Addition of spin angular momentum for three spin-1/2 particles, using the Kronecker products and Clebsch-Gordan coefficients

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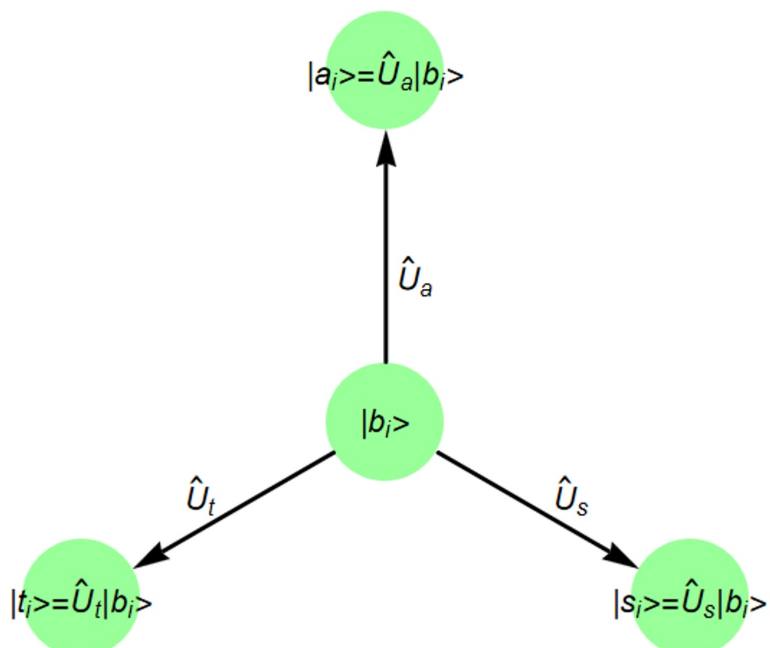
Here we discuss the eigenstates of three spin-1/2 particles by using two methods.

- **I. Clebsch-Gordan coefficient**
- **II. Method of Kronecker product**
- **III. Comparison of our results with Schiff's result**
- **IV. Comparison of our results with Tomonaga's result**
- **V. Conclusion**



The unitary operator Ua is obtained from both the Kronecker product and the Clebsch-Gordan coefficient (in this article). The unitary operator Us comes from the book of Schiff. The unitary operator Ut comes from the book of Tomonaga.
The ket |bi> is expressed as

$|a_i\rangle = \hat{U}_a |b_i\rangle$ (from the Kronecker product and Clebsch-Gordan)

$|b_1\rangle = |+z,+z,+z\rangle$, $|b_2\rangle = |+z,+z,-z\rangle$
$|b_3\rangle = |+z,-z,+z\rangle$, $|b_4\rangle = |+z,-z,-z\rangle$
$|b_5\rangle = |-z,+z,+z\rangle$, $|b_6\rangle = |+z,+z,-z\rangle$
$|b_7\rangle = |-z,-z,+z\rangle$, $|b_8\rangle = |-z,-z,-z\rangle$

$|a_i\rangle = \hat{U}_a |b_i\rangle$ (from the Kronecker product and Clebsch-Gordan)

$|s_i\rangle = \hat{U}_s |b_i\rangle$ (Schiff)

$\langle t_i | a_j\rangle = \langle b_i|\hat{U}_t^{+}\hat{U}_a|b_j\rangle$

$|t_i\rangle = \hat{U}_t |b_i\rangle$ (Tomonaga)

$\langle t_i | a_j\rangle = \langle b_i|\hat{U}_t^{+}\hat{U}_a|b_j\rangle$

Note that

det(Ua)=1, det(Us)=-1, det(Ut)=-1

REFERENCES

1.
L.I. Skiff, Quantum Mechanics, 3rd edition (McGraw-Hill, 1968) p.377

2.
Sin-ItiroTomonaga , “Angular Momentum and Spin,” Supplement to Quantum Mechanics I (1962) and II (1966) (North-Holland), edited by Susumu Kamebuchi, Yasuo Hara, and Takeyasu Kodera (Misuzu-Shobou, 1988, in Japanese). p.46.
https://en.wikipedia.org/wiki/Shin%27ichir%C5%8D_Tomonaga

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I. Clebsch-Gordan coefficient for three spin-1/2 particles

Addition of angular momentum with three S= 1/2 spins
D1/2 x D1/2 x D1/2
= (D1 + D0) x D1/2
=(D1 x D1/2)+D1/2
= (D3/2 + D1/2) + D1/2

```
Clear["Global`*"];
CCGG[{j1_, m1_}, {j2_, m2_}, {j_, m_}] := Module[{s1}, s1 = If[
    Abs[m1] ≤ j1 && Abs[m2] ≤ j2 && Abs[m] ≤ j, ClebschGordan[{j1, m1}, {j2, m2}, {j, m}], 0]];
```

```
CG2[j1_, j2_, j_, a1_, a2_] :=
 Table[Sum[CCGG[{j1, k1}, {j2, k2}, {j, k1 + k2}] × a1[j1, k1] × a2[j2, k2]
    KroneckerDelta[k1 + k2, m], {k1, -j1, j1}, {k2, -j2, j2}], {m, -j, j}]
```

