Young tableau

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For more than two identical particles, we can determine the character of the symmetry of the total wave functions using the law of the addition of the angular momentum with the use of the Clebsch-Gordan co-efficient (both for orbital angular momentum and spin angular momentum). As the number of identical particles increases, such a calculation becomes much more complicated than we expect. To determine the ground state of the system, we do not have to obtain the exact form of the wave function. To this end, we just apply the Hund's rule. Here we discuss the Young tabeleau. Using this scheme, we can easily determine the symmetric character of the excited states as well as the ground state among the symmetric state, the mixed state, and the anti-symmetric state.

1 Two spin 1/2 particles

First we consider the case of two identical spin 1/2 particles. Using the Clebsch-Gordan co-efficient, we get the states as follows.

$$D_{1/2} \times D_{1/2} = D_1 + D_0$$

(i) $j = 1$ (spin triplet): symmetric states

$$\left| j = 1, m = 1 \right\rangle = \left| + + \right\rangle,$$

$$\left| 1, 0 \right\rangle = \frac{1}{\sqrt{2}} (\left| + - \right\rangle + \left| - + \right\rangle),$$

$$\left| 1, -1 \right\rangle = \left| - - \right\rangle.$$

(ii) $j = 0$ (singlet): anti-symmetric state

$$\left| j = 0, m = 0 \right\rangle = \frac{1}{\sqrt{2}} [(\left| + - \right\rangle - \left| - + \right\rangle].$$

2 Young tableau-I

We use the Young's tableau for the above problem. The spin state of an individual electron is to be represented by a box. A single box represents a doublet

$\boxed{1}$ spin up $\left| + \right\rangle$

$\boxed{2}$ spin down $\left| - \right\rangle$

$\begin{array}{|c|c|}\hline & \\ \hline\end{array}$: $\underset{m=1}{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}}$, $\underset{m=0}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}}$, $\underset{m=-1}{\begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}}$ symmetric tableau (spin triplet; S = 1)

$\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$: $\underset{m = 0}{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}}$ antisymmetric tableau (spin singlet, S = 0)

((**Rule**))

We do not consider

$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}$

When we put boxes horizontally, symmetry is understood. So we deduce an important rule. Double counting is avoided if we require that the number (label) not decrease going from the left to the right. Similarly, to eliminate the unwanted symmetry states, we require the number (label) to increase as we go down.

General rule:

In drawing Young tableau, going from left to right the number cannot decrease; going down the number must increase.

3 Three electrons with spin 1/2

Next we consider the case of three identical spin 1/2 particles. Using the Clebsch-Gordan co-efficient, we get the states as follows.

$$D_{1/2} \text{ x } D_{1/2} \text{ x } D_{1/2} = (D_1+D_0) \text{ x } D_{1/2} = D_{3/2}+D_{1/2}+D_{1/2}$$

(i) $j = 3/2$

$$\left| j=\frac{3}{2}, m=\frac{3}{2}\right\rangle = \left|+++\right\rangle$$

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}[\left|++-\right\rangle + \left|+-+\right\rangle + \left|-++\right\rangle]$$

$$\left|\frac{3}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}[\left|+--\right\rangle + \left|-+-\right\rangle + \left|--+\right\rangle]$$

$$\left|\frac{3}{2},-\frac{3}{2}\right\rangle = \left|---\right\rangle$$

(ii) $j = 1/2$

$$\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}[-\left|-++\right\rangle + 2\left|++-\right\rangle - \left|+-+]\right\rangle$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}[\left|+--\right\rangle + \left|-+-\right\rangle - 2\left|--+\right\rangle]$$

(iii) $j = 1/2$

$$\left|j=\frac{1}{2},m=\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}[\left|+-+\right\rangle - \left|-++\right\rangle]$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}[\left|+--\right\rangle - \left|-+-\right\rangle],$$

4 Young tableaux II

We use the Young's tableau for the identical 3 spin 1/2 particles. The result is as follows. The symmetric state is denoted as

(i) $j = 3/2$

$\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array}$:symmetric state

j = 3/2, m = 3/2, 1/2, -1/2, -3/2

$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline \end{array}$, $\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline \end{array}$

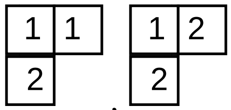
m=3/2 m=1/2 m=-1/2 m=-3/2

What about the totally antisymmetric states? We may try vertical tableau like

$\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array}$ $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 2 \\ \hline \end{array}$: forbidden state

But these are illegal, because the numbers must increase as we go down. So the anti-symmetric state is forbidden.

(ii) $j = 1/2$

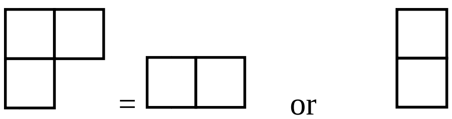


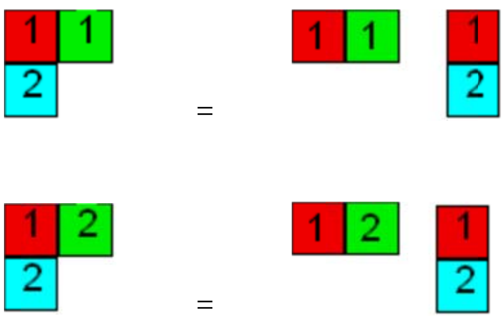
m=1/2 , m=-1/2

which is called the mixed state.

5 Mixed state

We define a mixed symmetry tableau. The mixed state is orthogonal to the symmetric state and anti-symmetric state.





