\rangle + 2|1,0\rangle|1,0\rangle + |1,-1\rangle|1,1\rangle}{\sqrt{6}} \quad (m=0)$$

$$\frac{|1,0\rangle|1,-1\rangle + |1,-1\rangle|1,0\rangle}{\sqrt{2}} \quad (m=-1)$$

$$|1,-1\rangle|1,-1\rangle \quad (m=-2)$$

Symmetric for the particle exchange

(ii) $j = 1$ ($|m| \leq 1$)



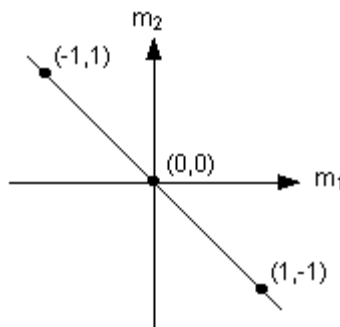
$$\frac{|1,1\rangle|1,0\rangle - |1,0\rangle|1,1\rangle}{\sqrt{2}} \quad (m = 1)$$

$$\frac{|1,1\rangle|1,-1\rangle - |1,-1\rangle|1,1\rangle}{\sqrt{2}} \quad (m = 0)$$

$$\frac{|1,0\rangle|1,-1\rangle - |1,-1\rangle|1,0\rangle}{\sqrt{2}} \quad (m = -1)$$

Anti-symmetric for the particle exchange

(iii) $j = 0$ ($m = 0$)



$$\frac{|1,1\rangle|1,-1\rangle + |1,0\rangle|1,0\rangle + |1,-1\rangle|1,1\rangle}{\sqrt{3}}$$