Addition of angular momentum with two S= 1/2 spins

j1=1/2, j2=1/2 D1/2 x D1/2 = D1+D0
leading to j = 1, 0

```
CG2[1 / 2, 1 / 2, 1, b1, b2] // TableForm
```

$$b1\left[\frac{1}{2}, -\frac{1}{2}\right] \times b2\left[\frac{1}{2}, -\frac{1}{2}\right]$$

$$\frac{b1\left[\frac{1}{2},\frac{1}{2}\right] \times b2\left[\frac{1}{2},-\frac{1}{2}\right]}{\sqrt{2}} + \frac{b1\left[\frac{1}{2},-\frac{1}{2}\right] \times b2\left[\frac{1}{2},\frac{1}{2}\right]}{\sqrt{2}}$$

$$b1\left[\frac{1}{2}, \frac{1}{2}\right] \times b2\left[\frac{1}{2}, \frac{1}{2}\right]$$

```
CG2[1 / 2, 1 / 2, 0, b1, b2] // TableForm
```

$$\frac{b1\left[\frac{1}{2},\frac{1}{2}\right] \times b2\left[\frac{1}{2},-\frac{1}{2}\right]}{\sqrt{2}} - \frac{b1\left[\frac{1}{2},-\frac{1}{2}\right] \times b2\left[\frac{1}{2},\frac{1}{2}\right]}{\sqrt{2}}$$

Addition of angular momentum with three S= 1/2 spins
D1/2 x D1/2 x D1/2
= (D1 + D0) x D1/2
=(D1 x D1/2)+D1/2
= (D3/2 + D1/2) + D1/2

(a) j1=1, j2=1/2 j = 3/2

```
j1 = 1;
j2 = 1 / 2;
j = 3 / 2;
Table[Sum[CCGG[{j1, k1}, {j2, k2}, {j, k1 + k2}] ×
    CG2[1 / 2, 1 / 2, j1, b1, b2][[k1 + j1 + 1]] × b3[j2, k2] KroneckerDelta[k1 + k2, m],
   {k1, -j1, j1}, {k2, -j2, j2}], {m, -j, j}] // Simplify
```

$$\left\{b1\left[\frac{1}{2}, -\frac{1}{2}\right] \times b2\left[\frac{1}{2}, -\frac{1}{2}\right] \times b3\left[\frac{1}{2}, -\frac{1}{2}\right], \frac{1}{\sqrt{3}}\left(b1\left[\frac{1}{2}, \frac{1}{2}\right] \times b2\left[\frac{1}{2}, -\frac{1}{2}\right] \times b3\left[\frac{1}{2}, -\frac{1}{2}\right] + b1\left[\frac{1}{2}, -\frac{1}{2}\right]\left(b2\left[\frac{1}{2}, \frac{1}{2}\right] \times b3\left[\frac{1}{2}, -\frac{1}{2}\right] + b2\left[\frac{1}{2}, -\frac{1}{2}\right] \times b3\left[\frac{1}{2}, \frac{1}{2}\right]\right)\right), \frac{1}{\sqrt{3}}\left(b1\left[\frac{1}{2}, -\frac{1}{2}\right] \times b2\left[\frac{1}{2}, \frac{1}{2}\right] \times b3\left[\frac{1}{2}, \frac{1}{2}\right] + b1\left[\frac{1}{2}, \frac{1}{2}\right]\left(b2\left[\frac{1}{2}, \frac{1}{2}\right] \times b3\left[\frac{1}{2}, -\frac{1}{2}\right] + b2\left[\frac{1}{2}, -\frac{1}{2}\right] \times b3\left[\frac{1}{2}, \frac{1}{2}\right]\right)\right), b1\left[\frac{1}{2}, \frac{1}{2}\right] \times b2\left[\frac{1}{2}, \frac{1}{2}\right] \times b3\left[\frac{1}{2}, \frac{1}{2}\right]\right\}$$

j=3/2

|+ + +> m=-3/2

$\frac{1}{\sqrt{3}}$[|- + +> + |+ + -> + |+ - +>]   m = 1/2

$\frac{1}{\sqrt{3}}$[|+ - -> + |- + -> + |- - +>]   m= - 1/2

|- - ->   m=-3/2

(b) j1=1, j2=1/2 j = 1/2

```
j1 = 1;
j2 = 1 / 2;
j = 1 / 2;
Table[Sum[CCGG[{j1, k1}, {j2, k2}, {j, k1 + k2}] ×
    CG2[1 / 2, 1 / 2, j1, b1, b2][[k1 + j1 + 1]] × b3[j2, k2] KroneckerDelta[k1 + k2, m],
   {k1, -j1, j1}, {k2, -j2, j2}], {m, -j, j}] // Simplify
```

$$\left\{\frac{1}{\sqrt{6}}\left(b1\left[\frac{1}{2},\frac{1}{2}\right]\times b2\left[\frac{1}{2},-\frac{1}{2}\right]\times b3\left[\frac{1}{2},-\frac{1}{2}\right]+ b1\left[\frac{1}{2},-\frac{1}{2}\right]\left(b2\left[\frac{1}{2},\frac{1}{2}\right]\times b3\left[\frac{1}{2},-\frac{1}{2}\right]-2\,b2\left[\frac{1}{2},-\frac{1}{2}\right]\times b3\left[\frac{1}{2},\frac{1}{2}\right]\right)\right),\right.$$
$$\left.\frac{1}{\sqrt{6}}\left(-b1\left[\frac{1}{2},-\frac{1}{2}\right]\times b2\left[\frac{1}{2},\frac{1}{2}\right]\times b3\left[\frac{1}{2},\frac{1}{2}\right]+ b1\left[\frac{1}{2},\frac{1}{2}\right]\left(2\,b2\left[\frac{1}{2},\frac{1}{2}\right]\times b3\left[\frac{1}{2},-\frac{1}{2}\right]-b2\left[\frac{1}{2},-\frac{1}{2}\right]\times b3\left[\frac{1}{2},\frac{1}{2}\right]\right)\right)\right\}$$

