# Three dimensional Green's function

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Here we discuss the concept of the 3D Green function, which is often used in the physics in particular in scattering problem in the quantum mechanics and electromagnetic problem.

## 1 Green’s function (summary)

$$L_1 y(\boldsymbol{r}_1) = -f(\boldsymbol{r}_1) \qquad \text{(self adjoint)}$$

The solution of this equation is given by

$$y(\boldsymbol{r}_1) = \int G(\boldsymbol{r}_1, \boldsymbol{r}_2) f(\boldsymbol{r}_2) d\boldsymbol{r}_2 + \varphi(\boldsymbol{r}_1),$$

where

$$L_1 = \nabla_1 \cdot [\boldsymbol{p}(r_1) \cdot \nabla_1] + q(\boldsymbol{r}_1),$$

$$L_1 G(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2),$$

and

$$L_1 \varphi(\boldsymbol{r}_1) = 0.$$

(a) $\boldsymbol{p}(\boldsymbol{r}_1) = \boldsymbol{1}$ and $q(\boldsymbol{r}_1) = 0$

$$L_1 = \nabla_1^{\ 2}$$

$$\nabla_1^{\ 2} G(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2)$$

The solution is

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{4\pi |\boldsymbol{r}_1 - \boldsymbol{r}_2|}.$$

(b) $\boldsymbol{p}(\boldsymbol{r}_1) = 1$ and $q(\boldsymbol{r}_1) = k^2$

$$L_1 = \nabla_1^{\ 2} + k^2,$$

$$(\nabla_1{}^2 + k^2)G(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2),$$

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{\exp(ik|\boldsymbol{r}_1 - \boldsymbol{r}_2|)}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|}.$$

The solution of $L_1 y(\boldsymbol{r}_1) = -f(\boldsymbol{r}_1)$ is

$$y(\boldsymbol{r}_1) = \int \frac{\exp(ik|\boldsymbol{r}_1 - \boldsymbol{r}_2|)}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|} f(\boldsymbol{r}_2) d\boldsymbol{r}_2 + \varphi(\boldsymbol{r}_1).$$

with

$$(\nabla_1{}^2 + k^2)\varphi(r_1) = 0$$

(c) $\boldsymbol{p}(\boldsymbol{r}_1) = 1$ and $q(\boldsymbol{r}_1) = -k^2$

$$L_1 = \nabla_1{}^2 - k^2,$$

$$(\nabla_1{}^2 - k^2)G(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2),$$

$$G(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{\exp(-k|\boldsymbol{r}_1 - \boldsymbol{r}_2|)}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|}.$$

The solution of $L_1 y(\mathbf{r}_1) = -f(\mathbf{r}_1)$ is

$$y(\boldsymbol{r}_1) = \int \frac{\exp(-k|\boldsymbol{r}_1 - \boldsymbol{r}_2|)}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|} f(\boldsymbol{r}_2) d\boldsymbol{r}_2 + \varphi(\boldsymbol{r}_1).$$

with

$$(\nabla_1{}^2 - k^2)\varphi(r_1) = 0$$

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Table

| Laplace | Helmholtz | Modified Helmholtz |
|---|---|---|
| $\nabla^2$ | $\nabla^2 + k^2$ | $\nabla^2 - k^2$ |

3D $\quad \frac{1}{4\pi}\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|} \quad \frac{1}{4\pi}\frac{\exp(ik|\mathbf{r}_1-\mathbf{r}_2|)}{|\mathbf{r}_1-\mathbf{r}_2|} \quad \frac{1}{4\pi}\frac{\exp(-k|\mathbf{r}_1-\mathbf{r}_2|)}{|\mathbf{r}_1-\mathbf{r}_2|}$

## 2 3D Green's function (Laplace)

We now consider the form of the Green's function (Laplace)

$$\nabla^2 G_0 = -\delta(\boldsymbol{r}).$$

The Fourier transform of $G(\boldsymbol{r})$ is defined as

$$G_0(\boldsymbol{q}) = \frac{1}{(2\pi)^{3/2}}\int d\boldsymbol{r} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{r}).$$

The inverse Fourier transform is also defined as

$$G_0(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}}\int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{q}),$$

where

$$\delta(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}.$$

Then

$$\begin{aligned}\nabla^2 G_0(\boldsymbol{r}) &= \frac{1}{(2\pi)^{3/2}}\int d\boldsymbol{q} \nabla^2 e^{i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{q}) \\ &= \frac{1}{(2\pi)^{3/2}}\int d\boldsymbol{q}(-q^2) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{q}) \\ &= -\frac{1}{(2\pi)^3}\int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\end{aligned},$$

or

$$G_0(\boldsymbol{q}) = \frac{1}{(2\pi)^{3/2}}\frac{1}{q^2}.$$

Then we have

$$G_0(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int d\boldsymbol{q}\frac{e^{i\boldsymbol{q}\cdot\boldsymbol{r}}}{q^2}.$$

For convenience, we assume that the direction of $\boldsymbol{r}$ is the $z$ axis. The angle between $\boldsymbol{r}$ and $\boldsymbol{q}$ is $\theta$.

$$\boldsymbol{q} \cdot \boldsymbol{r} = qr\cos\theta\,,$$

$$d\boldsymbol{q} = 2\pi q^2 dq \sin\theta d\theta\,,$$

$$G_0(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int\int 2\pi q^2 dq \sin\theta d\theta e^{\mathbf{i}qr\cos\theta}\frac{1}{q^2}\,.$$

Note

$$\int \sin\theta d\theta e^{\mathbf{i}qr\cos\theta} = -\frac{i}{qr}[e^{iqr} - e^{-iqr}]\,,$$

we have

$$\begin{aligned} G_0(\boldsymbol{r}) &= \frac{1}{(2\pi)^3}\int q^2 dq(-\frac{2\pi i}{qr})\frac{e^{iqr} - e^{-iqr}}{q^2} \\ &= \frac{-i}{(2\pi)^2 r}\int_{-\infty}^{\infty} dq\frac{e^{iqr}}{q} \end{aligned}$$

We calculate $I = \int_{-\infty}^{\infty} dq\frac{e^{iqr}}{q}$ by using the Cauchy's theorem.

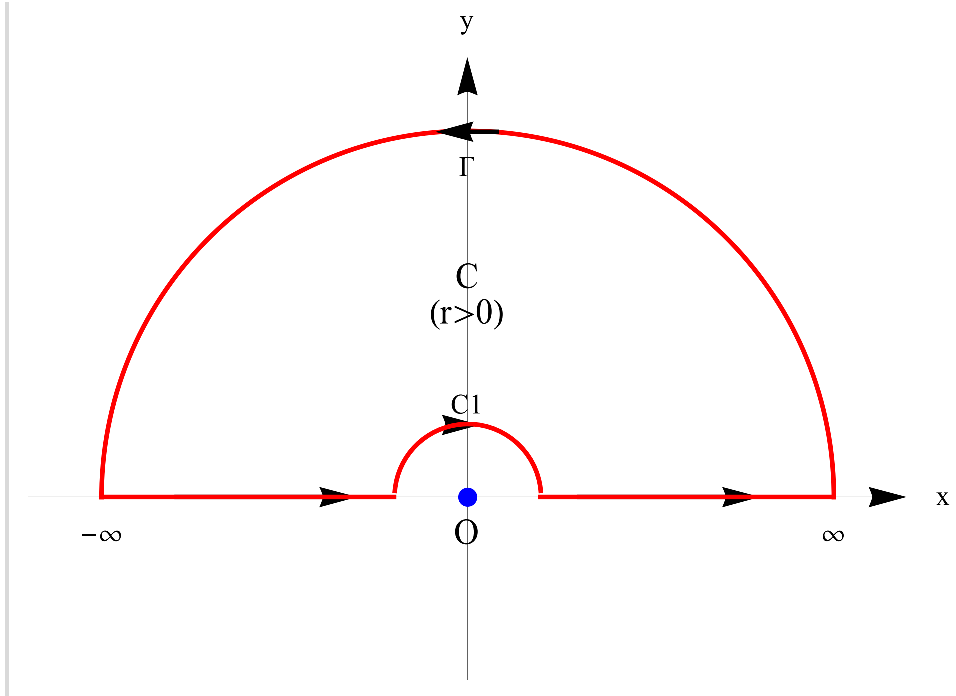



**Fig.** Upper half-plane contour for $r$>0. The semi circle $C_1$ (clock wise).

$$P\int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q} + \int_{C1} dz \frac{e^{izr}}{z} + \int_{\Gamma} dz \frac{e^{izr}}{z} = \oint_C dz \frac{e^{izr}}{z} = 0$$

or

$$P\int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q} = -\int_{C_1} dz \frac{e^{izr}}{z} = \pi i \operatorname{Re} s(z=0) = \pi i$$

since

$$\int_{\Gamma} dz \frac{e^{izr}}{z} = 0 \qquad \text{(Jordan's lemma, } r>0\text{).}$$