Symmetric for the particle exchange

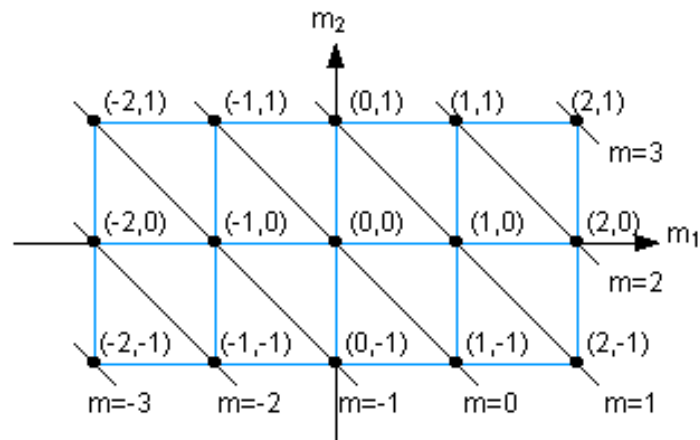
7 CG values: $j_1 = 2, j_2 = 1$

$$j_1 = 2, j_2 = 1 \quad (|m_1| \leq 2, |m_2| \leq 1)$$

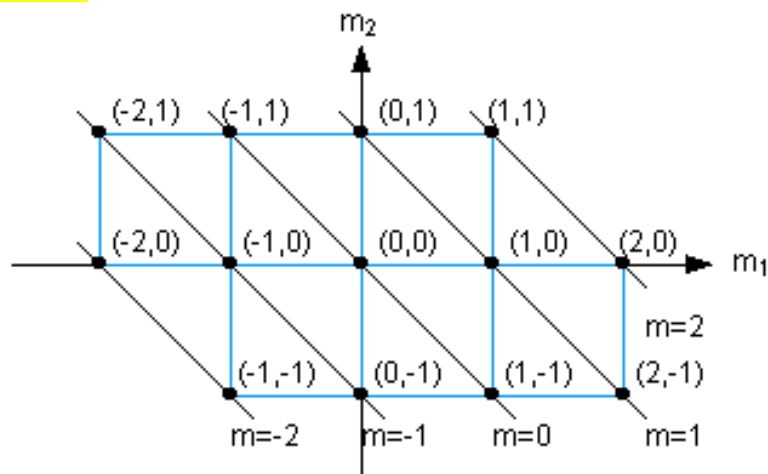
$$D_2 \times D_1 = D_3 + D_2 + D_1$$

(i) $j = 3 \quad (|m| \leq 3)$

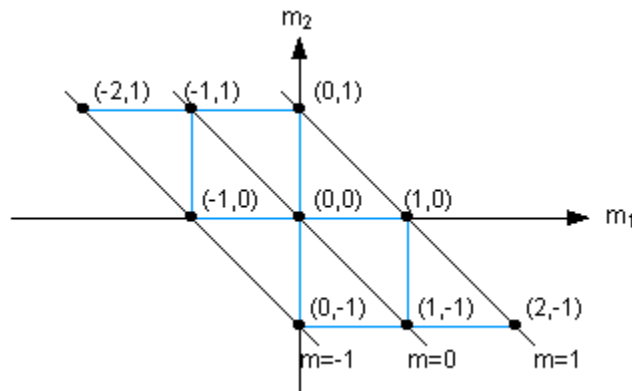
$$m = m_1 + m_2$$



(ii) $j = 2 \quad (|m| \leq 2)$



(iii) $j = 1 \quad (|m| \leq 1)$



8. Addition of orbital and spin angular momentum

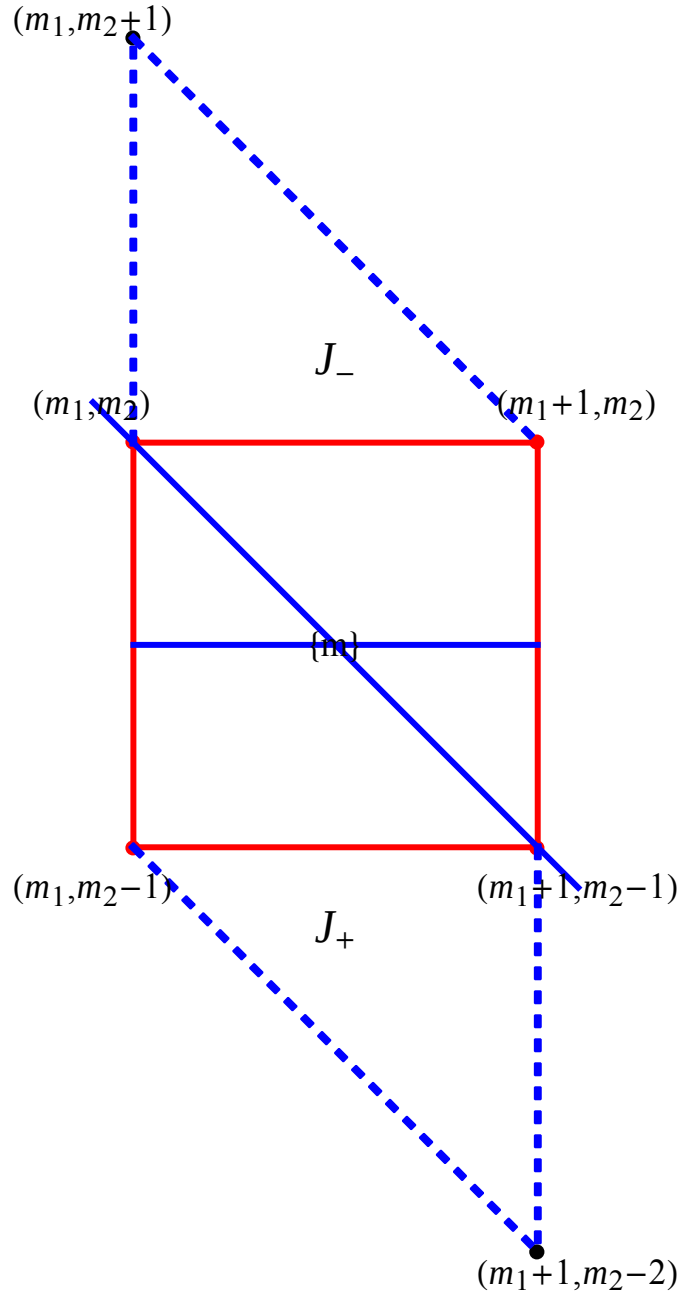
We consider one electron with the orbital angular momentum L (typically $l = 1$, p electron) and the spin angular momentum S . The total angular momentum J is an addition of L and S .

$$\hat{J} = \hat{L} + \hat{S}$$

$$D_l \times D_{1/2} = D_{l+1/2} + D_{l-1/2}$$

9. J . recursion

We use the J . recursion for the diagram below.



$$\begin{aligned}
 & \sqrt{(j+m+1)(j-m)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\
 &= \sqrt{(j_1+m_1+1)(j_1-m_1)} \langle j_1, j_2; m_1+1, m_2 | j_1, j_2; j, m+1 \rangle \\
 &+ \sqrt{(j_2+m_2+1)(j_2-m_2)} \langle j_1, j_2; m_1, m_2+1 | j_1, j_2; j, m+1 \rangle
 \end{aligned}$$

We note that

$$\langle j_1, j_2; m_1, m_2 + 1 | j_1, j_2; j, m + 1 \rangle = 0$$

Then we have

$$\begin{aligned} & \sqrt{(j+m+1)(j-m)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\ &= \sqrt{(j_1+m_1+1)(j_1-m_1)} \langle j_1, j_2; m_1+1, m_2 | j_1, j_2; j, m+1 \rangle \end{aligned}$$

We assume that

$$j_1 = l, \quad j_2 = 1/2, \quad j = l + 1/2$$

$$m_1 = m - 1/2, \quad m_2 = 1/2$$

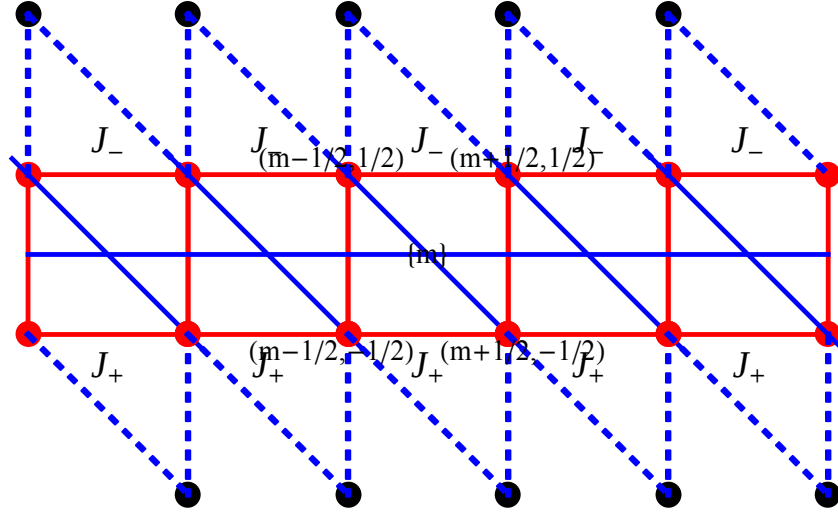
Then we have

$$\begin{aligned} & \sqrt{(l + \frac{1}{2} + m + 1)(l + \frac{1}{2} - m)} \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, \frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \right. \\ &= \sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})} \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, \frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m + 1 \right\rangle \right. \end{aligned}$$