(a)
We consider a mixed state,

$$|\psi_1\rangle = |+--\rangle + |--+\rangle = (|+\rangle_1|-\rangle_3 + |-\rangle_1|+\rangle_3)|-\rangle_2 \tag{1}$$

satisfies symmetry under 1↔3, but it is neither symmetric nor anti-symmetric with respect to 2 ↔ 3 (or 1 ↔ 2).

$$|\psi_2\rangle = |--+\rangle + |-+-\rangle = (|-\rangle_2|+\rangle_3 + |+\rangle_3|-\rangle_2)|-\rangle_1 \tag{2}$$

satisfies symmetry under 2↔3, but it is neither symmetric nor anti-symmetric with respect to 1 ↔ 2 (or 1 ↔ 3).

Subtraction: Eq.(1) - Eq.(2):

$$|\psi_1\rangle - |\psi_2\rangle = |+--\rangle - |-+-\rangle = (|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2)|-\rangle_3 \qquad (3)$$

This satisfies anti-symmetry under 1↔2, but no longer have the original symmetry under 1↔2.

This corresponds to

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}[|+--\rangle - |-+-\rangle]$$

which is obtained from the Clebsch-Gordan co-efficient.

(b)

$$|\psi_3\rangle = |+--\rangle - |--+\rangle = (|+\rangle_1|-\rangle_3 - |-\rangle_1|+\rangle_3)|-\rangle_2 \qquad (4)$$

This satisfies anti-symmetric under 1↔3.

$$|\psi_4\rangle = |-+-\rangle - |--+\rangle = (|+\rangle_2|-\rangle_3 - |-\rangle_2|+\rangle_3)|-\rangle_1 \qquad (5)$$

This satisfies anti-symmetric under 2↔3. Addition: Eq.(4) + Eq.(5):

$$|\psi_3\rangle + |\psi_4\rangle = |+--\rangle + |-+-\rangle - 2|--+\rangle \qquad (6)$$

which is the same as the state given by

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{6}}[|+--\rangle + |-+-\rangle - 2|--+]\rangle$$

which is obtained from the Clebsch-Gordan coefficient.

(c)

$$|\psi_5\rangle = |+-+\rangle + |++-\rangle = (|+\rangle_2|-\rangle_3 + |-\rangle_2|+\rangle_3)|+\rangle_1 . \tag{7}$$

This satisfies symmetric under 2↔3

$$|\psi_6\rangle = |++-\rangle + |-++\rangle = (|+\rangle_1|-\rangle_3 + |-\rangle_1|+\rangle_3)|+\rangle_2 . \tag{8}$$

This satisfies symmetric under 1↔3.

Addition: Eq.(7) - Eq.(8). We have

$$|\psi_5\rangle - |\psi_6\rangle = |+-+\rangle - |-++\rangle . \tag{9}$$

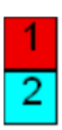
This satisfies anti-symmetric under 1↔2.

$$\left| j = \frac{1}{2}, m = \frac{1}{2} \right\rangle = \frac{1}{\sqrt{2}}[|+-+\rangle - |-++\rangle]$$

(d)

$$|\psi_7\rangle = |++-\rangle - |+-+\rangle = (|+\rangle_2|-\rangle_3 - |-\rangle_2|+\rangle_3)|+\rangle_1 . \tag{10}$$

This satisfies anti-symmetric under 2↔3



$$|\psi_8\rangle = |++-\rangle - |-++\rangle = (|+\rangle_1|-\rangle_3 - |-\rangle_1|+\rangle_3)|+\rangle_2 . \tag{11}$$

This satisfies anti-symmetric under 1↔3.

Subtraction: Eq.(10) - Eq.(11)

$$-|\psi_7\rangle + |\psi_8\rangle = |+-+\rangle - |-++\rangle . \qquad (12)$$

This satisfies antisymmetic under 1↔2.

Addition: Eq.(10) + Eq.(11)

$$|\psi_7\rangle + |\psi_8\rangle = 2|++-\rangle - |+-+\rangle - |-++\rangle$$

or

$$\left| j = \frac{1}{2}, m = \frac{1}{2} \right\rangle = \frac{1}{\sqrt{6}}[-|-++\rangle + 2|++-\rangle - |+-+\rangle]$$

6 4 electrons with spin 1/2

We consider the case of four identical spin 1/2 particles. Using the Clebsch-Gordan co-efficient, we get the states as follows.

$$D_{1/2} \text{ x } D_{1/2} \text{ x } D_{1/2} \text{ x } D_{1/2} = (D_{3/2} + D_{1/2} + D_{1/2}) \text{ x } D_{1/2}$$

$$= (D_2 + D_1) + (D_1 + D_0) + (D_1 + D_0)$$

(i) $j = 2$

$$|j = 2, m = 2\rangle = |++++\rangle$$

$$|2,1\rangle = \frac{1}{2}[|+++-\rangle + |++-+\rangle + |+-++\rangle + |-+++\rangle]$$

$$|2,0\rangle = \frac{1}{\sqrt{6}}[|-++-\rangle + |++--\rangle + |+-+-\rangle + |+--+\rangle + |-+-+\rangle + |--++\rangle]$$

$$|2,-1\rangle = \frac{1}{2}[|+---\rangle + |-+--\rangle + |--+-\rangle + |---+\rangle]$$

$$|2,-2\rangle = |----\rangle$$

(ii) $j = 1$

$$|j=1, m=1\rangle = -\frac{1}{2\sqrt{3}}[|-+++\rangle + |++-+\rangle + |+-++\rangle] + \frac{\sqrt{3}}{2}|+++-\rangle]$$

$$|1,0\rangle = \frac{1}{\sqrt{6}}[|-++-\rangle + |++--\rangle + |+-+-\rangle] - \frac{1}{\sqrt{6}}[|+--+\rangle + |--++\rangle + |-+-+\rangle]$$

$$|1,-1\rangle = \frac{1}{2\sqrt{3}}[|+---\rangle + |-+--\rangle + |--+-\rangle] - \frac{\sqrt{3}}{2}|---+\rangle]$$