j=1/2

$\frac{1}{\sqrt{6}}$[-|- + +> - |+ - +> + 2|+ + ->]   m = 1/2

$\frac{1}{\sqrt{6}}$[|+ - -> + |- + -> - 2|- - +>]   m= - 1/2

(c) j1=0, j2=1/2 j = 1/2

```
j1 = 0;
j2 = 1 / 2;
j = 1 / 2;
Table[Sum[CCGG[{j1, k1}, {j2, k2}, {j, k1 + k2}] ×
    CG2[1 / 2, 1 / 2, j1, b1, b2][[k1 + j1 + 1]] × b3[j2, k2] KroneckerDelta[k1 + k2, m],
   {k1, -j1, j1}, {k2, -j2, j2}], {m, -j, j}] // Simplify
```

$$\left\{\frac{\left(b1\left[\frac{1}{2},\frac{1}{2}\right]\times b2\left[\frac{1}{2},-\frac{1}{2}\right]-b1\left[\frac{1}{2},-\frac{1}{2}\right]\times b2\left[\frac{1}{2},\frac{1}{2}\right]\right)b3\left[\frac{1}{2},-\frac{1}{2}\right]}{\sqrt{2}},\right.$$
$$\left.\frac{\left(b1\left[\frac{1}{2},\frac{1}{2}\right]\times b2\left[\frac{1}{2},-\frac{1}{2}\right]-b1\left[\frac{1}{2},-\frac{1}{2}\right]\times b2\left[\frac{1}{2},\frac{1}{2}\right]\right)b3\left[\frac{1}{2},\frac{1}{2}\right]}{\sqrt{2}}\right\}$$

j=1/2

$\frac{1}{\sqrt{2}}$[|+ - +> - |- + +>]   m = 1/2

$\frac{1}{\sqrt{2}}$[|+ - -> - |- + ->]   m= - 1/2

These results are the exactly the same as those obtained from the Kronecker products (1)

$$|a_1\rangle = |j=3/2, m=3/2\rangle = |b_1\rangle$$

$$|a_2\rangle = |j=3/2, m=1/2\rangle = \frac{1}{\sqrt{3}}(|b_2\rangle + |b_3\rangle + |b_5\rangle)$$

These results are the exactly the same as those obtained from the Kronecker products (1)

$$|a_1\rangle = |j = 3/2, m = 3/2\rangle = |b_1\rangle$$

$$|a_2\rangle = |j = 3/2, m = 1/2\rangle = \frac{1}{\sqrt{3}}(|b_2\rangle + |b_3\rangle + |b_5\rangle)$$

$$|a_3\rangle = |j = 3/2, m = -1/2\rangle = \frac{1}{\sqrt{3}}(|b_4\rangle + |b_6\rangle + |b_7\rangle)$$

$$|a_4\rangle = |j = 3/2, m = -3/2\rangle = |b_8\rangle$$

$$|a_5\rangle = |j = 1/2, m = 1/2\rangle = \frac{1}{\sqrt{6}}(-|b_2\rangle + 2|b_3\rangle - |b_5\rangle)$$

$$|a_6\rangle = |j = 1/2, m = -1/2\rangle = \frac{1}{\sqrt{6}}(-|b_4\rangle + 2|b_6\rangle - |b_7\rangle)$$

$$|a_7\rangle = |j = 1/2, m = 1/2\rangle = \frac{1}{\sqrt{2}}(|b_2\rangle - |b_5\rangle)$$

$$|a_8\rangle = |j = 1/2, m = -1/2\rangle = \frac{1}{\sqrt{2}}(|b_4\rangle - |b_7\rangle)$$

II. Addition of spin angular momentum for three spin 1/2 particles using the Kronecker product

(D1/2×D1/2)×D1/2=(D1+D0)×D1/2=D3/2+D1/2+D1/2+ (S = 3/2 and S = 1/2)

```
Clear["Global`*"];
exp_* := exp /. {Complex[re_, im_] :> Complex[re, -im]};
ψ1 = (1
      0);
ψ2 = (0
      1);
σx = (0 1
      1 0);
σy = (0 -i
      i  0);
σz = (1  0
      0 -1);
I2 = IdentityMatrix[2];
ST1 = 1/2 (KroneckerProduct[σx, σx, I2] + KroneckerProduct[σy, σy, I2]
     + KroneckerProduct[σz, σz, I2] +
     KroneckerProduct[I2, σx, σx] + KroneckerProduct[I2, σy, σy]
     + KroneckerProduct[I2, σz, σz] +
     KroneckerProduct[σx, I2, σx] + KroneckerProduct[σy, I2, σy]
     + KroneckerProduct[σz, I2, σz]) + 9/4 KroneckerProduct[I2, I2, I2];
```

```
ST1 // MatrixForm
```

$$\begin{pmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{4} & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{7}{4} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{4} & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & \frac{7}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{7}{4} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{4} \end{pmatrix}$$

```
eq1 = Eigensystem[ST1] // Simplify
```