Then we have

$$G_0(\mathbf{r}) = \frac{1}{4\pi^2 ir}\pi i = \frac{1}{4\pi r}$$

or

$$G_0(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

((**Note**)) **Method II**

We show that we get the same result using a different contour (method II). Inside this contour, there is a simple pole at $z = 0$.

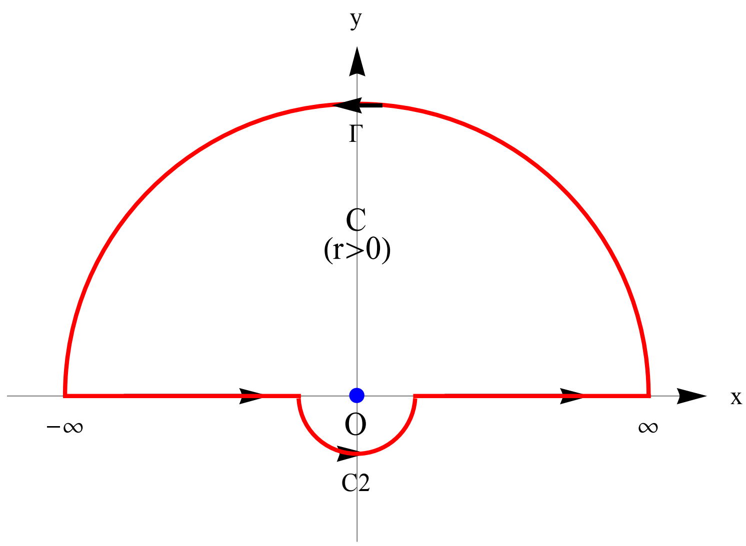



**Fig.** The contour $C_2$ has a counter clock-wise rotation.

$$P\int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q} + \int_{\Gamma} dz \frac{e^{izr}}{z} + \int_{C2} dz \frac{e^{izr}}{z} = \oint_C dz \frac{e^{izr}}{z} = 2\pi i \, \mathrm{Re}\, s(z=0)$$

Since $\int_{\Gamma} dz \frac{e^{izr}}{z} = 0$ from the Jordan's lemma, we have

$$\begin{aligned} P\int_{-\infty}^{\infty} dq \frac{e^{iqr}}{q} &= -\int_{C2} dz \frac{e^{izr}}{z} + 2\pi i \, \mathrm{Re}\, s(z=0) \\ &= -\pi i \, \mathrm{Re}\, s(z=0) + 2\pi i \, \mathrm{Re}\, s(z=0) \\ &= \pi i \, \mathrm{Re}\, s(z=0) = \pi i \end{aligned}$$

Then we have

$$G_0(\boldsymbol{r}) = \frac{1}{4\pi^2 i r}\pi i = \frac{1}{4\pi r}.$$

which is the same as that obtained in the method I.

## 3 Derivation of Green's function (vector analysis)

$$\nabla^2 \frac{1}{4\pi r} = -\delta(\boldsymbol{r}) ,$$

where

$$\boldsymbol{r} = (x, y, z) , \qquad r = \sqrt{x^2 + y^2 + z^2} .$$

We consider a sphere with radius $\varepsilon$ ( $\varepsilon \to 0$)

$$\int d\boldsymbol{r} \nabla \cdot \nabla \frac{1}{r} = \int d\boldsymbol{r} \Delta \frac{1}{r} = \int d\boldsymbol{a} \cdot \nabla \frac{1}{r} = \int da (\boldsymbol{n} \cdot \nabla \frac{1}{r}) ,$$

where

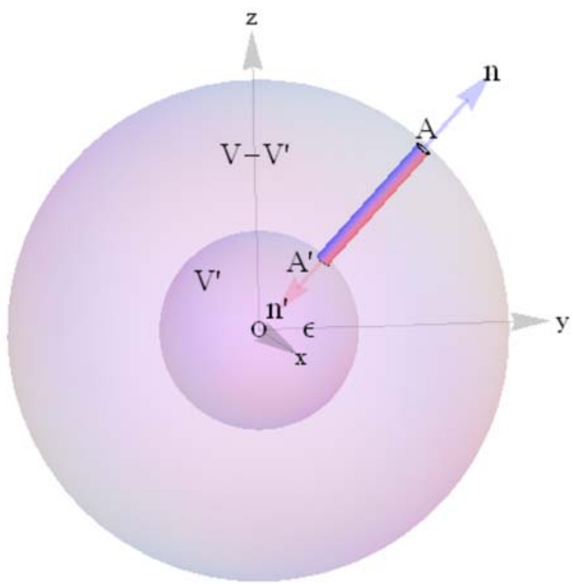
$$r = \sqrt{x^2 + y^2 + z^2} , \quad \boldsymbol{n} = \frac{\boldsymbol{r}}{r} = \boldsymbol{e}_r = (\frac{x}{r}, \frac{y}{r}, \frac{z}{r}) , \qquad d\boldsymbol{a} = \boldsymbol{n} da$$

and

$$\nabla \frac{1}{r} = -\frac{\boldsymbol{r}}{r^3} , \qquad \boldsymbol{n} \cdot \nabla \frac{1}{r} = \hat{r} \cdot (-\frac{\boldsymbol{r}}{r^3}) = -\frac{1}{r^2}$$

$\nabla \cdot \nabla (\frac{1}{r}) = 0$ except at the origin.

We now consider the volume integral over the whole volume (*V* - *V*') between the surface *A* and the surface of sphere *A*' (volume *V*', radius $\varepsilon \to 0$) . We note that the outer surface and the inner surface are connected to an appropriate cylinder.




Since $\nabla \cdot \nabla(\frac{1}{r}) = 0$ over the whole volume *V* - *V'* we have

Using the Gauss's law, we get

$$\int_{V-V'} d\boldsymbol{r} \nabla \cdot \nabla \frac{1}{r} = \int_{V-V'} d\boldsymbol{r} \nabla^2 \frac{1}{r}$$

$$= \int_A da(\boldsymbol{n} \cdot \nabla \frac{1}{r}) + \int_{A'} da'(\boldsymbol{n}' \cdot \nabla \frac{1}{r}) = 0$$

or

$$\int_A da(\boldsymbol{n} \cdot \nabla \frac{1}{r}) = -\int_{A'} da'(\boldsymbol{n}' \cdot \nabla \frac{1}{r}) = \int_{A'} da'(\boldsymbol{n} \cdot \nabla \frac{1}{r})$$

where $\boldsymbol{n}' = -\boldsymbol{n} = -\hat{r}$ and d$\boldsymbol{r}$ is over the volume integral. Then we have

$$\int_A da(\boldsymbol{n} \cdot \nabla \frac{1}{r}) = \int da(-\frac{1}{r^2}) = -4\pi\varepsilon^2 \frac{1}{\varepsilon^2} = -4\pi = -4\pi \int d\boldsymbol{r} \delta(\boldsymbol{r})$$