(iii) $j = 1$

$$|j=1, m=1\rangle = \frac{1}{\sqrt{6}}[-|-+++\rangle - |+-++\rangle + 2|++-+\rangle]$$

$$|1,0\rangle = \frac{1}{2\sqrt{3}}[-|-++-\rangle + 2|++--\rangle - |+-+-\rangle + |+--+\rangle - 2|--++\rangle + |-+-+\rangle]$$

$$|1,-1\rangle = \frac{1}{\sqrt{6}}[|+---\rangle + |-+--\rangle - 2|--+-\rangle]$$

$$|j=0, m=0\rangle = \frac{1}{2\sqrt{3}}[-|-++-\rangle + 2|++--\rangle - |+-+-\rangle$$
$$-|+--+\rangle - 2|--++\rangle + |-+-+\rangle]$$

(iv) $j = 1$

$$|j=1, m=1\rangle = \frac{1}{\sqrt{2}}[|+-++\rangle - |-+++\rangle]$$

$$|1,0\rangle = \frac{1}{2}[|+-+-\rangle - |-++-\rangle + |+--+\rangle - |-+-+\rangle]$$

$$|1,-1\rangle = \frac{1}{\sqrt{2}}[|+---\rangle - |-+--\rangle]$$

(v) $j = 0$

$$|\, j=0, m=0\rangle = \frac{1}{2}[|+-+-\rangle - |-++-\rangle - |+--+\rangle + |-+-+\rangle]$$

7 Young tableaux III

We apply the Young's tableau for the 4 identical spin 1/2 particles The results are as follows. Only j = 2 state is symmetric upon the interchange of the positions.

(i) j = 2 symmetric state

$$\underset{m=2}{\begin{ytableau} 1 & 1 & 1 & 1 \end{ytableau}}, \quad \underset{m=1}{\begin{ytableau} 1 & 1 & 1 & 2 \end{ytableau}}, \quad \underset{m=0}{\begin{ytableau} 1 & 1 & 2 & 2 \end{ytableau}}, \quad \underset{m=-1}{\begin{ytableau} 1 & 2 & 2 & 2 \end{ytableau}}$$

$$\underset{m=-2}{\begin{ytableau} 2 & 2 & 2 & 2 \end{ytableau}}$$

(ii) j = 1 mixed state

$$\underset{m=1}{\begin{ytableau} 1 & 1 & 1 \\ 2 \end{ytableau}}, \quad \underset{m=0}{\begin{ytableau} 1 & 1 & 2 \\ 2 \end{ytableau}}, \quad \underset{m=-1}{\begin{ytableau} 1 & 2 & 2 \\ 2 \end{ytableau}}$$

(iii) j = 0 mixed state

$$\underset{m=0}{\begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau}}$$

8 Simplified model for spin 1/2

Now we introduce a simple way to build a Young diagram.

(a) Two spin 1/2 particles

$$\begin{ytableau} \; \end{ytableau} \otimes \begin{ytableau} \; \end{ytableau}$$

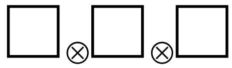
2 x 2 = 4 states

$$\begin{ytableau} \; \end{ytableau} \otimes \begin{ytableau} \; \end{ytableau} = \underset{\text{triplet}}{\begin{ytableau} \; & \; \end{ytableau}} \oplus \underset{\text{singlet}}{\begin{ytableau} \; \\ \; \end{ytableau}}$$

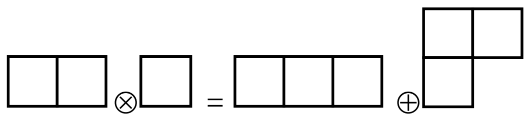
2 x 2 = 3 + 1 states

$D_{1/2} \times D_{1/2} = D_1 + D_0$

(b) Three spin 1/2 particles



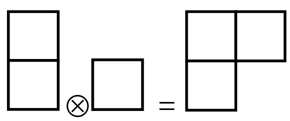
2 x 2 x 2 = 8



triplet doublet quartet doublet

3 x 2 = 4+2

$D_1 \times D_{1/2} = D_{3/2} + D_{1/2}$

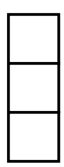


singlet doublet doublet

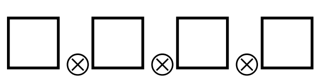
1 x 2 = 2

$D_0 \times D_{1/2} = D_{1/2}$

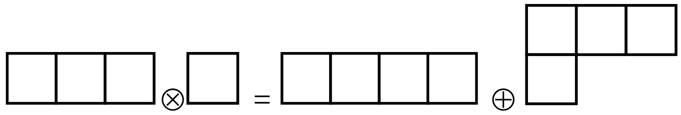
((**Note**))

 is forbidden.