$$\left\{\left\{\frac{15}{4}, \frac{15}{4}, \frac{15}{4}, \frac{15}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right\}, \{\{0,0,0,0,0,0,0,1\}, \{0,0,0,1,0,1,1,0\}, \{0,1,1,0,1,0,0,0\}, \{1,0,0,0,0,0,0,0\}, \{0,0,0,-1,0,0,1,0\}, \{0,0,0,-1,0,1,0,0\}, \{0,-1,0,0,1,0,0,0\}, \{0,-1,1,0,0,0,0,0\}\}\right\}$$

```
ξ1 = eq1[[2, 4]]; ξ2 = eq1[[2, 3]];
ξ3 = eq1[[2, 2]];
ξ4 = eq1[[2, 1]];
ξ5 = eq1[[2, 5]];
ξ6 = eq1[[2, 6]];
ξ7 = eq1[[2, 7]];
ξ8 = eq1[[2, 8]];
```

```
eq2 = Orthogonalize[{ξ1, ξ2, ξ3, ξ4, ξ5, ξ6, ξ7, ξ8}]
```

$$\left\{\{1,0,0,0,0,0,0,0\}, \left\{0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, 0,0,0\right\}, \left\{0,0,0,\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right\}, \{0,0,0,0,0,0,0,1\}, \left\{0,0,0,-\frac{1}{\sqrt{2}}, 0,0,\frac{1}{\sqrt{2}}, 0\right\}, \left\{0,0,0,-\frac{1}{\sqrt{6}}, 0, \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}, 0\right\}, \left\{0, -\frac{1}{\sqrt{2}}, 0,0,\frac{1}{\sqrt{2}}, 0,0,0\right\}, \left\{0, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}, 0, -\frac{1}{\sqrt{6}}, 0,0,0\right\}\right\}$$

Eigenstates

$$|a_i\rangle = \hat{U}_a |b_i\rangle$$

$$|b_i\rangle = \hat{U}_a^{+} |a_i\rangle$$

```
a1 = eq2[[1]];
a2 = eq2[[2]];
a3 = eq2[[3]];
a4 = eq2[[4]];
a5 = eq2[[8]];
a6 = eq2[[6]];
a7 = -eq2[[7]];
a8 = -eq2[[5]];
```

$$Sz = \frac{1}{2} (\text{KroneckerProduct}[\sigma z, I2, I2] + \text{KroneckerProduct}[I2, \sigma z, I2] + \text{KroneckerProduct}[I2, I2, \sigma z])$$

$$\left\{\left\{\frac{3}{2}, 0, 0, 0, 0, 0, 0, 0\right\}, \left\{0, \frac{1}{2}, 0, 0, 0, 0, 0, 0\right\}, \left\{0, 0, \frac{1}{2}, 0, 0, 0, 0, 0\right\}, \left\{0, 0, 0, -\frac{1}{2}, 0, 0, 0, 0\right\}, \left\{0, 0, 0, 0, \frac{1}{2}, 0, 0, 0\right\}, \left\{0, 0, 0, 0, 0, -\frac{1}{2}, 0, 0\right\}, \left\{0, 0, 0, 0, 0, 0, -\frac{1}{2}, 0\right\}, \left\{0, 0, 0, 0, 0, 0, 0, -\frac{3}{2}\right\}\right\}$$

```
Sz // MatrixForm
```

$$\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

D3/2 (j = 3/2, m = 3/2, 1/2,-1/2, -3/2)

$$ST1.a1 - \frac{15}{4} a1$$

{0, 0, 0, 0, 0, 0, 0, 0}

$$Sz.a1 - \frac{3}{2} a1$$

{0, 0, 0, 0, 0, 0, 0, 0}

$$ST1.a2 - \frac{15}{4} a2$$

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.a2 - 1/2 a2
```
{0, 0, 0, 0, 0, 0, 0, 0}

```
ST1.a3 - 15/4 a3
```
{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.a3 + 1/2 a3
```
{0, 0, 0, 0, 0, 0, 0, 0}

```
ST1.a4 - 15/4 a4
```
{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.a4 + 3/2 a4
```
{0, 0, 0, 0, 0, 0, 0, 0}

D1/2 (j=1/2, m=1/2, -1/2)

```
ST1.a5 - 3/4 a5 // Simplify
```
{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.a5 - 1/2 a5
```
{0, 0, 0, 0, 0, 0, 0, 0}

```
ST1.a6 - 3/4 a6 // Simplify
```
{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.a6 + 1/2 a6
```
{0, 0, 0, 0, 0, 0, 0, 0}

D1/2 (j = 1/2, m = 1/2, -1/2)

```
ST1.a7 - 3/4 a7 // Simplify
```
{0, 0, 0, 0, 0, 0, 0, 0}

$$Sz.a7-\frac{1}{2}a7$$

{0, 0, 0, 0, 0, 0, 0, 0}

$$ST1.a8-\frac{3}{4}\ a8\ //\ Simplify$$

{0, 0, 0, 0, 0, 0, 0, 0}

$$Sz.a8+\frac{1}{2}a8$$

{0, 0, 0, 0, 0, 0, 0, 0}

{a1.a2, a2.a3, a3.a4, a4.a5, a5.a6, a6.a7, a7.a8}

{0, 0, 0, 0, 0, 0, 0}

a1 // MatrixForm

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

a2 // MatrixForm

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

a3 // MatrixForm

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}$$

a4 // MatrixForm

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

a5 // MatrixForm

$$\begin{pmatrix} 0 \\ -\frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ 0 \\ -\frac{1}{\sqrt{6}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

a6 // Simplify // MatrixForm

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{6}} \\ 0 \end{pmatrix}$$

a7 // Simplify // MatrixForm

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
a8 // Simplify // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Determination of unitary operator Ua
Ua: Unitary operator
UaT: Transpose of unitary operator
UaH: Hermitain conjugate of Ua