Using the Gauss's law, we have

$$\int_A da(\boldsymbol{n}\cdot\nabla\frac{1}{r}) = \int_V d\boldsymbol{r}(\nabla\cdot\nabla\frac{1}{r}) = -4\pi\int_V d\boldsymbol{r}\delta(\boldsymbol{r}),$$

or

$$\Delta\frac{1}{r} = -4\pi\delta(\boldsymbol{r}).$$

or

$$\boxed{\Delta(\frac{1}{4\pi r}) = -\delta(\boldsymbol{r})}.$$

((**Mathematica**))

```
Clear["Gobal`"];

Needs["VectorAnalysis`"]

SetCoordinates[Cartesian[x, y, z]]

Cartesian[x, y, z]

r1 = {x, y, z}; r = √r1.r1

√(x^2 + y^2 + z^2)

Grad[1/r] // Simplify

{-x/(x^2 + y^2 + z^2)^(3/2), -y/(x^2 + y^2 + z^2)^(3/2), -z/(x^2 + y^2 + z^2)^(3/2)}

Laplacian[1/r] // Simplify

0
```

## 4 3D Green's function (Helmholtz)

We now consider the form of the Green's function (Helmholtz)

$$(\Delta + k^2)G_0(\boldsymbol{r}) = -\delta(\boldsymbol{r})$$

The Fourier transform of $G_0(\boldsymbol{r})$ is defined as

$$G_0(\boldsymbol{q}) = \frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{r} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{r})$$

The inverse Fourier transform is also defined as

$$G_0(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{q})$$

where

$$\delta(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$$

Then

$$\begin{aligned}
(\Delta + k^2) G_0(\boldsymbol{r}) &= \frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{q} (\Delta + k^2) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{q}) \\
&= \frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{q} (-q^2 + k^2) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{q}) \\
&= -\delta(\boldsymbol{r}) = -\frac{1}{(2\pi)^3} \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}
\end{aligned}$$

or

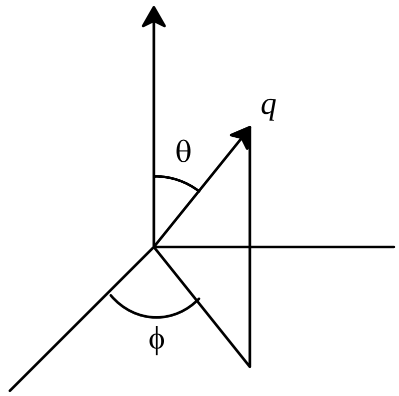
$$\int d\boldsymbol{q} (-q^2 + k^2) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{q}) = -\frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$$

or

$$G_0(\boldsymbol{q}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{q^2 - k^2}, \qquad G_0^{(\pm)}(\boldsymbol{q}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{q^2 - (k^2 \pm i\varepsilon)}$$

where $\varepsilon > 0$. Thus the Green's function is rewritten as

$$G_0(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \frac{1}{q^2 - k^2}, \qquad G_0^{(\pm)}(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \frac{1}{q^2 - (k^2 \pm \varepsilon)}$$




For convenience, we assume that the direction of $\boldsymbol{r}$ is the $z$ axis. The angle between $\boldsymbol{r}$ and $\boldsymbol{q}$ is $\theta$.

$$\boldsymbol{q}\cdot\boldsymbol{r} = qr\cos\theta$$

$$d\boldsymbol{q} = 2\pi q^2 dq\sin\theta d\theta$$

$$G_0(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int\int 2\pi q^2 dq\sin\theta d\theta e^{iqr\cos\theta}\frac{1}{q^2-k^2}$$

Note

$$\int \sin\theta d\theta e^{iqr\cos\theta} = -\frac{i}{qr}[e^{iqr} - e^{-iqr}],$$

we have

$$G_0(\boldsymbol{r}) = G_0\ (r) = \frac{1}{(2\pi)^3}\int q^2 dq(-\frac{2\pi i}{qr})\frac{e^{iqr}-e^{-iqr}}{q^2-k^2}$$

since $G_0(\boldsymbol{r})$ depends only on $r$.

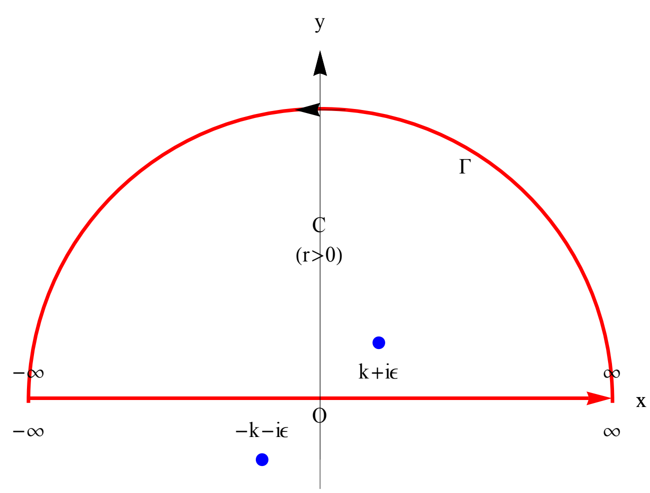
or

$$\begin{aligned}
G_0(r) &= \frac{1}{4\pi^2 ir}\int_0^\infty qdq\frac{e^{iqr}-e^{-iqr}}{q^2-k^2} \\
&= \frac{1}{4\pi^2 ir}\int_{-\infty}^\infty qdq\frac{e^{iqr}}{q^2-k^2} \\
&= \frac{1}{8\pi^2 ir}\int_{-\infty}^\infty e^{iqr}dq(\frac{1}{q-k}+\frac{1}{q+k})
\end{aligned}$$

We consider the shift the position of the simple poles by $\pm i\varepsilon$ in the complex plane, where $\varepsilon \to 0$ and $\varepsilon>0$. This shift is significant to the calculation, since the contour $C$ is in the upper half plane. In other words, the position of the simple poles from the real axis to the upper half plane or to the lower half plane by $i\varepsilon$.

**(i) Retarded Green's function**

$$G_0^{(+)}(r) = \frac{1}{8\pi^2 ir}\int_{-\infty}^{\infty} e^{iqr}dq\left(\frac{1}{q-k-i\varepsilon}+\frac{1}{q+k+i\varepsilon}\right)$$
$$= \frac{1}{4\pi^2 ir}\int_{-\infty}^{\infty} qdq\left(\frac{e^{iqr}}{q^2-k^2-i\varepsilon}\right)$$




Since $r>0$, the path of integration can be closed by an infinite semicircle in the upper half-plane (Jordan's lemma).

$$G_0^{(+)}(r) = \frac{1}{8\pi^2 ir}\oint_C e^{izr}dz\left(\frac{1}{z-k-i\varepsilon}+\frac{1}{z+k+i\varepsilon}\right)$$
$$= \frac{1}{8\pi^2 ir}2\pi i \,\mathrm{Res}(q=k+i\varepsilon) = \frac{1}{4\pi r}e^{ikr}$$

(retarded Green's function).