or

$$\begin{aligned} & \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, \frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \right. \\ &= \sqrt{\frac{l + m + \frac{1}{2}}{l + m + \frac{3}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, \frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m + 1 \right\rangle \right. \end{aligned}$$

This procedure can be continued until $m = l + 1/2$.



Then we have

$$\begin{aligned}
 & \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, \frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \right. \\
 &= \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}} \sqrt{\frac{l+m+\frac{3}{2}}{l+m+\frac{5}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{3}{2}, \frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m + 2 \right\rangle \right. \\
 &= \sqrt{\frac{l+m+\frac{1}{2}}{l+m+\frac{3}{2}}} \sqrt{\frac{l+m+\frac{3}{2}}{l+m+\frac{5}{2}}} \sqrt{\frac{l+m+\frac{5}{2}}{l+m+\frac{7}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{5}{2}, \frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m + 3 \right\rangle \right. \\
 &= \dots\dots\dots \\
 &= \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}} \left\langle j_1 = l, j_2 = \frac{1}{2}; l, \frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, l + \frac{1}{2} \right\rangle \right.
 \end{aligned}$$

We choose

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; l, \frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, l + \frac{1}{2} \right\rangle = 1$$

Then we have

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, \frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle = \sqrt{\frac{l + m + \frac{1}{2}}{2l + 1}}$$

10 J_+ recursion

We use the J_+ recursion for the diagram.

$$\begin{aligned} & \sqrt{(j - m + 1)(j + m)} \langle j_1, j_2; m_1, m_2 | j_1, j_2; j, m \rangle \\ &= \sqrt{(j_1 - m_1 + 1)(j_1 + m_1)} \langle j_1, j_2; m_1 - 1, m_2 | j_1, j_2; j, m - 1 \rangle \\ &+ \sqrt{(j_2 - m_2 + 1)(j_2 + m_2)} \langle j_1, j_2; m_1, m_2 - 1 | j_1, j_2; j, m - 1 \rangle \end{aligned}$$

We assume that

$$j_1 = l, \quad j_2 = 1/2, \quad j = l + 1/2$$

$$m_1 = m + 1/2, \quad m_2 = -1/2$$

Then we have

$$\begin{aligned} & \sqrt{(l - m + \frac{2}{3})(l + m + \frac{1}{2})} \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, -\frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \right. \\ &= \sqrt{(l - m + \frac{1}{2})(l + m + \frac{1}{2})} \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, -\frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m - 1 \right\rangle \right. \end{aligned}$$

or

$$\begin{aligned} & \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, -\frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \right. \\ &= \sqrt{\frac{l - m + \frac{1}{2}}{l - m + \frac{3}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, -\frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m - 1 \right\rangle \right. \end{aligned}$$

This procedure can be continued until $m = -l - 1/2$.

$$\begin{aligned}
& \left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, -\frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle \right. \\
&= \sqrt{\frac{l-m+\frac{1}{2}}{l-m+\frac{3}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{1}{2}, -\frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m-1 \right\rangle \right. \\
&= \sqrt{\frac{l-m+\frac{1}{2}}{l-m+\frac{3}{2}}} \sqrt{\frac{l-m+\frac{3}{2}}{l-m+\frac{5}{2}}} \left\langle j_1 = l, j_2 = \frac{1}{2}; m - \frac{3}{2}, -\frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m-2 \right\rangle \right. \\
&= \dots\dots\dots \\
&= \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}} \left\langle j_1 = l, j_2 = \frac{1}{2}; -l, -\frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, -l - \frac{1}{2} \right\rangle \right.
\end{aligned}$$

We choose

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; -l, -\frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, -l - \frac{1}{2} \right\rangle = 1$$

Then we have

$$\left\langle j_1 = l, j_2 = \frac{1}{2}; m + \frac{1}{2}, -\frac{1}{2} \left| j_1 = l, j_2 = \frac{1}{2}; j = l + \frac{1}{2}, m \right\rangle = \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}$$

11 Derivation of Clebsch-Gordan coefficient)

We assume that

$$|j = l + 1/2, m\rangle = \alpha |m_l = m - 1/2, m_s = 1/2\rangle + \beta |m_l = m + 1/2, m_s = -1/2\rangle$$

$$|j = l - 1/2, m\rangle = \gamma |m_l = m - 1/2, m_s = 1/2\rangle + \delta |m_l = m + 1/2, m_s = -1/2\rangle$$

where α , β , γ , and δ are real. The normalization condition:

$$\alpha^2 + \beta^2 = 1$$

$$\gamma^2 + \delta^2 = 1$$

The condition of the orthogonality:

$$\alpha\gamma + \beta\delta = 0$$

There are four unknown parameters and three equations. So we need one more equation. In Appendix, we show that