```
UaT = {a1, a2, a3, a4, a5, a6, a7, a8} // Simplify;
```

```
Ua = Transpose[UaT] // Simplify; UaH = UaT* // Simplify; Ua // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
UaH // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{6}} & 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

```
UaH.Ua // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
Det[Ua]
```

1

Basis;

$$|b_1\rangle = |+z,+z,+z\rangle, \quad |b_2\rangle = |+z,+z,-z\rangle$$
$$|b_3\rangle = |+z,-z,+z\rangle, \quad |b_4\rangle = |+z,-z,-z\rangle$$
$$|b_5\rangle = |-z,+z,+z\rangle, \quad |b_6\rangle = |+z,+z,-z\rangle$$
$$|b_7\rangle = |-z,-z,+z\rangle, \quad |b_8\rangle = |-z,-z,-z\rangle$$

$$|a_i\rangle = \hat{U}_a |b_i\rangle$$
$$|b_i\rangle = \hat{U}_a^{+} |a_i\rangle$$

These results are the exactly the same as those obtained from the Kronecker products (1)

$$|a_1\rangle = |j=3/2, m=3/2\rangle = |b_1\rangle$$
$$|a_2\rangle = |j=3/2, m=1/2\rangle = \frac{1}{\sqrt{3}}(|b_2\rangle + |b_3\rangle + |b_5\rangle)$$
$$|a_3\rangle = |j=3/2, m=-1/2\rangle = \frac{1}{\sqrt{3}}(|b_4\rangle + |b_6\rangle + |b_7\rangle)$$
$$|a_4\rangle = |j=3/2, m=-3/2\rangle = |b_8\rangle$$
$$|a_5\rangle = |j=1/2, m=1/2\rangle = \frac{1}{\sqrt{6}}(-|b_2\rangle + 2|b_3\rangle - |b_5\rangle)$$
$$|a_6\rangle = |j=1/2, m=-1/2\rangle = \frac{1}{\sqrt{6}}(-|b_4\rangle + 2|b_6\rangle - |b_7\rangle)$$
$$|a_7\rangle = |j=1/2, m=1/2\rangle = \frac{1}{\sqrt{2}}(|b_2\rangle - |b_5\rangle)$$
$$|a_8\rangle = |j=1/2, m=-1/2\rangle = \frac{1}{\sqrt{2}}(|b_4\rangle - |b_7\rangle)$$

```
b1 = KroneckerProduct[ψ1, ψ1, ψ1]; b1 // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
b2 = KroneckerProduct[ψ1, ψ1, ψ2]; b2 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
b3 = KroneckerProduct[ψ1, ψ2, ψ1]; b3 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
b4 = KroneckerProduct[ψ1, ψ2, ψ2]; b4 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
b5 = KroneckerProduct[ψ2, ψ1, ψ1]; b5 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
b6 = KroneckerProduct[ψ2, ψ1, ψ2]; b6 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

```
b7 = KroneckerProduct[ψ2, ψ2, ψ1]; b7 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

```
b8 = KroneckerProduct[ψ2, ψ2, ψ2]; b8 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

It is confirmed that a11=a1, a22 = a2, a33 = a3, a44=a4, a55=a5, a66=a6, a77=a7, and a88 = a8.

$$|a_i\rangle = \hat{U}_a |b_i\rangle$$

$$|b_i\rangle = \hat{U}_a^{+} |a_i\rangle$$

```
a11 = UaH.b1 // Simplify; a11 // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

a22 = UaH.b2 // Simplify; a22 // MatrixForm

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

a33 = UaH.b3 // Simplify; a33 // MatrixForm

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

a44 = UaH.b4 // Simplify; a44 // MatrixForm

$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

a55 = UaH.b5 // Simplify; a55 // MatrixForm

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

```
a66 = UaH.b6 // Simplify; a66 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \end{pmatrix}$$