This corresponds to the outgoing spherical wave. Formally we get

$$G_0^{(+)}(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}.$$

Here we note that

$$\frac{1}{q-k-i\varepsilon}+\frac{1}{q+k+i\varepsilon}=\frac{1}{q-(k+i\varepsilon)}+\frac{1}{q+(k+i\varepsilon)}$$

$$=\frac{2q}{q^2-(k+i\varepsilon)^2}$$

$$=\frac{2q}{q^2-(k^2+2ik\varepsilon)}$$

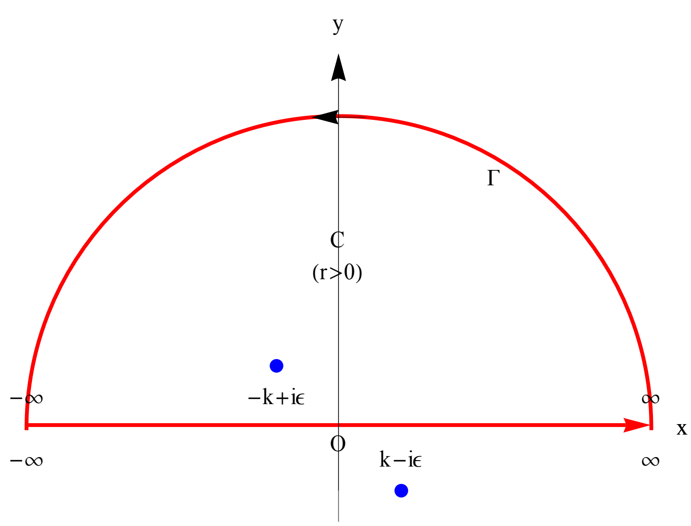
$$=\frac{2q}{q^2-k^2-i\delta}$$

where $\delta = 2\varepsilon k$ (>0)

**(ii) Advanced Green's function**

$$G_0^{(-)}(r)=\frac{1}{8\pi^2 ir}\int_{-\infty}^{\infty} e^{iqr}dq(\frac{1}{q-k+i\varepsilon}+\frac{1}{q+k-i\varepsilon})$$

$$=\frac{1}{4\pi^2 ir}\int_{-\infty}^{\infty} qdq(\frac{e^{iqr}}{q^2-k^2+i\varepsilon})$$



$$G_0^{(-)}(r) = \frac{1}{8\pi^2 ir}\oint_C e^{izr}dz\left(\frac{1}{z-k+i\varepsilon} + \frac{1}{z+k-i\varepsilon}\right)$$
$$= \frac{1}{8\pi^2 ir}2\pi i \,\mathrm{Re}\,s(q=-k+i\varepsilon)$$
$$= \frac{1}{4\pi r}e^{-ikr}$$

(Advanced Green's function)

This corresponds to the incoming spherical wave. We note that

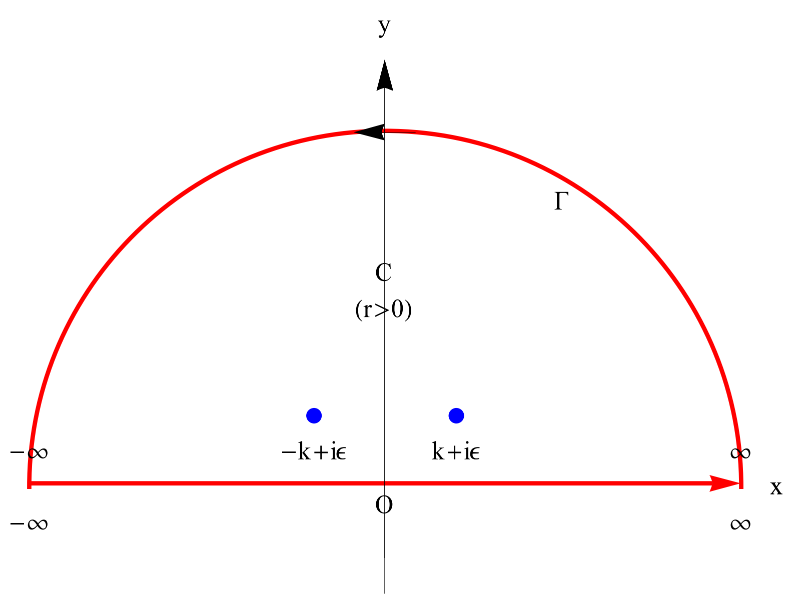
$$\frac{1}{q-k+i\varepsilon} + \frac{1}{q+k-i\varepsilon} = \frac{1}{q-(k-i\varepsilon)} + \frac{1}{q+(k-i\varepsilon)}$$
$$= \frac{2q}{q^2-(k-i\varepsilon)^2}$$
$$= \frac{2q}{q^2-(k^2-2i\varepsilon k)}$$
$$= \frac{2q}{q^2-k^2+i\delta}$$

where $\delta = 2\varepsilon k$ (>0).

((**Note**))

There are two more types of Green's function which depends on the shift of poles

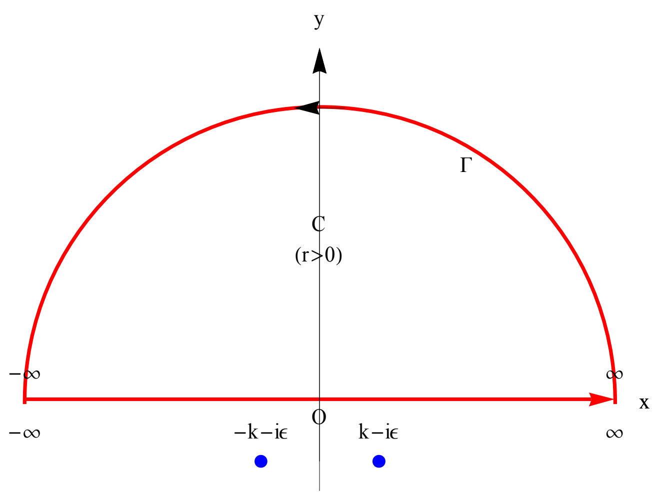
(iii)



$$G_0(r) = \frac{1}{8\pi^2 ir}\oint_C e^{izr} dz(\frac{1}{z-k-i\varepsilon} + \frac{1}{z+k-i\varepsilon})$$
$$= \frac{1}{8\pi^2 ir} 2\pi i[\mathrm{Re}\, s(q = k + i\varepsilon) + \mathrm{Re}\, s(q = -k + i\varepsilon)]$$
$$= \frac{1}{4\pi r}(e^{-ikr} + e^{ikr})$$

which is the superposition of incoming spherical wave and outgoing spherical wave.

(iv)




There is no pole inside the contour $C$. then we have

$$G_0(r) = \frac{1}{8\pi^2 ir}\oint_C e^{izr} dz(\frac{1}{z-k+i\varepsilon} + \frac{1}{z+k+i\varepsilon})$$
$$= 0$$

## 5 Calculation of $\nabla^2 G_0^{(\pm)}(r)$

$$\nabla^2(fg) = \nabla\cdot\nabla(fg) = \nabla\cdot[f\nabla g + g\nabla f] = 2\nabla g\cdot\nabla f + f\nabla^2 g + g\nabla^2 f$$
$$\nabla\cdot(\phi\boldsymbol{A}) = \boldsymbol{A}\cdot\nabla\phi + \phi\nabla\cdot\boldsymbol{A} \qquad f$$
$$\nabla(fg) = f\nabla g + g\nabla f$$

We calculate $\nabla^2 G_0^{(\pm)}(r)$.

$$\nabla^2 G_0^{(\pm)}(r) = \nabla^2 (\frac{1}{4\pi r} e^{\pm ikr})$$
$$= e^{\pm ikr}\nabla^2 (\frac{1}{4\pi r}) + (\frac{1}{4\pi r})\nabla^2 (e^{\pm ikr}) + 2\nabla(\frac{1}{4\pi r})\cdot\nabla(e^{\pm ikr})$$

Here

$$\nabla^2 (e^{\pm ikr}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r} e^{\pm ikr})$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}[(\pm ikr^2)e^{\pm ikr}] = (\pm\frac{2ik}{r} - k^2)e^{\pm ikr}$$

$$\nabla(\frac{1}{4\pi r}) = \boldsymbol{e}_r \frac{\partial}{\partial r}(\frac{1}{4\pi r}) = -\frac{\boldsymbol{e}_r}{4\pi}\frac{1}{r^2},$$

$$\nabla(e^{\pm ikr}) = \boldsymbol{e}_r \frac{\partial}{\partial r}(e^{\pm ikr}) = \boldsymbol{e}_r(\pm ikr)e^{\pm ikr}.$$