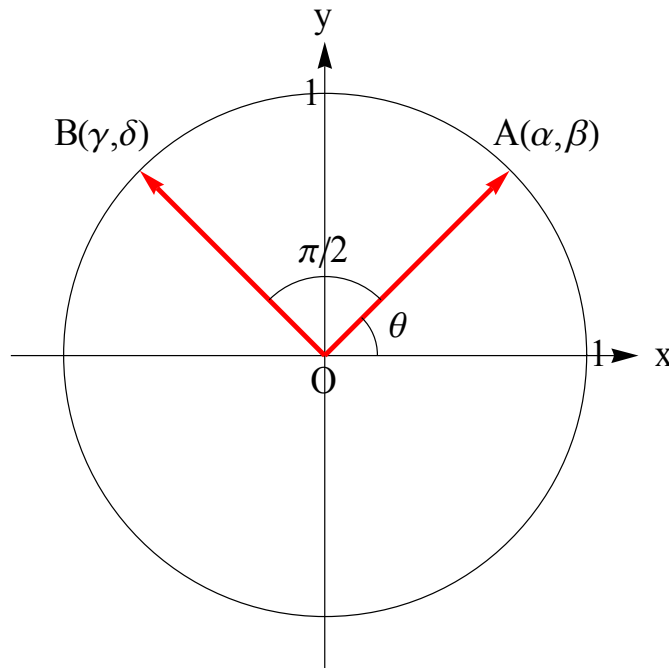
$$\frac{\alpha}{\beta} = \sqrt{\frac{l+m+\frac{1}{2}}{l-m+\frac{1}{2}}}.$$

We want to determine the values of β , γ , and δ . To this end, we assume that

$$\alpha = \cos \theta,$$

$$\beta = \sin \theta$$

$$\gamma = \cos(\theta + \pi/2) = -\sin \theta, \delta = \sin(\theta + \pi/2) = \cos \theta$$



Since

$$\alpha^2 + \beta^2 = 1$$

we have

$$\beta = \sqrt{1 - \frac{l+m+\frac{1}{2}}{2l+1}} = \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}$$

Then we get

$$\alpha = \delta = \sqrt{\frac{l+m+1/2}{2l+1}}, \quad \beta = -\gamma = \sqrt{\frac{l-m+1/2}{2l+1}}$$

The final result is

$$\begin{aligned} |j=l+1/2, m\rangle &= \sqrt{\frac{l+m+1/2}{2l+1}} |m_l = m-1/2, m_s = 1/2\rangle \\ &\quad + \sqrt{\frac{l-m+1/2}{2l+1}} |m_l = m+1/2, m_s = -1/2\rangle \\ |j=l-1/2, m\rangle &= -\sqrt{\frac{l-m+1/2}{2l+1}} |m_l = m-1/2, m_s = 1/2\rangle \\ &\quad + \sqrt{\frac{l+m+1/2}{2l+1}} |m_l = m+1/2, m_s = -1/2\rangle \end{aligned}$$

or

$$\begin{aligned} |j=l+1/2, m\rangle &= \begin{pmatrix} \sqrt{\frac{l+m+1/2}{2l+1}} \\ \sqrt{\frac{l-m+1/2}{2l+1}} \end{pmatrix}, \\ |j=l-1/2, m\rangle &= \begin{pmatrix} -\sqrt{\frac{l-m+1/2}{2l+1}} \\ \sqrt{\frac{l+m+1/2}{2l+1}} \end{pmatrix} \end{aligned}$$

We note that

$$\begin{aligned} |m_l = m-1/2, m_s = 1/2\rangle &= \alpha |j=l+1/2, m\rangle - \beta |j=l-1/2, m\rangle \\ |m_l = m+1/2, m_s = -1/2\rangle &= \beta |j=l+1/2, m\rangle + \alpha |j=l-1/2, m\rangle \end{aligned}$$

12 Mathematica: The use of Standard Conventions of Edmonds

It is much easier to calculate the Clebsch-Gordan co-efficient by using Mathematica.

(i) $\text{ClebschGordan}[\{l, m-1/2\}, \{1/2, 1/2\}, \{l+1/2, m\}]$

$$\alpha = -\frac{(-1)^{2(m+l)}\sqrt{1+2m+2l}}{\sqrt{2+4l}}$$

Since l is an integer and m is an half-integer, we have $(-1)^{2(m+l)} = -1$. Then α is simplified as

$$\alpha = \frac{\sqrt{1+2m+2l}}{\sqrt{2+4l}}$$

(ii) $\text{ClebschGordan}[\{l, m+1/2\}, \{1/2, -1/2\}, \{l+1/2, m\}]$

$$\beta = -\frac{(-1)^{2(m+l)}\sqrt{1-2m+2l}}{\sqrt{2+4l}}$$

or

$$\beta = \frac{\sqrt{1-2m+2l}}{\sqrt{2+4l}}$$

(iii) $\text{ClebschGordan}[\{l, m-1/2\}, \{1/2, 1/2\}, \{l-1/2, m\}]$

$$\gamma = -\frac{\sqrt{1-2m+2l}}{\sqrt{2+4l}}.$$

(iv) $\text{ClebschGordan}[\{l, m+1/2\}, \{1/2, -1/2\}, \{l-1/2, m\}]$

$$\delta = \frac{\sqrt{1+2m+2l}}{\sqrt{2+4l}}.$$

13. Clebsch-Gordan coefficient with l and $s = 1$

$$D_l \times D_1 = D_{l+1} + D_l + D_{l-1}$$

(i) $\text{ClebschGordan}[\{l, m-1\}, \{1, 1\}, \{l+1, m\}]$

$$\frac{(-1)^{2(m+l)} \sqrt{m+l} \sqrt{1+m+l}}{\sqrt{2} \sqrt{1+l} \sqrt{1+2l}}.$$