(c) Four particles with 1/2

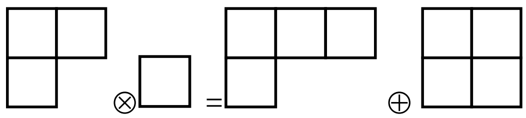


2 x 2 x 2 x 2 = 16 states



quartet

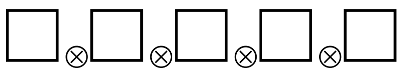
$D_{3/2}$ x $D_{1/2}$ =D_2 + D_1



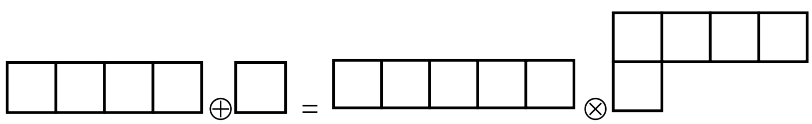
doublet

$D_{1/2}$ x $D_{1/2}$ =D_1 + D_0

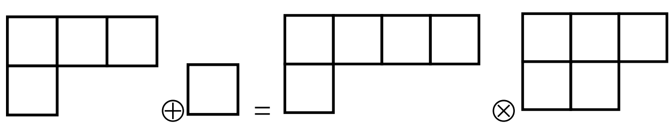
(d) 5 spin 1/2 particles



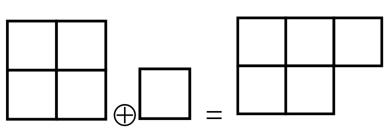
2 x 2 x 2 x 2 x 2 = 32



D_2 x $D_{1/2}$ =$D_{5/2}$ + $D_{3/2}$



D_1 x $D_{1/2}$ = $D_{3/2}$+ $D_{1/2}$



D_0 x $D_{1/2}$ =$D_{1/2}$

9 Particles with *l* = 1; *m* = 1, 0, -1 (*p* electrons)

The labels 1, 2, and 3 may stand for the magnetic quantum number of *p*-orbitals (*l*= 1 particle).

$\square$: $\boxed{1}$, $\boxed{2}$, $\boxed{3}$

m=1 m=0 m=-1

10 Two particles with spin 1: 3x3 = 9 states

For j = 1

$$\begin{array}{|c|}\hline \ \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \ \\ \hline\end{array} = \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}, \quad 3 \text{ x } 3 = 6 + 3$$

D_1, D_1 D_2, D_0 D_1

The horizontal tableau has six states: the tableau is to be broken down into j =- 2 (multiplicity 5) and j = 0 (multiplicity1); both of which are symmetric.

The vertical tableau corresponds to an antisymmetric j = 1 state.

(i) Symmetric

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} : \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 3 \\ \hline\end{array},$$

6 states (j = 2 and 0)

m=2 m=1 m=0 m=0 m=-1 m=-2

(ii) Anti-symmetric

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} : \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}, \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array},$$

3 states (j=1)

m=1 m=0 m=-1

11 Three particles with *l* = 1

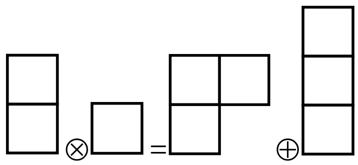
The Young's diagram for a system of three particles are obtained by adding to the diagrams (1) one cell in every possible way. The results may be written as the symbolic equations,

3 x 3 x 3 = 27 states

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \ \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$$

: 6 x 3=7+3 +5+3

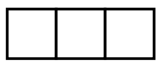
D_2, D_0 x D_1 D_3, D_1 D_2, D_1

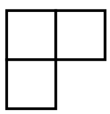


: 3 x 3=8 +1

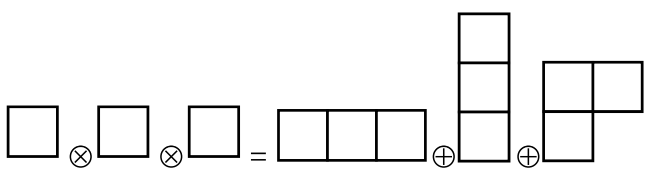
D1 x D1 D_2, D_1 D_0

((**Note**))

 contains both j = 3 (7 states) and j = 1 (3 states).

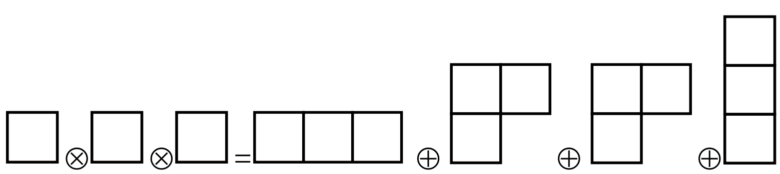
As for  with eight possibilities altogether, the argument is more involved, but we note that this 8 cannot be broken into 7 + 1 because 7 is totally symmetric, while 1 is totally anti-symmetric when we know that 8 is of mixed symmetry. So the only possibility is 8 = 5 + 3 - in other words j = 2 and j = 1.

Finally, therefore



$D_1 \text{ x } D_1 \text{ x } D_1 = D_3 + 2D_2 + 3D_1 + D_0$

or



States 3 3 3 7+3 5+3 5+3 1

In terms of angular momentum states, we have

j = 3 (7 states)	once	(totally symmetric)
j = 2 (5 states))	twice	(both mixed symmetry)
j = 1 (3 states)	three times	(one totally symmetric, two mixed symmetry)
j = 0 (1 state)	once	(totally antisymmetric).

$$\underset{m\,=\,3}{\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}},\quad \underset{m\,=\,2}{\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}},\quad \underset{m\,=\,1}{\begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array}}$$

$$\underset{m\,=\,1}{\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}},\quad \underset{m\,=0}{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}}\quad \underset{m\,=\,-1}{\begin{array}{|c|c|c|}\hline 1&3&3\\\hline\end{array}}$$

$$\underset{m\,=\,0}{\begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array}},\quad \underset{m\,=\,-1}{\begin{array}{|c|c|c|}\hline 2&2&3\\\hline\end{array}},\quad \underset{m\,=\,-2}{\begin{array}{|c|c|c|}\hline 2&3&3\\\hline\end{array}}$$

$$\underset{m\,=\,-3}{\begin{array}{|c|c|c|}\hline 3&3&3\\\hline\end{array}}$$

$$\underset{m\,=\,0}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}}$$

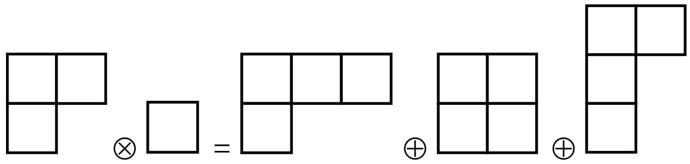
$$\underset{m\,=2}{\begin{array}{|c|c|}\hline 1&1\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}}\quad \underset{m\,=\,1}{\begin{array}{|c|c|}\hline 1&1\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}}\quad \underset{m\,=\,1}{\begin{array}{|c|c|}\hline 1&2\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}}\quad \underset{m\,=\,0}{\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}}$$

$$\underset{m\,=\,0}{\begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}}\quad \underset{m\,=\,-1}{\begin{array}{|c|c|}\hline 1&3\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}}\quad \underset{m\,=\,-1}{\begin{array}{|c|c|}\hline 2&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}}\quad \underset{m\,=\,-2}{\begin{array}{|c|c|}\hline 2&3\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}}$$