Then

$$\nabla^2 G_0^{(\pm)}(r) = e^{\pm ikr}\nabla^2 (\frac{1}{4\pi r}) + (\frac{1}{4\pi r})(\pm\frac{2ik}{r} - k^2)e^{\pm ikr} - \frac{1}{2\pi r^2}(\pm ikr)e^{\pm ikr},$$

or

$$\nabla^2 G_0^{(\pm)}(r) = -\delta(\boldsymbol{r}) - k^2(\frac{1}{4\pi r})e^{\pm ikr} = -\delta(\boldsymbol{r}) - k^2 G^{(\pm)}(r),$$

or

$$(\nabla^2 + k^2)G_0^{(\pm)}(r) = -\delta(\boldsymbol{r}).$$

## 6 Derivation of Green's function (Helmholtz)

We assume that $G(r)$ is only dependent on r.

$$(\nabla^2 + k^2)G(r) = -\delta(\boldsymbol{r}),$$

where

$$\nabla^2 G(r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial G(r)}{\partial r}) = \frac{1}{r}\frac{\partial}{\partial r^2}(rG(r)).$$

Then

$$\frac{1}{r}\frac{\partial}{\partial r^2}(rG(r)) + k^2 G(r) = -\delta(\mathbf{r})\,.$$

For $r \neq 0$, we have

$$\frac{\partial}{\partial r^2}(rG(r)) + k^2 rG(r) = 0\,.$$

Then we have

$$G_{\pm}(r) = A_{\pm}\frac{e^{\pm ikr}}{r}$$

where $A_{\pm}$ is constant. We show that

$$A_{\pm} = \frac{1}{4\pi}\,.$$

We use the formula

$$\nabla^2(fg) = 2\nabla g\cdot\nabla f + f\nabla^2 g + g\nabla^2 f\,,$$

with

$$f = \frac{1}{r}, \qquad \text{and} \qquad g = A_{\pm}e^{\pm ikr}\,.$$

Then

$$\begin{aligned}
\nabla^2(G_{\pm}) &= 2A_{\pm}\nabla e^{\pm ikr}\cdot\nabla\frac{1}{r} + \frac{A_{\pm}}{r}\nabla^2 e^{\pm ikr} + A_{\pm}e^{\pm ikr}\nabla^2\frac{1}{r} \\
&= A_{\pm}e^{\pm ikr}[2\boldsymbol{e}_r(\pm ikr)\cdot(-\boldsymbol{e}_r\frac{1}{r^2}) + \frac{1}{r}(\pm\frac{2ik}{r} - k^2) - 4\pi\delta(\boldsymbol{r})] \\
&= A_{\pm}e^{\pm ikr}[-\frac{k^2}{r} - 4\pi\delta(\boldsymbol{r})]
\end{aligned}$$

or

$$(\nabla^2 + k^2)G_\pm = -4\pi\delta(\boldsymbol{r})A_\pm e^{\pm ikr} = -4\pi\delta(\boldsymbol{r})A_\pm = -\delta(\boldsymbol{r}) .$$

Then we have

$$A_\pm = \frac{1}{4\pi} .$$

Note that

$$\nabla^2(e^{\pm ikr}) = (\pm\frac{2ik}{r} - k^2)e^{\pm ikr}$$

$$\nabla(\frac{1}{r}) = \boldsymbol{e}_r \frac{\partial}{\partial r}(\frac{1}{r}) = -\boldsymbol{e}_r \frac{1}{r^2} ,$$

$$\nabla(e^{\pm ikr}) = \boldsymbol{e}_r(\pm ikr)e^{\pm ikr}$$

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\boldsymbol{r})$$

## 7 Derivation of the Green's function from the Green's theorem

$$\int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)d\tau = \int_S (\psi\nabla\phi - \phi\nabla\psi)\cdot d\boldsymbol{a}$$

((**Proof**)) In the Gauss's theorem, we put

$$\boldsymbol{A} = \psi\nabla\phi$$

Then we have

$$I_1 = \int_V \nabla\cdot\boldsymbol{A}d\tau = \int_V \nabla\cdot(\psi\nabla\phi)d\tau = \int_S (\psi\nabla\phi)\cdot d\boldsymbol{a} .$$

Noting that

$$\nabla\cdot(\psi\nabla\phi) = \psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi ,$$

we have

$$I_1 = \int_V (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)d\tau = \int_S (\psi\nabla\phi)\cdot d\boldsymbol{a} .$$

By replacing $\psi \leftrightarrow \phi$, we also have

$$I_1 = \int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)d\tau = \int_S (\phi\nabla\psi)\cdot d\boldsymbol{a} ,$$

Thus we find the Green's theorem

$$I_1 - I_2 = \int_V (\psi\nabla^2\phi - \phi\nabla^2\psi)d\tau = \int_S (\psi\nabla\phi - \phi\nabla\psi)\cdot d\boldsymbol{a} ,$$

or

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d\tau = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot \boldsymbol{n}da ,$$

We now consider the formula

$$\int_V [\phi(\boldsymbol{r}')\nabla'^2\psi(\boldsymbol{r}') - \psi(\boldsymbol{r}')\nabla'^2\phi(\boldsymbol{r}')]d^3\boldsymbol{r}' = \int_S [\phi(\boldsymbol{r}')\nabla'\psi(\boldsymbol{r}') - \psi(\boldsymbol{r}')\nabla'\phi(\boldsymbol{r}')]\cdot \boldsymbol{n}da' ,$$

where $\boldsymbol{r}$ is the observation point and $\boldsymbol{r}'$ is the integration variable. Here we choose

$$\nabla'^2\phi(\boldsymbol{r}') = -4\pi\rho(\boldsymbol{r}') ,$$

$$\psi = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = G(\boldsymbol{r},\boldsymbol{r}') ,$$

$$\nabla'^2 G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}') ,$$

Then we have

$$\int_V [-\delta(\boldsymbol{r}-\boldsymbol{r}')\phi(\boldsymbol{r}') + \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\rho(\boldsymbol{r}')]d^3\boldsymbol{r}' = \int_S [\phi(\boldsymbol{r}')\nabla'(\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}) - \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\phi(\boldsymbol{r}')]\cdot \boldsymbol{n}da'$$

If the point $\boldsymbol{r}$ lies in the volume $V$, we obtain

$$\phi(\boldsymbol{r}) = \int_V \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\rho(\boldsymbol{r}')}{\varepsilon_0}d^3\boldsymbol{r}' + \frac{1}{4\pi}\int_S [\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\phi(\boldsymbol{r}') - \phi(\boldsymbol{r}')\nabla'(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|})]\cdot \boldsymbol{n}da'$$

## 8 3D Green's function (modified Helmholtz)

We now consider the form of the Green's function (modified Helmholtz)

$$(\Delta - k^2)G_0(\boldsymbol{r}) = -\delta(\boldsymbol{r}) .$$

The Fourier transform of $G_0(\boldsymbol{r})$ is defined as

$$G_0(\boldsymbol{q}) = \frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{r} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{r}) .$$

The inverse Fourier transform is also defined as

$$G_0(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{q}) ,$$

where

$$\delta(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} .$$

Then

$$\begin{aligned} \frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{q} (\Delta - k^2) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{q}) &= \frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{q} (-q^2 - k^2) e^{i\boldsymbol{q}\cdot\boldsymbol{r}} G_0(\boldsymbol{q}) \\ &= -\delta(\boldsymbol{r}) = -\frac{1}{(2\pi)^3} \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \end{aligned} ,$$

or

$$G_0(\boldsymbol{q}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{q^2 + k^2} .$$

Thus the Green's function is rewritten as

$$G_0(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int d\boldsymbol{q} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} \frac{1}{q^2 + k^2}$$

$$G_0(\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int\int 2\pi q^2 dq \sin\theta d\theta e^{iqr\cos\theta} \frac{1}{q^2 + k^2} .$$