(ii) ClebschGordan[$\{l, m\}, \{1, 0\}, \{l+1, m\}$]

$$\frac{(-1)^{2(m+l)} \sqrt{1-m+l} \sqrt{1+m+l}}{\sqrt{1+l} \sqrt{1+2l}}.$$

(iii) ClebschGordan[$\{l, m+1\}, \{1, -1\}, \{l+1, m\}$]

$$\frac{(-1)^{2(m+l)} \sqrt{-m+l} \sqrt{1-m+l}}{\sqrt{2} \sqrt{1+l} \sqrt{1+2l}}.$$

(iv) ClebschGordan[$\{l, m-1\}, \{1, 1\}, \{l, m\}$]

$$-\frac{\sqrt{m+l} \sqrt{1-m+l}}{\sqrt{2} \sqrt{l} \sqrt{1+2l}}.$$

(v) ClebschGordan[$\{l, m\}, \{1, 0\}, \{l, m\}$]

$$\frac{m}{\sqrt{l} \sqrt{1+2l}}$$

(vi) ClebschGordan[$\{l, m+1\}, \{1, -1\}, \{l, m\}$]

$$\frac{(-1)^{2(m+l)} \sqrt{-m+l} \sqrt{1+m+l}}{\sqrt{2} \sqrt{l} \sqrt{1+2l}}$$

(vii) ClebschGordan[$\{l, m-1\}, \{1, 1\}, \{l-1, m\}$]

$$\frac{\sqrt{-m+l} \sqrt{1-m+l}}{\sqrt{2} \sqrt{l} \sqrt{1+2l}}$$

(viii) ClebschGordan[$\{l, m\}, \{1, 0\}, \{l-1, m\}$]

$$-\frac{\sqrt{-m+l} \sqrt{m+l}}{\sqrt{l} \sqrt{1+2l}}$$

(ix) ClebschGordan[$\{l, m+1\}, \{1, -1\}, \{l-1, m\}$]

$$\frac{\sqrt{m+l}\sqrt{1+m+l}}{\sqrt{2}\sqrt{l}\sqrt{1+2l}}$$

4. Typical example: Na D lines

- (a) For the electron with 3 s state ($l = 0, s = 1/2$)

$$D_0 \times D_{1/2} = D_{1/2}$$

Thus we have $j = 1/2$. The state is described by $^2S_{1/2}$.

$$|j = 1/2, m = 1/2\rangle = |m_l = 0, m_s = 1/2\rangle$$

$$|j = 1/2, m = -1/2\rangle = |m_l = 0, m_s = -1/2\rangle$$

- (b) For the electron with 3 p state ($l = 1, s = 1/2$)

$$D_1 \times D_{1/2} = D_{3/2} + D_{1/2}$$

Thus we have $j = 3/2$ and $j = 1/2$. The state is described by $^2P_{3/2}$ and $^2P_{1/2}$

- (i) $j = 3/2$

$$|j = 3/2, m = -3/2\rangle = |m_l = -1, m_s = -1/2\rangle$$

$$|j = 3/2, m = -1/2\rangle = \sqrt{\frac{2}{3}}|m_l = 0, m_s = -1/2\rangle + \frac{1}{\sqrt{3}}|m_l = -1, m_s = 1/2\rangle$$

$$|j = 3/2, m = 1/2\rangle = \frac{1}{\sqrt{3}}|m_l = 1, m_s = -1/2\rangle + \sqrt{\frac{2}{3}}|m_l = 0, m_s = 1/2\rangle$$

$$|j = 3/2, m = 3/2\rangle = |m_l = 1, m_s = 1/2\rangle$$

- (ii) $j = 1/2$

$$|j = 1/2, m = -1/2\rangle = \frac{1}{\sqrt{3}}|m_l = 0, m_s = -1/2\rangle - \sqrt{\frac{2}{3}}|m_l = -1, m_s = 1/2\rangle$$

$$|j = 1/2, m = 1/2\rangle = \sqrt{\frac{2}{3}}|m_l = 1, m_s = -1/2\rangle - \frac{1}{\sqrt{3}}|m_l = 0, m_s = 1/2\rangle$$