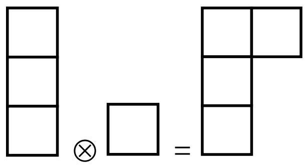
12 Four particles with *l*= 1 (Landau) $(p)^4$

The Young's diagram for a system of four particles are obtained by adding to the diagrams each cell in every possible way. The results may be written as the symbolic equations,

$$\underset{D_3,\ D_1}{\begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline\end{array}} \otimes \underset{D_1}{\begin{array}{|c|}\hline \ \\\hline\end{array}} = \underset{D_4,\ D_3,\ D_2}{\begin{array}{|c|c|c|c|}\hline \ &\ &\ &\ \\\hline\end{array}} \oplus \underset{D_2,\ D_1,\ D_0}{\begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline \ &\multicolumn{2}{c}{}\\\cline{1-1}\end{array}}$$

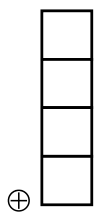


D_1, D_2 D_1 $D_1, D_2, D_3,$ D_0, D_2 D_1



D_0 D_1 D_1

((**Note**))

 is forbidden.

1 1 1 1, m = 4	1 1 1 2, m = 3	1 1 1 3, m = 2
1 1 2 2, m = 2	1 1 2 3, m = 1	1 1 3 3, m = 0
1 2 2 2, m = 1	1 2 2 3, m = 0	1 2 3 3, m = -1
1 3 3 3, m = -2	2 2 2 2, m = 0	2 2 2 3, m = -1
2 2 3 3, m = -2	2 3 3 3, m = -3	3 3 3 3, m = -4

13. Symmetric states and antisymmetric states for two identical particles

The spin states of the two identical particles, each of spin s, can be separated into symmetric states and antisymmetric states. For s = 1/2, the $(2s+1)(2s+1)=4$ states consist of three symmetric states and 1 antisymmetric one. In general if one has two variables, each taking on n values, the number of anti-symmetrical combinations is $\frac{1}{2}n(n-1)$, and the number of symmetrical ones is $\frac{1}{2}n(n-1)+n=\frac{1}{2}n(n+1)$ correctly adding to n^2. Thus the fraction of spin states are symmetrical or anti-symmetric is (n = 2s+1)

$$\frac{\frac{1}{2}n(n+1)}{n^2}=\frac{n+1}{2n}=\frac{s+1}{2s+1}>\frac{1}{2} \qquad \text{for the symmetric states}$$

$$\frac{\frac{1}{2}n(n-1)}{n^2}=\frac{n-1}{2n}=\frac{s}{2s+1}<\frac{1}{2} \qquad \text{for the anti-symmetric states}$$

where s is an integer for boson, and is a half integer for fermion (**Schwinger**). This concept will be applied to the scattering of identical particles.

Here we show the proof of this theorem using the Young's tableau.

(a) Two identical particles with s = 1/2

The total number of states is 3 x 3 = 9

$$\boxed{1} \qquad \boxed{2}$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle, \left|\frac{1}{2},-\frac{1}{2}\right\rangle,$$

Symmetric states (3 states)

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array}$$

Anti-symmetric state (1 state)

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

(b) Two identical particles with *s* = 1

The total number of states is 3 x 3 = 9

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$

$|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$

Symmetric states (6 states)

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 3 \\ \hline \end{array}$$

Anti-symmetric states (3 states)

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}$$

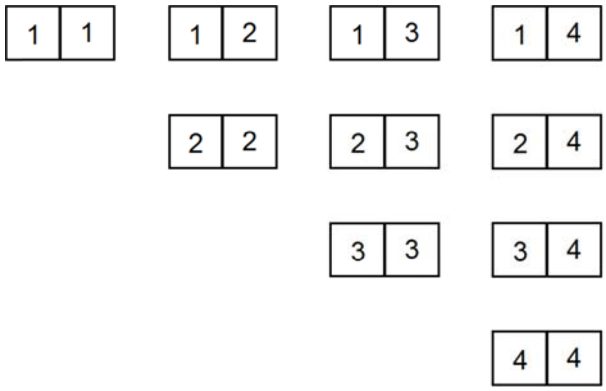
(c) Two identical particles with spin *s* = 3/2

The total number of states is 4 x 4 = 16.

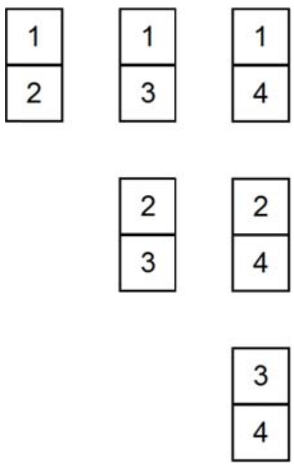
$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$

$\left|\frac{3}{2},\frac{3}{2}\right\rangle$, $\left|\frac{3}{2},\frac{1}{2}\right\rangle$, $\left|\frac{3}{2},-\frac{1}{2}\right\rangle$, $\left|\frac{3}{2},-\frac{3}{2}\right\rangle$,

Symmetric states (10 states)



Anti-symmetric states (6 states)



(d) Two identical particles with spin s = 2.
The total number of states is 5 x 5 = 25.