```
a77 = UaH.b7 // Simplify; a77 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{6}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

```
a88 = UaH.b8 // Simplify; a88 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

III. Results from L.I. Schiff, Quantum Mechanics

L.I. Skiff, Quantum Mechanics, 3rd edition (McGraw-Hill, 1968) p.377

Leonard Isaac Schiff was born in Fall River, Massachusetts, on March 29, 1915 and died on January 21, 1971, in Stanford, California. He was a physicist best known for his book *Quantum Mechanics*, originally published in 1949 (a second edition appeared in 1955 and a third in 1968).
https://en.wikipedia.org/wiki/Leonard_I._Schiff

SPIN FUNCTIONS FOR THREE ELECTRONS

In the treatment of exchange scattering from helium given in Sec. 43, we shall require eigenfunctions of the total spin of three electrons that are analogous to those given in Eqs. (41.6) for two electrons. We can regard three electrons as $1 + 2$ electrons, in the sense that we can combine an electron ($s = \frac{1}{2}$) with the triplet two-electron function ($s = 1$) and with the singlet function ($s = 0$). In the first case, the results on addition of angular momenta, given in Sec. 28, show that we should get two groups of three-electron spin functions that correspond to $s = \frac{1}{2}$ and $s = \frac{3}{2}$; in the second case we should get a single group that corresponds to $s = \frac{1}{2}$. We thus expect one *quartet* group of spin states ($s = \frac{3}{2}$) and two distinct *doublet* groups of spin states ($s = \frac{1}{2}$), or a total of $4 + 2 + 2 = 8$ individual three-electron spin states. These must be expressible as linear combinations of the $2^3 = 8$ products of one-electron spin functions.

Again, these combinations are obtained from Eq. (28.1), with the Clebsch-Gordan coefficients now being given by (28.13); the two-electron functions required (for electrons 2 and 3) are those of (41.6). We thus obtain

$$
\begin{array}{lcc}
 & (\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3)^2 & S_{1z} + S_{2z} + S_{3z} \\
(+++) & \frac{15}{4}\hbar^2 & \frac{3}{2}\hbar \\
3^{-\frac{1}{2}}[(++-) + (+-+) + (-++)] & \frac{15}{4}\hbar^2 & \frac{1}{2}\hbar \\
3^{-\frac{1}{2}}[(-+-) + (--+) + (+--)] & \frac{15}{4}\hbar^2 & -\frac{1}{2}\hbar \\
(---) & \frac{15}{4}\hbar^2 & -\frac{3}{2}\hbar \\
6^{-\frac{1}{2}}[2(-++) - (++-) - (+-+)] & \frac{3}{4}\hbar^2 & \frac{1}{2}\hbar \\
6^{-\frac{1}{2}}[(-+-) + (--+) - 2(+--)] & \frac{3}{4}\hbar^2 & -\frac{1}{2}\hbar \\
2^{-\frac{1}{2}}[(++-) - (+-+)] & \frac{3}{4}\hbar^2 & \frac{1}{2}\hbar \\
2^{-\frac{1}{2}}[(-+-) - (--+)] & \frac{3}{4}\hbar^2 & -\frac{1}{2}\hbar
\end{array}
\tag{41.9}
$$

The orthonormality and the indicated eigenvalues can be verified directly. The first four (quartet) states are symmetric in the interchange of any pair of particles. The division of the four doublet states into two pairs is such that the first pair is symmetric in the interchange of particles 2 and 3, and the second pair is antisymmetric in 2 and 3. The symmetry with respect to interchanges of the other two pairs is characterized by the 2×2 matrices mentioned below Eq. (40.7); these matrices operate on either pair of doublet spin states that have the same m value (see Prob. 8).

Here we discuss the equivalence between our result and the expressions given by Schiff.

The following unitary operator is defined as Us (the index s comes from the name of Schiff.

$$\text{Us} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{-1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{-1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{-2}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{2}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix};$$

UsT=Transpose[Us], UsH=Us^+

$Us^+Us=1$

```
UsT = Transpose[Us];
```

```
UsH = Transpose[Us*];
```

```
UsH.Us // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$|a_i\rangle = \hat{U}_a |b_i\rangle$

$|b_i\rangle = \hat{U}_a^{+} |a_i\rangle$

Eigenkets (Schiff)

$|s_i\rangle = \hat{U}_s |b_i\rangle$

$|s_1\rangle,\ |s_2\rangle,\ |s_3\rangle,\ |s_4\rangle,$

$|s_5\rangle,\ |s_6\rangle,\ |s_7\rangle,\ |s_8\rangle$

```
s1 = UsT[[1]]; s1 // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
s2 = UsT[[2]]; s2 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
s3 = UsT[[3]]; s3 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}$$

```
s4 = UsT[[4]]; s4 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

```
s5 = UsT[[5]]; s5 // MatrixForm
```

$$\begin{pmatrix} 0 \\ -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
s6 = UsT[[6]]; s6 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sqrt{\frac{2}{3}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ 0 \end{pmatrix}$$

```
s7 = UsT[[7]]; s7 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
s8 = UsT[[8]]; s8 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

D3/2 (j = 3/2, s1, s2, s3, s4),
D1/2 (j = 1/2, s5, s6)
D1/2 (j = 1/2, s7, s8)

$$\texttt{ST1.s1} - \frac{15}{4}\,\texttt{s1 // Simplify}$$

```
{0, 0, 0, 0, 0, 0, 0, 0}
```

$$\texttt{Sz.s1} - \frac{3}{2}\,\texttt{s1}$$

```
{0, 0, 0, 0, 0, 0, 0, 0}
```

```
ST1.s2 - 15/4 s2 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.s2 - 1/2 s2 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
ST1.s3 - 15/4 s3 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.s3 + 1/2 s3 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
ST1.s4 - 15/4 s4 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.s4 + 3/2 s4 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

D1/2 (j=1/2, m=1/2, -1/2)

```
ST1.s5 - 3/4 s5 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.s5 - 1/2 s5 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
ST1.s6 - 3/4 s6 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.s6 + 1/2 s6 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

D1/2 (j = 1/2, m = 1/2, -1/2)

```
ST1.s7 - 3/4 s7 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.s7 - 1/2 s7 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
ST1.s8 - 3/4 s8 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.s8 + 1/2 s8 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

|si>=Us|bi>, |bi>=UsH|si>

```
UsH.s1 // Simplify
```

{1, 0, 0, 0, 0, 0, 0, 0}

```
UsH.s2 // Simplify
```

{0, 1, 0, 0, 0, 0, 0, 0}

```
UsH.s3 // Simplify
```

{0, 0, 1, 0, 0, 0, 0, 0}

```
UsH.s4 // Simplify
```

{0, 0, 0, 1, 0, 0, 0, 0}

```
UsH.s5 // Simplify
```

{0, 0, 0, 0, 1, 0, 0, 0}

```
UsH.s6 // Simplify
```

{0, 0, 0, 0, 0, 1, 0, 0}

```
UsH.s7 // Simplify
```

{0, 0, 0, 0, 0, 0, 1, 0}

```
UsH.s8 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 1}