Note

$$\int \sin\theta d\theta e^{iqr\cos\theta} = -\frac{i}{qr} [e^{iqr} - e^{-iqr}] ,$$

we have

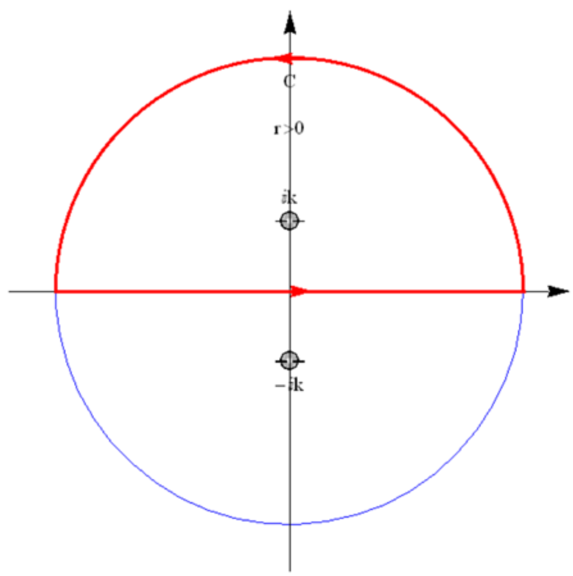
$$G_0(\boldsymbol{r}) = G_0(r) = \frac{1}{(2\pi)^3}\int q^2 dq(-\frac{2\pi i}{qr})\frac{e^{iqr}-e^{-iqr}}{q^2+k^2},$$

since $G_0(\boldsymbol{r})$ depends only on $r$. This equation can be rewritten as

$$G_0(r) = \frac{1}{4\pi^2 ir}\int_0^\infty qdq\frac{e^{iqr}-e^{-iqr}}{q^2+k^2} = \frac{1}{4\pi^2 ir}\int_{-\infty}^\infty qdq\frac{e^{iqr}}{q^2+k^2}$$

This function has a single pole at

$$q = \pm ik$$




Since $r$>0, the path of integration can be closed by an infinite semicircle in the upper half-plane (Jordan's lemma).

$$G_0(r) = \frac{1}{4\pi^2 ir}\oint_C qdq\frac{e^{iqr}}{q^2+k^2} = \frac{1}{4\pi^2 ir}2\pi i\,\mathrm{Re}\,s(q=ik) = \frac{1}{4\pi r}e^{-kr}$$

## 9 Derivation of Green's function (modified Helmholtz)

$$(\nabla^2 - k^2)G(r) = -\delta(\boldsymbol{r})$$

$$\nabla^2 G(r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial G(r)}{\partial r}) = \frac{1}{r}\frac{\partial}{\partial r^2}(rG(r))$$

$$\frac{1}{r}\frac{\partial}{\partial r^2}(rG(r)) - k^2 G(r) = -\delta(\mathbf{r})$$

For $r \neq 0$,

$$\frac{\partial}{\partial r^2}(rG(r)) - k^2 rG(r) = 0$$

Then we have

$$G_{\pm}(r) = A_{\pm}\frac{e^{\pm kr}}{r}$$

Here we show that

$$A_{\pm} = \frac{1}{4\pi}.$$

((**Proof**))

$$\begin{aligned}
\nabla^2(G_{\pm}) &= 2A_{\pm}\nabla e^{\pm kr}\cdot\nabla\frac{1}{r} + \frac{A_{\pm}}{r}\nabla^2 e^{\pm kr} + A_{\pm}e^{\pm kr}\nabla^2\frac{1}{r} \\
&= A_{\pm}e^{\pm kr}[2\boldsymbol{e}_r(\pm kr)\cdot(-\boldsymbol{e}_r\frac{1}{r^2}) + \frac{1}{r}(\pm\frac{2k}{r} + k^2) - 4\pi\delta(\boldsymbol{r})] \\
&= A_{\pm}e^{\pm kr}[\frac{k^2}{r} - 4\pi\delta(\boldsymbol{r})]
\end{aligned}$$

or

$$(\nabla^2 - k^2)G_{\pm} = -4\pi\delta(\boldsymbol{r})A_{\pm}e^{\pm kr} = -4\pi\delta(\boldsymbol{r})A_{\pm} = -\delta(\boldsymbol{r}).$$

Then we have

$$A_{\pm} = \frac{1}{4\pi}.$$

Note that

$$\nabla^2(e^{\pm kr}) = (\pm\frac{2k}{r} + k^2)e^{\pm kr}$$

$$\nabla(\frac{1}{r}) = \boldsymbol{e}_r \frac{\partial}{\partial r}(\frac{1}{r}) = -\boldsymbol{e}_r \frac{1}{r^2},$$

$$\nabla(e^{\pm kr}) = \boldsymbol{e}_r(\pm kr)e^{\pm kr}$$

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\boldsymbol{r})$$

## 10 Classical electrodynamics

Using

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho,$$

and

$$\boldsymbol{E} = -\nabla\phi,$$

we obtain a Poisson equation

$$\nabla^2\phi = -4\pi\rho.$$

The solution of $\phi$ can be obtained using the Green's function

$$\phi = 4\pi\int G(\boldsymbol{r},\boldsymbol{r}')\rho(r')d^3\boldsymbol{r}',$$

$$\nabla^2\phi = 4\pi\int \nabla^2 G(\boldsymbol{r},\boldsymbol{r}')\rho(r')d^3\boldsymbol{r}' = -4\pi\int \delta(\boldsymbol{r}-\boldsymbol{r}')\rho(r')d^3\boldsymbol{r}' - 4\pi\rho,$$

where

$$\nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}'),$$

with

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}.$$

Then we have

$$\phi(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}'.$$

## 11 Ampere's law and Biot-Savart law

We start with the Maxwell's equation

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} ,$$

$$\nabla \cdot \boldsymbol{B} = 0 .$$

$\boldsymbol{B}$ is expressed in terms of the vector potential $\boldsymbol{A}$ as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} ,$$

$$\nabla \times \boldsymbol{B} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \frac{4\pi}{c}\boldsymbol{J} ,$$

Here we choose a Coulomb gauge

$$\nabla \cdot \boldsymbol{A} = 0 .$$

Then we get

$$\nabla^2 \boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J} .$$

The solution of $\boldsymbol{A}$ is obtained using the Green's function as

$$\boldsymbol{A}(\boldsymbol{r}) = \int G(\boldsymbol{r},\boldsymbol{r}')\frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{r}')d^3\boldsymbol{r}' ,$$

$$\nabla^2 \boldsymbol{A} = \int \nabla^2 G(\boldsymbol{r},\boldsymbol{r}')\frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{r}')d^3\boldsymbol{r}' = -\frac{4\pi}{c}\boldsymbol{J} ,$$

where

$$\nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}') ,$$

with

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} .$$

Then we have

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{c}\int \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3\boldsymbol{r}' .$$

$\boldsymbol{B}$ can be calculated as

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{1}{c}\int\nabla\times(\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|})d^3\boldsymbol{r}'.$$

Since

$$\nabla\times(\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}) = -\boldsymbol{J}(\boldsymbol{r}')\times\nabla\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\boldsymbol{J}(\boldsymbol{r}')\times(-\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}),$$

we have the Bio-Savart law,

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} = \frac{1}{c}\int\frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d^3\boldsymbol{r}'.$$

We note that

$$\nabla\times\boldsymbol{B} = \frac{1}{c}\int\nabla\times[\frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}]d^3\boldsymbol{r}'.$$

Here

$$\begin{aligned}\nabla\times[\frac{\boldsymbol{J}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}] &= \boldsymbol{J}(\boldsymbol{r}')[\nabla\cdot\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}] - [\boldsymbol{J}(\boldsymbol{r}')\cdot\nabla]\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \\ &= 4\pi J(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}') - [\boldsymbol{J}(\boldsymbol{r}')\cdot\nabla]\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\end{aligned}$$