$|2,2\rangle$, $|2,1\rangle$, $|2,0\rangle$, $|2,-1\rangle$, $|2,-2\rangle$

Symmetric states (15 states)

$\boxed{1}\boxed{1}$ $\boxed{1}\boxed{2}$ $\boxed{1}\boxed{3}$ $\boxed{1}\boxed{4}$ $\boxed{1}\boxed{5}$

$\boxed{2}\boxed{2}$ $\boxed{2}\boxed{3}$ $\boxed{2}\boxed{4}$ $\boxed{2}\boxed{5}$

$\boxed{3}\boxed{3}$ $\boxed{3}\boxed{4}$ $\boxed{3}\boxed{5}$

$\boxed{3}\boxed{4}$ $\boxed{4}\boxed{5}$

$\boxed{5}\boxed{5}$

Antisymmetric states (10 states)

$\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}$ $\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}$ $\begin{array}{|c|}\hline 1\\ \hline 4\\ \hline\end{array}$ $\begin{array}{|c|}\hline 1\\ \hline 5\\ \hline\end{array}$

$\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}$ $\begin{array}{|c|}\hline 2\\ \hline 4\\ \hline\end{array}$ $\begin{array}{|c|}\hline 2\\ \hline 5\\ \hline\end{array}$

$\begin{array}{|c|}\hline 3\\ \hline 4\\ \hline\end{array}$ $\begin{array}{|c|}\hline 3\\ \hline 5\\ \hline\end{array}$

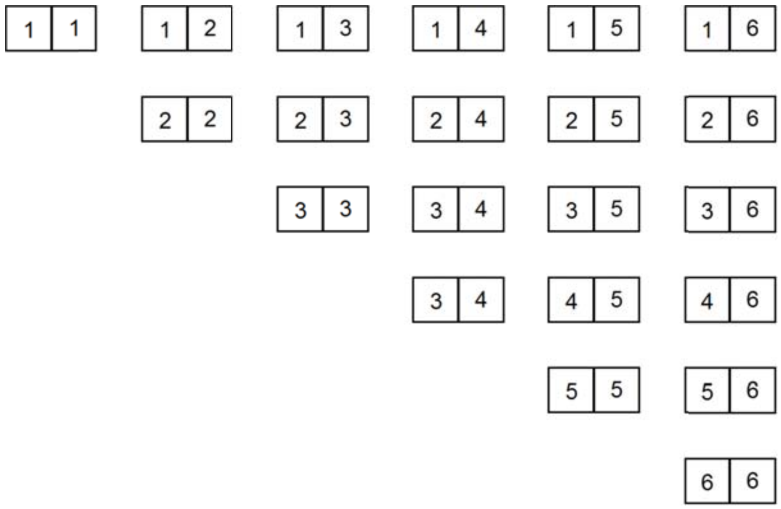
$\begin{array}{|c|}\hline 4\\ \hline 5\\ \hline\end{array}$

(e) Two identical particles with spin s = 5/2.
The total number of states is 6 x 6 = 36.

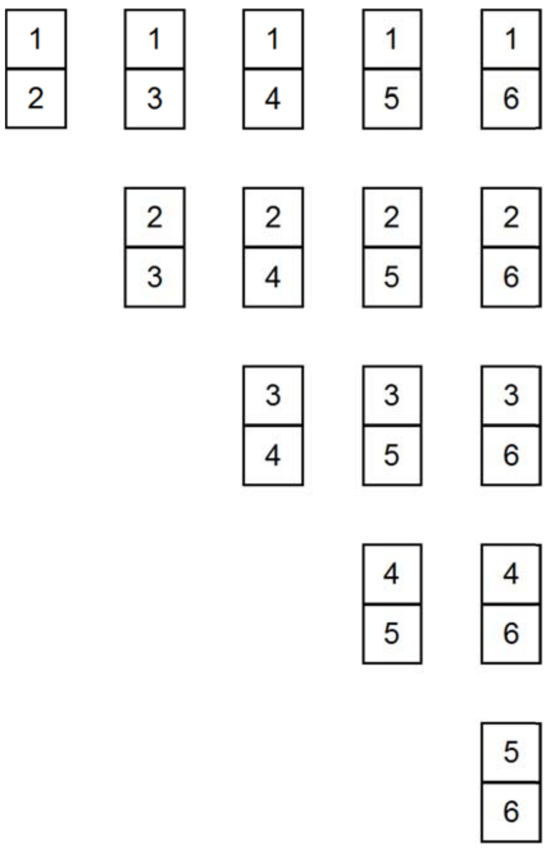
$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$

$$\left|\frac{5}{2},\frac{5}{2}\right\rangle, \left|\frac{5}{2},\frac{3}{2}\right\rangle, \left|\frac{5}{2},\frac{1}{2}\right\rangle, \left|\frac{5}{2},-\frac{1}{2}\right\rangle, \left|\frac{5}{2},-\frac{3}{2}\right\rangle, \left|\frac{5}{2},-\frac{5}{2}\right\rangle$$

Symmetric states (21 states)



Antisymmetric states (15 states)



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APPENDIX

Orbital angular momentum and spin angular momentum for three and four spin s = 1/2 particles and three l = 1 particles

A.1. Definition of symmetrizer and antisymmetrizer

For the three particles (N = 3), we define the symmetrizer and anti-symmetrizer as

$$\hat{S} = \frac{1}{6}[\hat{1} + \hat{P}_{12} + \hat{P}_{23} + \hat{P}_{31} + \hat{P}_{123} + \hat{P}_{132}]$$

and

$$\hat{A} == \frac{1}{6}[\hat{1} - \hat{P}_{12} - \hat{P}_{23} - \hat{P}_{31} + \hat{P}_{123} + \hat{P}_{132}]$$

where

$$\hat{P}_{123} = \hat{P}_{12}\hat{P}_{23}, \qquad \hat{P}_{132} = \hat{P}_{12}\hat{P}_{13}$$

$$\hat{S} + \hat{A} = \frac{1}{3}(\hat{1} + \hat{P}_{123} + \hat{P}_{132}) \neq \hat{1}$$