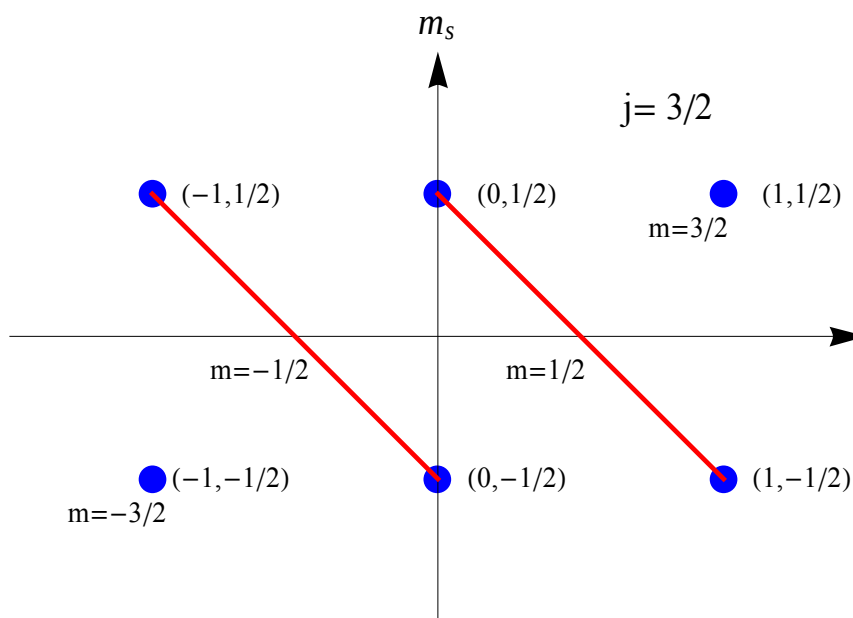


Fig. Clebsch-Gordan diagram for $|j, m\rangle$ with $j = 3/2$. $m = 3/2, 1/2, -1/2$, and $-3/2$.

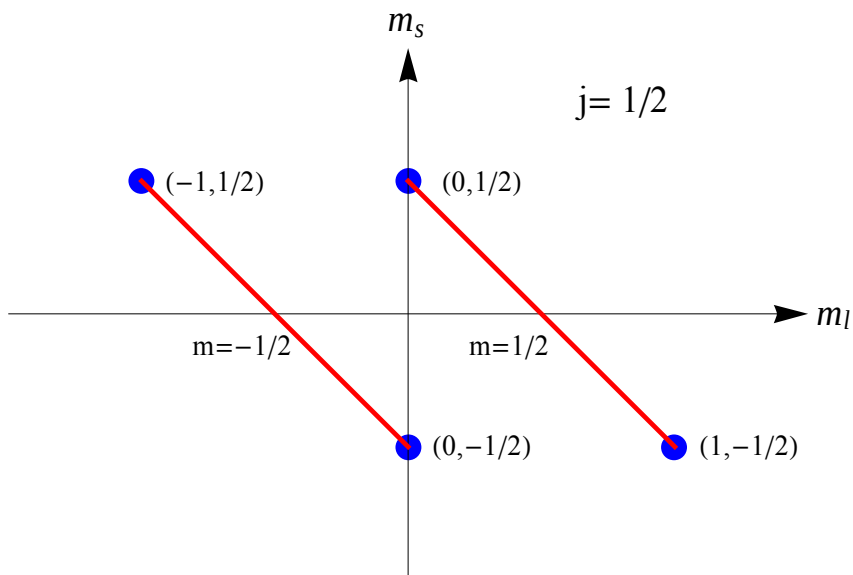


Fig. Clebsch-Gordan diagram for $|j, m\rangle$ with $j = 1/2$. $m = 1/2$ and $-1/2$.

5. Mathematica

ClebschGordan

`ClebschGordan[{j1, m1}, {j2, m2}, {j, m}]`

gives the Clebsch-Gordan coefficient for the decomposition of $|j, m\rangle$ in terms of $|j_1, m_1\rangle |j_2, m_2\rangle$.

▼ Details

- The Clebsch-Gordan coefficients vanish except when $m = m_1 + m_2$ and the j_i satisfy a triangle inequality.
- The parameters of `ClebschGordan` can be integers, half-integers, or symbolic expressions.
- *Mathematica* uses the standard conventions of Edmonds for the phase of the Clebsch-Gordan coefficients.

We can use the Mathematica to obtain the Clebsch-Gordan co-efficient. We show some typical programs using Mathematica.

5.1 $j_1 = 1/2$ and $j_2 = 1/2$

Clebsch-Gordan coefficient

```

Clear["Global`*"];

CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]]

CG[{j_, m_}, j1_, j2_] :=
Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1, m1]
  b[j2, m - m1], {m1, -j1, j1}]

```

j1 = 1/2 and j2 = 1/2

j = 1

j1 = 1 / 2; j2 = 1 / 2;

CG[{1, 1}, j1, j2]

$$a\left[\frac{1}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

CG[{1, 0}, j1, j2]

$$\frac{a\left[\frac{1}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{2}} + \frac{a\left[\frac{1}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{2}}$$

CG[{1, -1}, j1, j2]

$$a\left[\frac{1}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]$$

j = 0

j1 = 1 / 2; j2 = 1 / 2;

CG[{0, 0}, j1, j2]

$$\frac{a\left[\frac{1}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{2}} - \frac{a\left[\frac{1}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{2}}$$

5.2 $j_1 = 1$ and $j_2 = 1/2$

```

Clear["Global`*"];

CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]

CG[{j_, m_}, j1_, j2_] :=
Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1, m1]
  b[j2, m - m1], {m1, -j1, j1}]

j1 = 1; j2 = 1 / 2;