Then we have

$$\begin{aligned}\nabla\times\boldsymbol{B} &= \frac{1}{c}\int[4\pi J(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}') - (\boldsymbol{J}(\boldsymbol{r}')\cdot\nabla)\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}]d^3\boldsymbol{r}' \\ &= \frac{4\pi}{c}J(\boldsymbol{r}) + \frac{1}{c}\int(\boldsymbol{J}(\boldsymbol{r}')\cdot\nabla)\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}{}^3\,\boldsymbol{r}' \\ &= \frac{4\pi}{c}J(\boldsymbol{r}) - \frac{1}{c}\int(\boldsymbol{J}(\boldsymbol{r}')\cdot\nabla')\frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d^3\boldsymbol{r}'\end{aligned}$$

Note that

$$(\boldsymbol{J}(\boldsymbol{r}')\cdot\nabla')\frac{(\boldsymbol{r}-\boldsymbol{r}')_x}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \nabla'\cdot[\frac{(\boldsymbol{r}-\boldsymbol{r}')_x}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(r')] - \frac{(\boldsymbol{r}-\boldsymbol{r}')_x}{|\boldsymbol{r}-\boldsymbol{r}'|^3}(\nabla'\cdot\boldsymbol{J}(\boldsymbol{r}')).$$

For the steady current, $\nabla'\cdot\boldsymbol{J}(\boldsymbol{r}') = 0$. Then

$$\nabla\times\boldsymbol{B}=\frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{r})-\frac{1}{c}\int_V\{\hat{x}\nabla'\cdot[\frac{(\boldsymbol{r}-\boldsymbol{r}')_x}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(r')]+\hat{y}\nabla'\cdot[\frac{(\boldsymbol{r}-\boldsymbol{r}')_y}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(r')]d^3\boldsymbol{r}'+\hat{z}\nabla'\cdot[\frac{(\boldsymbol{r}-\boldsymbol{r}')_z}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(r')]\}d^3\boldsymbol{r}'$$

$$=\frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{r})-\frac{1}{c}\int_A\{\hat{x}[\frac{(\boldsymbol{r}-\boldsymbol{r}')_x}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(r')]+\hat{y}[\frac{(\boldsymbol{r}-\boldsymbol{r}')_y}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(r')]+\hat{z}[\frac{(\boldsymbol{r}-\boldsymbol{r}')_z}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(r')]\}d\boldsymbol{a}'$$

$$=\frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{r})$$

where we use the Gauss's law and we assume that $J(\boldsymbol{r}') = 0$ on the surface $A$ (large enough to include all the currents).

((**Note**)) formula

$$\nabla\times(f\boldsymbol{A})=f(\nabla\times\boldsymbol{A})-A\times\nabla f\ ,$$

$$\nabla\cdot(f\boldsymbol{A})=f(\nabla\cdot\boldsymbol{A})+\boldsymbol{A}\cdot\nabla f\ ,$$

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B})=(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}+\boldsymbol{A}(\nabla\cdot\boldsymbol{B})-\boldsymbol{B}(\nabla\cdot\boldsymbol{A}),$$

$$\nabla\frac{1}{r}=-\frac{\boldsymbol{r}}{r^3},$$

$$\nabla\cdot\nabla\frac{1}{r}=\nabla^2\frac{1}{r}=-\nabla\cdot\frac{\boldsymbol{r}}{r^3}=-4\pi\delta(\boldsymbol{r}-\boldsymbol{r}').$$

## 12 Electromagnetism and d"Alembertian operator

### 12.1 Maxwell's equation (in cgs units)

The Maxwell’s equations are given by

$$\nabla\cdot\boldsymbol{E}=4\pi\rho$$

$$\nabla\cdot\boldsymbol{B}=0$$

$$\nabla\times\boldsymbol{E}=-\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}$$

$$\nabla\times\boldsymbol{B}=\frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}+\frac{4\pi}{c}\boldsymbol{J}$$

where $c$ is the velocity of light

### 12.2 Vector potential *A* and scalar potential $\phi$

$$\boldsymbol{B}=\nabla\times\boldsymbol{A},$$

since $\nabla\cdot\boldsymbol{B}=0$.

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \boldsymbol{A},$$

or

$$\nabla \times (\boldsymbol{E} + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}) = 0,$$

or

$$\boldsymbol{E} + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} = -\nabla\phi.$$

Then we have

$$\boxed{\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\phi}, \quad \text{and} \quad \boxed{\boldsymbol{B} = \nabla \times \boldsymbol{A}}.$$

We now calculate

$$\begin{aligned}\nabla \times \boldsymbol{B} &= \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \\ &= \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial}{\partial t}(-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\phi)\end{aligned}$$

or

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J} + \nabla(\nabla \cdot \boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t})$$

Similarly we have

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho$$

or

$$\nabla \cdot (-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\phi) = 4\pi\rho$$

or

$$-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A} = 4\pi\rho$$

or

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\phi = -4\pi\rho - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t})$$

## 12.3 Gauge transformation

We have a gauge transformation

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi \,,$$

$$\phi' = \phi - \frac{1}{c}\frac{\partial\chi}{\partial t} \,,$$

where

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\phi \qquad \text{and} \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}$$

Let us calculate

$$-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}' - \nabla\phi' = -\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{A} + \nabla\chi) - \nabla(\phi - \frac{1}{c}\frac{\partial\chi}{\partial t}) = -\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} - \nabla\phi = \boldsymbol{E} \,,$$

$$\nabla\times\boldsymbol{A}' = \nabla\times(\boldsymbol{A} + \nabla\chi) = \nabla\times\boldsymbol{A} \,.$$

Therefore $(\boldsymbol{A}',\phi')$ and $(\boldsymbol{A},\phi)$ gives the same expression for ***E*** and ***B***.

We adopt the Lorentz gauge

$$\nabla\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0 \qquad \text{(Lorentz gauge)}$$

Then we have

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J}$$

and

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\phi = -4\pi\rho$$

The d'Alemberetian operator or simply d'Alembertian is the differential operator

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

Using this notation, we get the following expression

$$\Box\phi = -4\pi\rho, \qquad \text{and} \qquad \Box\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J}$$

The Green's function associated with the d"Alembertian satisfies the differential equation

$$\Box G(\mathbf{r},t) = -\delta(\mathbf{r})\delta(t).$$

## 12.4. Fourier transform

Fourier transform

$$\boldsymbol{A}(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{A}(\mathbf{r},t)e^{i\omega t}dt$$

$$\boldsymbol{A}(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{A}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

$$\boldsymbol{J}(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{J}(\mathbf{r},t)e^{i\omega t}dt$$

$$\boldsymbol{J}(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{J}(\mathbf{r},\omega)e^{-i\omega t}d\omega$$

$$\phi(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(\mathbf{r},t)e^{i\omega t}dt$$

$$\phi(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(\mathbf{r},\omega)e^{-i\omega t}d\omega,$$

$$\rho(\mathbf{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\rho(\mathbf{r},t)e^{i\omega t}dt$$

$$\rho(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\rho(\mathbf{r},\omega)e^{-i\omega t}d\omega,$$

## 13 Retarded vector potential $A(r, t)$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{J}$$

where

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{A}(\boldsymbol{r},\omega)e^{-i\omega t}d\omega$$

$$\boldsymbol{J}(\boldsymbol{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{J}(\boldsymbol{r},\omega)e^{-i\omega t}d\omega$$

Then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\boldsymbol{A}(\boldsymbol{r},\omega)e^{-i\omega t}d\omega = -\frac{4\pi}{c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{J}(\boldsymbol{r},\omega)e^{-i\omega t}d\omega$$

or

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\nabla^2 + \frac{\omega^2}{c^2})\boldsymbol{A}(\boldsymbol{r},\omega)e^{-i\omega t}d\omega = -\frac{4\pi}{c}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\boldsymbol{J}(\boldsymbol{r},\omega)e^{-i\omega t}d\omega$$

or

$$(\nabla^2 + \frac{\omega^2}{c^2})\boldsymbol{A}(\boldsymbol{r},\omega) = -\frac{4\pi}{c}\boldsymbol{J}(\boldsymbol{r},\omega)$$