```
UaH.Us // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

```
UsH.Ua // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

```
Det[Us] // Simplify
```

-1

((Summary))

$|s_i\rangle = \hat{U}_s |b_i\rangle$ (Schiff)

$|a_i\rangle = \hat{U}_a |b_i\rangle$ (from the Kronecker product)

$\langle s_i | a_j\rangle = \langle b_i | \hat{U}_s{}^+ \hat{U}_a | b_j\rangle$

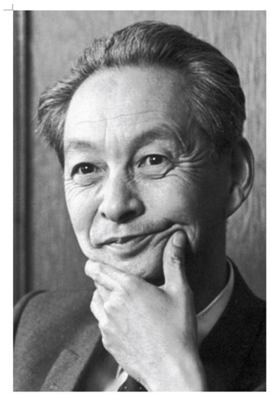
$$\hat{U}_s{}^+\hat{U}_a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\cdots}{2} & 0 & -\frac{\cdots}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$|s_1\rangle = |a_1\rangle$$
$$|s_2\rangle = |a_2\rangle$$
$$|s_3\rangle = |a_3\rangle$$
$$|s_4\rangle = |a_4\rangle$$
$$|s_5\rangle = -\frac{1}{2}|a_5\rangle - \frac{\sqrt{3}}{2}|a_7\rangle$$
$$|s_6\rangle = \frac{1}{2}|a_6\rangle + \frac{\sqrt{3}}{2}|a_8\rangle$$
$$|s_7\rangle = -\frac{\sqrt{3}}{2}|a_5\rangle + \frac{1}{2}|a_7\rangle$$
$$|s_8\rangle = -\frac{\sqrt{3}}{2}|a_6\rangle + \frac{1}{2}|a_8\rangle$$

III. Result from Tomonaga:

Sin-itiro Tomonaga (朝永 振一郎, March 31, 1906 – July 8, 1979), was a Japanese physicist, influential in the development of quantum electrodynamics, work for which he was jointly awarded the Nobel Prize in Physics in 1965 along with Richard Feynman and Julian Schwinger.



https://en.wikipedia.org/wiki/Shin%27ichir%C5%8D_Tomonaga

Picture of Prof. Sin-itiro Tomonaga (1906-1979)

Sin-ItiroTomonaga
Angular Momentum and Spin, Supplement to Quantum Mechanics I (1962) and II (1966) (North-Holland), edited by Susumu Kamebuchi, Yasuo Hara, and Takeyasu Kodera
(Misuzu-Shobou, 1988, in Japanese), Page: 46

表 14　3 個のスピンを合成した時の可能なスピン関数.
最上欄に s^2 の固有値，左はじの列に s_z の固有値を示した.

	$s^2=\frac{3}{2}\left(\frac{3}{2}+1\right)$ $s=\frac{3}{2}$	$s^2=\frac{1}{2}\left(\frac{1}{2}+1\right)$ $s=\frac{1}{2}$	$s^2=\frac{1}{2}\left(\frac{1}{2}+1\right)$ $s=\frac{1}{2}$
$s_z=\frac{3}{2}$	$\alpha(1)\alpha(2)\alpha(3)$		
$s_z=\frac{1}{2}$	$\beta(1)\alpha(2)\alpha(3)+\alpha(1)\beta(2)\alpha(3)$ $+\alpha(1)\alpha(2)\beta(3)$	$\frac{1}{2}\alpha(1)\beta(2)\alpha(3)+\frac{1}{2}\beta(1)\alpha(2)\alpha(3)$ $-\alpha(1)\alpha(2)\beta(3)$	$\alpha(1)\beta(2)\alpha(3)-\beta(1)\alpha(2)\alpha(3)$
$s_z=-\frac{1}{2}$	$\beta(1)\beta(2)\alpha(3)+\alpha(1)\beta(2)\beta(3)$ $+\beta(1)\alpha(2)\beta(3)$	$\frac{1}{2}\beta(1)\alpha(2)\beta(3)+\frac{1}{2}\alpha(1)\beta(2)\beta(3)$ $-\beta(1)\beta(2)\alpha(3)$	$\beta(1)\alpha(2)\beta(3)-\alpha(1)\beta(2)\beta(3)$
$s_z=-\frac{3}{2}$	$\beta(1)\beta(2)\beta(3)$		

Here we discuss the equivalence between our result and the expressions given by TomonagaThe following unitary operator is defined as Ut (t comes from the name of Tomonaga.

$$\text{Ut} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{-2}{\sqrt{6}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{-2}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix};$$

```
Det[Ut]
```

```
-1
```

UtT=Transpose[Ut],
UtH=Ut^+
Ut^+Ut=1

```
UtT = Transpose[Ut];
```

```
UtH = Transpose[Ut*];
```

```
UtH.Ut // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$|t_i\rangle = \hat{U}_t |b_i\rangle$ (Tomonaga)

$|a_i\rangle = \hat{U}_a |b_i\rangle$ (from the Kronecker product)

$\langle t_i | a_j \rangle = \langle b_i | \hat{U}_t^{+} \hat{U}_a | b_j \rangle$

```
t1 = UtT⟦1⟧; t1 // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
t2 = UtT⟦2⟧; t2 // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
t3 = UtT⟦3⟧; t3 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}$$