```

j = 3/2;

```
CG[{3 / 2, 3 / 2}, j1, j2]
```

$$a[1, 1] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

```
CG[{3 / 2, 1 / 2}, j1, j2]
```

$$\frac{a[1, 1] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{3}} + \sqrt{\frac{2}{3}} a[1, 0] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

```
CG[{3 / 2, -1 / 2}, j1, j2]
```

$$\sqrt{\frac{2}{3}} a[1, 0] b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{a[1, -1] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{3}}$$

```
CG[{3 / 2, -3 / 2}, j1, j2]
```

$$a[1, -1] b\left[\frac{1}{2}, -\frac{1}{2}\right]$$

j = 1/2;

```
CG[{1 / 2, 1 / 2}, j1, j2]
```

$$\sqrt{\frac{2}{3}} a[1, 1] b\left[\frac{1}{2}, -\frac{1}{2}\right] - \frac{a[1, 0] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{3}}$$

```
CG[{1 / 2, -1 / 2}, j1, j2]
```

$$\frac{a[1, 0] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{3}} - \sqrt{\frac{2}{3}} a[1, -1] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

5.3 $j_1 = 1$ and $j_2 = 1$

```

Clear["Global`*"];

CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]

CG[{j_, m_}, j1_, j2_] :=
Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1, m1]
  b[j2, m - m1], {m1, -j1, j1}]

j1 = 1; j2 = 1;

```

j=2;

CG[{2, 2}, j1, j2]

a[1, 1] b[1, 1]

CG[{2, 1}, j1, j2]

$$\frac{a[1, 1] b[1, 0]}{\sqrt{2}} + \frac{a[1, 0] b[1, 1]}{\sqrt{2}}$$

$$\mathbf{CG}[\{2, 0\}, j1, j2]$$

$$\frac{a[1, 1] b[1, -1]}{\sqrt{6}} + \sqrt{\frac{2}{3}} a[1, 0] b[1, 0] + \frac{a[1, -1] b[1, 1]}{\sqrt{6}}$$

$$\mathbf{CG}[\{2, -1\}, j1, j2]$$

$$\frac{a[1, 0] b[1, -1]}{\sqrt{2}} + \frac{a[1, -1] b[1, 0]}{\sqrt{2}}$$

$$\mathbf{CG}[\{2, -2\}, j1, j2]$$

$$a[1, -1] b[1, -1]$$

j = 1;

$$\mathbf{CG}[\{1, 1\}, j1, j2]$$

$$\frac{a[1, 1] b[1, 0]}{\sqrt{2}} - \frac{a[1, 0] b[1, 1]}{\sqrt{2}}$$

$$\mathbf{CG}[\{1, 0\}, j1, j2]$$

$$\frac{a[1, 1] b[1, -1]}{\sqrt{2}} - \frac{a[1, -1] b[1, 1]}{\sqrt{2}}$$

$$\mathbf{CG}[\{1, -1\}, j1, j2]$$

$$\frac{a[1, 0] b[1, -1]}{\sqrt{2}} - \frac{a[1, -1] b[1, 0]}{\sqrt{2}}$$

j = 0;

$$\mathbf{CG}[\{0, 0\}, j1, j2]$$

$$\frac{a[1, 1] b[1, -1]}{\sqrt{3}} - \frac{a[1, 0] b[1, 0]}{\sqrt{3}} + \frac{a[1, -1] b[1, 1]}{\sqrt{3}}$$

APPENDIX-I

Mathematica

■

j1=3/2 and j2=1/2

```
Clear["Global`*"];

CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] :=
Module[{s1},
  s1 = If[Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j,
    ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]

ψ[j_, m_, j1_, j2_] := Sum[CCGG[{j1, m1}, {j2, m - m1}, {j, m}] a[j1, m1] b[j2, m - m1],
  {m1, -j1, j1}]

ψ[2, 2, 3/2, 1/2]
a[3/2, 3/2] b[1/2, 1/2]

ψ[2, 1, 3/2, 1/2]
1/2 a[3/2, 3/2] b[1/2, -1/2] + 1/2 √3 a[3/2, 1/2] b[1/2, 1/2]
```

$$\psi[2, 0, 3/2, 1/2]$$

$$\frac{a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{2}} + \frac{a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{2}}$$

$$\psi[2, -1, 3/2, 1/2]$$

$$\frac{1}{2} \sqrt{3} a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{1}{2} a\left[\frac{3}{2}, -\frac{3}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

$$\psi[2, -2, 3/2, 1/2]$$

$$a\left[\frac{3}{2}, -\frac{3}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]$$

$$\psi[2, 1, 3/2, 1/2]$$

$$\frac{1}{2} a\left[\frac{3}{2}, \frac{3}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{1}{2} \sqrt{3} a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

$$\psi[2, 0, 3/2, 1/2]$$

$$\frac{a\left[\frac{3}{2}, \frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right]}{\sqrt{2}} + \frac{a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]}{\sqrt{2}}$$

$$\psi[2, -1, 3/2, 1/2]$$

$$\frac{1}{2} \sqrt{3} a\left[\frac{3}{2}, -\frac{1}{2}\right] b\left[\frac{1}{2}, -\frac{1}{2}\right] + \frac{1}{2} a\left[\frac{3}{2}, -\frac{3}{2}\right] b\left[\frac{1}{2}, \frac{1}{2}\right]$$