A.2. Spin state of three particles with spin 1/2

$$\begin{aligned} D_{1/2} \times D_{1/2} \times D_{1/2} &= (D_1 + D_0) \times D_{1/2} \\ &= D_{3/2} + 2D_{1/2} \end{aligned}$$

$$\alpha = \left|\frac{1}{2}, \frac{1}{2}\right\rangle = \left|+\right\rangle, \qquad \alpha = \left|\frac{1}{2}, \frac{1}{2}\right\rangle = \left|+\right\rangle$$

j = 3/2 (symmetric), j = 1/2, j = 1/2,

$$\left| j = 3/2, m = 3/2 \right\rangle = \alpha\alpha\alpha$$

$$\left| j = 3/2, m = 1/2 \right\rangle = \frac{1}{\sqrt{3}}(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha)$$

$$\left| j = 3/2, m = -1/2 \right\rangle = \frac{1}{\sqrt{3}}(\alpha\beta\beta + \beta\alpha\beta + \beta\beta\alpha)$$

$$\left| j = 3/2, m = -3/2 \right\rangle = \beta\beta\beta$$

$$\left| j = 1/2, m = 1/2 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\alpha\beta - \beta\alpha\alpha)$$

$$\left| j=1/2, m=-1/2 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\beta\beta - \beta\beta\alpha)$$

$$\left| j=1/2, m=1/2 \right\rangle = \frac{1}{\sqrt{6}}(\alpha\alpha\beta - 2\alpha\beta\alpha + \beta\alpha\alpha)$$

$$\left| j=1/2, m=-1/2 \right\rangle = \frac{1}{\sqrt{6}}(\alpha\beta\beta - 2\beta\alpha\beta + \beta\beta\alpha)$$

A.3. Three particles with the angular momentum $l=1$.

We now consider the state of two particles with the angular momentum $\hbar$.

$$D_1 \times D_1 \times D_1 = (D_2 + D_1 + D_0) \times D_1 = D_3 + 2D_2 + 3D_1 + D_0$$

$$\alpha = \left|1,1\right\rangle, \qquad \beta = \left|1,0\right\rangle, \qquad \gamma = \left|1,-1\right\rangle$$

j = 3, 2, 1, and 0

$$\left| j=3, m=3 \right\rangle = \alpha\alpha\alpha$$

$$\left| j=2, m=2 \right\rangle = \frac{1}{\sqrt{3}}(\alpha\alpha\beta + \alpha\beta\alpha + \beta\alpha\alpha)$$

$$\left| j=3, m=1 \right\rangle = \frac{1}{\sqrt{15}}(\alpha\alpha\gamma + 2\alpha\beta\beta + \alpha\gamma\alpha + 2\beta\alpha\beta + 2\beta\beta\alpha + \gamma\alpha\alpha)$$

$$\left| j=3, m=0 \right\rangle = \frac{1}{\sqrt{10}}(\alpha\beta\gamma + \alpha\gamma\beta + \beta\alpha\gamma + 2\beta\beta\beta + \beta\gamma\alpha + \gamma\alpha\beta + \gamma\beta\alpha)$$

$$\left| j=3, m=-1 \right\rangle = \frac{1}{\sqrt{15}}(\alpha\gamma\gamma + 2\beta\beta\gamma + 2\beta\gamma\beta + \gamma\alpha\gamma + 2\gamma\beta\beta + \gamma\gamma\alpha)$$

$$\left| j=3, m=-2 \right\rangle = \frac{1}{\sqrt{3}}(\beta\gamma\gamma + \gamma\beta\gamma + \gamma\gamma\beta)$$

$$\left| j=3, m=-3 \right\rangle = \gamma\gamma\gamma$$

$$\left| j=2, m=2 \right\rangle = \frac{1}{\sqrt{6}}(\alpha\alpha\beta - 2\alpha\beta\alpha + \beta\alpha\alpha)$$

$$\left| j=2, m=2 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\alpha\beta - \beta\alpha\alpha)$$

$$\left| j=2, m=1 \right\rangle = \frac{1}{2\sqrt{3}}(2\alpha\alpha\gamma + \alpha\beta\beta - \alpha\gamma\alpha + \beta\alpha\beta - 2\beta\beta\alpha - \gamma\alpha\alpha)$$

$$\left| j=2, m=1 \right\rangle = \frac{1}{2}(\alpha\beta\beta + \alpha\gamma\alpha - \beta\alpha\beta - \gamma\alpha\alpha)$$

$$\left| j=2, m=0 \right\rangle = \frac{1}{2\sqrt{2}3}(\alpha\beta\gamma + 2\alpha\gamma\beta - \beta\alpha\gamma + \beta\gamma\alpha - 2\gamma\alpha\beta - \gamma\beta\alpha)$$

$$\left| j=2, m=0 \right\rangle = \frac{1}{2}(\alpha\beta\gamma + \beta\alpha\gamma - \beta\gamma\alpha - \gamma\beta\alpha)$$

$$\left| j=2, m=-1 \right\rangle = \frac{1}{2\sqrt{3}}(2\alpha\gamma\gamma + \beta\beta\gamma + \beta\gamma\beta - \gamma\alpha\gamma - 2\gamma\beta\beta - \gamma\gamma\alpha)$$

$$\left| j=2, m=-1 \right\rangle = \frac{1}{2}(\beta\beta\gamma - \beta\gamma\beta + \gamma\alpha\gamma - \gamma\gamma\alpha)$$