```
t4 = UtT⟦4⟧; t4 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

```
t5 = UtT⟦5⟧; t5 // MatrixForm
```

$$\begin{pmatrix} 0 \\ -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
t6 = UtT⟦6⟧; t6 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{6}} \\ 0 \\ \frac{1}{\sqrt{6}} \\ -\sqrt{\frac{2}{3}} \\ 0 \end{pmatrix}$$

```
t7 = UtT⟦7⟧; t7 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

```
t8 = UtT[[8]]; t8 // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}$$

$|t_i\rangle = \hat{U}_t |b_i\rangle$ (Tomonaga)

$|a_i\rangle = \hat{U}_a |b_i\rangle$ (from the Kronecker product)

$\langle t_i | a_j \rangle = \langle b_i | \hat{U}_t^+ \hat{U}_a | b_j \rangle$

D3/2 (j = 3/2, t1, t2, t3, t4),

D1/2 (j = 1/2, t5, t6)

D1/2 (j = 1/2, t7, t8)

$\texttt{ST1.t1} - \frac{15}{4}\texttt{ t1 // Simplify}$

{0, 0, 0, 0, 0, 0, 0, 0}

$\texttt{Sz.t1} - \frac{3}{2}\texttt{ t1}$

{0, 0, 0, 0, 0, 0, 0, 0}

$\texttt{ST1.t2} - \frac{15}{4}\texttt{ t2 // Simplify}$

{0, 0, 0, 0, 0, 0, 0, 0}

$\texttt{Sz.t2} - \frac{1}{2}\texttt{ t2 // Simplify}$

{0, 0, 0, 0, 0, 0, 0, 0}

$\texttt{ST1.t3} - \frac{15}{4}\texttt{ t3 // Simplify}$

{0, 0, 0, 0, 0, 0, 0, 0}

$\texttt{Sz.t3} + \frac{1}{2}\texttt{ t3 // Simplify}$

{0, 0, 0, 0, 0, 0, 0, 0}

```
ST1.t4 - 15/4 t4 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.t4 + 3/2 t4 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

D1/2 (j=1/2, m=1/2, -1/2)

```
ST1.t5 - 3/4 t5 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.t5 - 1/2 t5 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
ST1.t6 - 3/4 t6 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.t6 + 1/2 t6 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

D1/2 (j = 1/2, m = 1/2, -1/2)

```
ST1.t7 - 3/4 t7 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.t7 - 1/2 t7 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
ST1.t8 - 3/4 t8 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

```
Sz.t8 + 1/2 t8 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 0}

|ti>=Ut|bi>, |bi>=UtH|ai>

```
UtH.t1 // Simplify
```

{1, 0, 0, 0, 0, 0, 0, 0}

```
UtH.t2 // Simplify
```

{0, 1, 0, 0, 0, 0, 0, 0}

```
UtH.t3 // Simplify
```

{0, 0, 1, 0, 0, 0, 0, 0}

```
UtH.t4 // Simplify
```

{0, 0, 0, 1, 0, 0, 0, 0}

```
UtH.t5 // Simplify
```

{0, 0, 0, 0, 1, 0, 0, 0}

```
UtH.t6 // Simplify
```

{0, 0, 0, 0, 0, 1, 0, 0}

```
UtH.t7 // Simplify
```

{0, 0, 0, 0, 0, 0, 1, 0}

```
UtH.t8 // Simplify
```

{0, 0, 0, 0, 0, 0, 0, 1}

```
UaH.Ut // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

```
UtH.Ua // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

```
Det[Ut] // Simplify
-1
```

((Summary))

$$|t_i\rangle = \hat{U}_t |b_i\rangle \qquad \text{(Tomonaga)}$$

$$|a_i\rangle = \hat{U}_a |b_i\rangle \qquad \text{(from the Kronecker product)}$$

$$\langle t_i | a_j\rangle = \langle b_i | \hat{U}_t^{+} \hat{U}_a | b_j\rangle$$

$$\hat{U}_t^{+}\hat{U}_a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{pmatrix}$$

$$|t_1\rangle = |a_1\rangle$$

$$|t_2\rangle = |a_2\rangle$$

$$|t_3\rangle = |a_3\rangle$$

$$|t_4\rangle = |a_4\rangle$$

$$|t_5\rangle = \frac{1}{2}|a_5\rangle + \frac{\sqrt{3}}{2}|a_7\rangle$$

$$|t_6\rangle = \frac{1}{2}|a_6\rangle + \frac{\sqrt{3}}{2}|a_8\rangle$$

$$|t_7\rangle = -\frac{\sqrt{3}}{2}|a_5\rangle + \frac{1}{2}|a_7\rangle$$

$$|t_8\rangle = \frac{\sqrt{3}}{2}|a_6\rangle - \frac{1}{2}|a_8\rangle$$

V. CONCLUSION

The eigenkets and eigenvalues of the three spin=1/2 particles are derived from the two methods, (i) Kronecker product and (ii) Clebsch-Gordan coefficient. The same results are derived from these methods. We also check the validity of the expressions reported by L.I. Schiff in his famous book, Quantum Mechanics.

The Schiff's results for the eigenkets and eigenvalues are slightly different from our results, since the unitary operators used in Schiff is different from our unitary operator. det(U)=1 for our

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The Schiff's results for the eigenkets and eigenvalues are slightly different from our results, since the unitary operators used in Schiff is different from our unitary operator. det(U)=1 for our result, and det(U_s)=-1 for the Schiff's case. Nevertheless, we confirm the equivalence between our results and the Schiff's result.

⋮

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