APPENDIX II

Claculation of the ratio α/β

$$|j = l + 1/2, m\rangle = \alpha |m_l = m - 1/2, m_s = 1/2\rangle + \beta |m_l = m + 1/2, m_s = -1/2\rangle$$

$$|j = l - 1/2, m\rangle = \gamma |m_l = m - 1/2, m_s = 1/2\rangle + \delta |m_l = m + 1/2, m_s = -1/2\rangle$$

$$\hat{J}_- |j, m\rangle = \hbar \sqrt{(j+m)(j-m+1)} |j, m-1\rangle$$

$$\hat{J}_+ |j, m\rangle = \hbar \sqrt{(j-m)(j+m+1)} |j, m+1\rangle$$

$$\hat{S}_+ |-\rangle = \hbar |+\rangle$$

$$\hat{S}_-|+\rangle = \hbar|-\rangle$$

$$\hat{J}_+|j, m\rangle = \hbar\sqrt{(j-m)(j+m+1)}|j, m+1\rangle$$

$$\hat{J}^2|j, m\rangle = \hbar^2 j(j+1)|j, m\rangle$$

We demand that

$$\hat{J}^2\left|j = l + \frac{1}{2}, m\right\rangle = \hbar^2 j(j+1)\left|j = l + \frac{1}{2}, m\right\rangle = \hbar^2 \left(l + \frac{1}{2}\right)\left(l + \frac{3}{2}\right)\left|j = l + \frac{1}{2}, m\right\rangle$$

where

$$\left|j = l + \frac{1}{2}, m\right\rangle = \alpha\left|m_l = m - \frac{1}{2}, m_s = \frac{1}{2}\right\rangle + \beta\left|m_l = m + \frac{1}{2}, m_s = -\frac{1}{2}\right\rangle.$$

Here we note that

$$\hat{J} = \hat{L} + \hat{S}$$

$$\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + 2\hat{L} \cdot \hat{S} = \hat{L}^2 + \hat{S}^2 + 2\hat{L}_z \cdot \hat{S}_z + (\hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+)$$

Then we get

$$\begin{aligned} \hat{J}^2 \alpha \left| m_l = m - \frac{1}{2}, m_s = \frac{1}{2} \right\rangle &= \hbar^2 \alpha \left[l(l+1) + \frac{3}{4} + \left(m - \frac{1}{2}\right) + \hat{L}_+ \hat{S}_- \right] \left| m_l = m - \frac{1}{2}, m_s = \frac{1}{2} \right\rangle \\ &= \hbar^2 \alpha \left[l(l+1) + \frac{3}{4} + \left(m - \frac{1}{2}\right) \right] \left| m_l = m - \frac{1}{2}, m_s = \frac{1}{2} \right\rangle \\ &\quad + \hbar^2 \alpha \sqrt{\left(l + m + \frac{1}{2}\right)\left(l - m + \frac{1}{2}\right)} \left| m_l = m + \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle \\ \hat{J}^2 \beta \left| m_l = m + \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle &= \hbar^2 \beta \left[l(l+1) + \frac{3}{4} - \left(m + \frac{1}{2}\right) + \hat{L}_- \hat{S}_+ \right] \left| m_l = m + \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle \\ &= \hbar^2 \beta \left[l(l+1) + \frac{3}{4} - \left(m + \frac{1}{2}\right) \right] \left| m_l = m + \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle \\ &\quad + \hbar^2 \beta \sqrt{\left(l + m + \frac{1}{2}\right)\left(l - m + \frac{1}{2}\right)} \left| m_l = m - \frac{1}{2}, m_s = \frac{1}{2} \right\rangle \end{aligned}$$

or

$$\begin{aligned}
& \hbar^2(l + \frac{1}{2})(l + \frac{3}{2})[\alpha \left| m_l = m - \frac{1}{2}, m_s = \frac{1}{2} \right\rangle + \beta \left| m_l = m + \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle] \\
&= \hbar^2 \alpha [l(l+1) + \frac{3}{4} + (m - \frac{1}{2})] \left| m_l = m - \frac{1}{2}, m_s = \frac{1}{2} \right\rangle \\
&+ \hbar^2 \alpha \sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})} \left| m_l = m + \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle \\
&+ \hbar^2 \beta [l(l+1) + \frac{3}{4} - (m + \frac{1}{2})] \left| m_l = m + \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle \\
&+ \hbar^2 \beta \sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})} \left| m_l = m - \frac{1}{2}, m_s = \frac{1}{2} \right\rangle
\end{aligned}$$

or

$$\begin{aligned}
& [\alpha \sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})} - \beta(l + m + \frac{1}{2})] \left| m_l = m - 1/2, m_s = \frac{1}{2} \right\rangle \\
&+ [\beta \sqrt{(l + m + \frac{1}{2})(l - m + \frac{1}{2})} - \alpha(l - m + \frac{1}{2})] \left| m_l = m + 1/2, m_s = -\frac{1}{2} \right\rangle \\
&= 0
\end{aligned}$$

Then we have

$$\frac{\alpha}{\beta} = \sqrt{\frac{l + m + \frac{1}{2}}{l - m + \frac{1}{2}}}$$