$$\left| j=2, m=-2 \right\rangle = \frac{1}{\sqrt{6}}(\beta\gamma\gamma - 2\gamma\beta\gamma + \gamma\gamma\beta)$$

$$\left| j=2, m=-2 \right\rangle = \frac{1}{\sqrt{2}}(\beta\gamma\gamma - \gamma\gamma\beta)$$

$$\left| j=1, m=1 \right\rangle = \frac{1}{2\sqrt{15}}(\alpha\alpha\gamma - 3\alpha\beta\beta + 6\alpha\gamma\alpha + 2\beta\alpha\beta - 3\beta\beta\alpha + \gamma\alpha\alpha)$$

$$\left| j=1, m=1 \right\rangle = \frac{1}{2}(\alpha\alpha\gamma - \alpha\beta\beta + \beta\beta\alpha - \gamma\alpha\alpha$$

$$\left| j=1, m=1 \right\rangle = \frac{1}{\sqrt{3}}(\alpha\alpha\gamma - \beta\alpha\beta + \gamma\alpha\alpha)$$

$$\left| j=1, m=0 \right\rangle = \frac{1}{2\sqrt{10}}(\alpha\beta\gamma + \alpha\gamma\beta - 4\beta\alpha\gamma + 2\beta\beta\beta - 4\beta\gamma\alpha + \gamma\alpha\beta + \gamma\beta\alpha)$$

$$\left| j=1, m=0 \right\rangle = \frac{1}{2\sqrt{6}}(\alpha\beta\gamma - 3\alpha\gamma\beta + 2\beta\beta\beta - 3\gamma\alpha\beta + \gamma\beta\alpha)$$

$$\left| j=1, m=0 \right\rangle = \frac{1}{\sqrt{3}}(\alpha\beta\gamma - \beta\beta\beta + \gamma\beta\alpha)$$

$$\left| j=1, m=-1 \right\rangle = \frac{1}{2\sqrt{15}}(\alpha\gamma\gamma - 3\beta\beta\gamma + 2\beta\gamma\beta + 6\gamma\alpha\gamma - 3\gamma\beta\beta + \gamma\gamma\alpha)$$

$$\left| j=1, m=-1 \right\rangle = \frac{1}{2}(\alpha\gamma\gamma - \beta\beta\gamma + \gamma\beta\beta - \gamma\gamma\alpha)$$

$$\left| j=1, m=-1 \right\rangle = \frac{1}{\sqrt{3}}(\alpha\gamma\gamma - \beta\gamma\beta + \gamma\gamma\alpha)$$

$$\left| j=0, m=0 \right\rangle = \frac{1}{\sqrt{6}}(\alpha\beta\gamma - \alpha\gamma\beta - \beta\alpha\gamma + \beta\gamma\alpha + \gamma\alpha\beta - \gamma\beta\alpha)$$

A.4. Four particles with the angular momentum $S = 1/2$.

$$\left| j=2, m=2 \right\rangle = \alpha\alpha\alpha\alpha$$

$$\left| j=2, m=1 \right\rangle = \frac{1}{2}(\alpha\alpha\alpha\beta + \alpha\alpha\beta\alpha + \alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha)$$

$$\left| j=2, m=0 \right\rangle = \frac{1}{\sqrt{6}}(\alpha\alpha\beta\beta + \alpha\beta\alpha\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta + \beta\alpha\beta\alpha + \beta\beta\alpha\alpha)$$

$$\left| j=2, m=-1 \right\rangle = \frac{1}{2}(\alpha\beta\beta\beta + \beta\alpha\beta\beta + \beta\beta\alpha\beta + \beta\beta\beta\alpha)$$

$$\left| j=2, m=-2 \right\rangle = \beta\beta\beta\beta$$

$$\left| j=1, m=1 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\alpha\alpha\beta - \beta\alpha\alpha\alpha)$$

$$\left| j=1, m=0 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\alpha\beta\beta - \beta\beta\alpha\alpha)$$

$$\left| j=1, m=-1 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\beta\beta\beta - \beta\beta\beta\alpha)$$

$$\left| j=1, m=1 \right\rangle = \frac{1}{\sqrt{6}}(\alpha\alpha\alpha\beta - 2\alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha)$$

$$\left| j=1, m=0 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\beta\alpha\beta - \beta\alpha\beta\alpha)$$

$$\left| j=1, m=-1 \right\rangle = \frac{1}{\sqrt{6}}(\alpha\beta\beta\beta - 2\beta\beta\alpha\beta + \beta\beta\beta\alpha)$$

$$\left| j=1, m=1 \right\rangle = \frac{1}{2\sqrt{3}}(\alpha\alpha\alpha\beta - 3\alpha\alpha\beta\alpha + \alpha\beta\alpha\alpha + \beta\alpha\alpha\alpha)$$

$$\left| j=1, m=0 \right\rangle = \frac{1}{\sqrt{2}}(\alpha\beta\beta\alpha - \beta\alpha\alpha\beta)$$

$$\left| j=1, m=-1 \right\rangle = \frac{1}{2\sqrt{3}}(\alpha\beta\beta\beta - 3\beta\alpha\beta\beta + \beta\beta\alpha\beta + \beta\beta\beta\alpha)$$

$$\left| j=0, m=0 \right\rangle = \frac{1}{2}(\alpha\alpha\beta\beta - \alpha\beta\beta\alpha - \beta\alpha\alpha\beta + \beta\beta\alpha\alpha)$$

$$\left| j=0, m=0 \right\rangle = \frac{1}{2\sqrt{3}}(\alpha\alpha\beta\beta - 2\alpha\beta\alpha\beta + \alpha\beta\beta\alpha + \beta\alpha\alpha\beta - 2\beta\alpha\beta\alpha + \beta\beta\alpha\alpha)$$