We can solve this using the Green's function (Helmholtz)

$$\boldsymbol{A}(\boldsymbol{r},\omega) = \int G(\boldsymbol{r},\boldsymbol{r}')\mu_0\boldsymbol{J}(\boldsymbol{r}',\omega)d^3\boldsymbol{r}'$$

with

$$(\nabla^2 + \frac{\omega^2}{c^2})G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}')$$

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{\exp[i\frac{\omega}{c}|\boldsymbol{r}-\boldsymbol{r}'|]}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

Thus we have

$$\boldsymbol{A}(\boldsymbol{r},\omega)=\frac{4\pi}{c}\int d^3\boldsymbol{r}'\frac{\exp[i\frac{\omega}{c}|\boldsymbol{r}-\boldsymbol{r}'|]}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{J}(\boldsymbol{r}',\omega)$$

or

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{4\pi}{c}\int d^3\boldsymbol{r}'\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\frac{1}{\sqrt{2\pi}}\int d\omega\exp[-i\omega(t-\frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|)]\boldsymbol{J}(\boldsymbol{r}',\omega)$$
$$=\frac{1}{c}\int d^3\boldsymbol{r}'\frac{\boldsymbol{J}(\boldsymbol{r}',t-\frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

The retarded time is defined as

$$t_r=t-\frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|.$$

## 14 Retarded potential $\phi(\boldsymbol{r},t)$

$$(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})\phi=-4\pi\rho$$

where

$$\phi(\boldsymbol{r},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(\boldsymbol{r},\omega)e^{-i\omega t}d\omega$$

$$\rho(\boldsymbol{r},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\rho(\boldsymbol{r},\omega)e^{-i\omega t}d\omega$$

Then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})\phi(\boldsymbol{r},\omega)e^{-i\omega t}d\omega=-4\pi\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\rho(\boldsymbol{r},\omega)e^{-i\omega t}d\omega$$

or

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\nabla^2+\frac{\omega^2}{c^2})\phi(\boldsymbol{r},\omega)e^{-i\omega t}d\omega=-4\pi\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\rho(\boldsymbol{r},\omega)e^{-i\omega t}d\omega$$

or

$$(\nabla^2 + \frac{\omega^2}{c^2})\phi(\boldsymbol{r},\omega) = -4\pi\rho(\boldsymbol{r},\omega)$$

We can solve this using the Green's function (Helmholtz)

$$\phi(\boldsymbol{r},\omega) = \int G(\boldsymbol{r},\boldsymbol{r}')4\pi\phi(\boldsymbol{r}',\omega)d^3\boldsymbol{r}'$$

with

$$(\nabla^2 + \frac{\omega^2}{c^2})G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}')$$

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{\exp[i\frac{\omega}{c}|\boldsymbol{r}-\boldsymbol{r}'|]}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

Thus we have

$$\phi(\boldsymbol{r},\omega) = \int d^3\boldsymbol{r}' \frac{\exp[i\frac{\omega}{c}|\boldsymbol{r}-\boldsymbol{r}'|]}{|\boldsymbol{r}-\boldsymbol{r}'|}\rho(\boldsymbol{r}',\omega)$$

or

$$\begin{aligned}\phi(\boldsymbol{r},t) &= \int d^3\boldsymbol{r}' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{1}{\sqrt{2\pi}}\int d\omega \exp[-i\omega(t-\frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|)]\rho(\boldsymbol{r}',\omega) \\ &= \int d^3\boldsymbol{r}' \frac{\rho(\boldsymbol{r}',t-\frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}\end{aligned}$$

## 15. Jefimenko's equation

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} - \nabla\phi \qquad \text{and} \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}$$

For simplicity, we define

$$t_r = t - \frac{1}{c}|\boldsymbol{r}-\boldsymbol{r}'|, \qquad \text{and} \qquad \boldsymbol{R} = \boldsymbol{r}-\boldsymbol{r}'$$

$$\boldsymbol{E}(\boldsymbol{r},t)=\int[\frac{\rho(\boldsymbol{r}',t_r)\boldsymbol{R}}{R^3}+\frac{\dot{\rho}(\boldsymbol{r}',t_r)}{cR^2}\boldsymbol{R}-\frac{\dot{\boldsymbol{J}}(r',t_r)}{c^2R}]d^3\boldsymbol{r}'$$

and

$$\boldsymbol{B}(\boldsymbol{r},t)=\frac{1}{c}\int[\frac{J(\boldsymbol{r}',t_r)}{R^3}+\frac{\dot{\boldsymbol{J}}(\boldsymbol{r}',t_r)}{cR^2}]\times\boldsymbol{R}d^3\boldsymbol{r}'\,.$$

## 16 Green's function of the d'Alembertian

We consider the equation

$$(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})\phi=-4\pi\rho$$

The Green's function of the d'Alembertian is defined as

$$(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})G(\boldsymbol{r},t)=-\delta(\boldsymbol{r})\delta(t)$$

Here we introduce the Fourier transforms,

$$G(\boldsymbol{r},t)=\frac{1}{(\sqrt{2\pi})^4}\int d\boldsymbol{k}\int_{-\infty}^{\infty}G(\boldsymbol{k},\omega)e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}d\omega$$

$$\delta(\boldsymbol{r})\delta(t)=\frac{1}{(2\pi)^4}\int e^{i\boldsymbol{k}\cdot\boldsymbol{r}}d\boldsymbol{k}\int_{-\infty}^{\infty}e^{-i\omega t}d\omega$$

Then

$$\frac{1}{(2\pi)^2}\iint(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})G(\boldsymbol{k},\omega)e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}d\boldsymbol{k}d\omega=-\frac{1}{(2\pi)^4}\iint e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}d\boldsymbol{k}d\omega$$

or

$$(-\boldsymbol{k}^2+\frac{\omega^2}{c^2})G(\boldsymbol{k},\omega)=-\frac{1}{(2\pi)^2}\,,$$

The solution to this equation is

$$G(\boldsymbol{k},\omega)=\frac{1}{(2\pi)^2}\frac{1}{\dfrac{\omega^2}{c^2}-\boldsymbol{k}^2}.$$

The inverse Fourier transform:

$$G(\boldsymbol{r},t)=\frac{1}{(2\pi)^2}\frac{1}{(2\pi)^2}\iint d\boldsymbol{k}d\omega\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}}{\dfrac{\omega^2}{c^2}-\boldsymbol{k}^2}$$

For convenience, we assume that the direction of $\boldsymbol{r}$ is the $z$ axis. The angle between $\boldsymbol{r}$ and $\boldsymbol{k}$ is $\theta$.

$$\boldsymbol{k}\cdot\boldsymbol{r}=kr\cos\theta,$$

$$d\boldsymbol{k}=2\pi k^2dk\sin\theta d\theta,$$

$$G(\boldsymbol{r},t)=\frac{1}{(2\pi)^4}\iint d\boldsymbol{k}d\omega\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}}{\dfrac{\omega^2}{c^2}-\boldsymbol{k}^2}$$

$$G(\boldsymbol{r},t)=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}e^{-i\omega t}d\omega\int_0^{\infty}2\pi k^2dk\int_0^{\pi}\sin\theta d\theta e^{ikr\cos\theta}\frac{1}{\dfrac{\omega^2}{c^2}-k^2},$$

Since

$$\int\sin\theta d\theta e^{ikr\cos\theta}=-\frac{i}{kr}(e^{ikr}-e^{-ikr}),$$

$$\begin{aligned}G(\boldsymbol{r},t)&=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}e^{-i\omega t}d\omega\int_0^{\infty}2\pi k^2dk\frac{1}{\dfrac{\omega^2}{c^2}-k^2}(-\frac{i}{kr})(e^{ikr}-e^{-ikr})\\&=\frac{1}{(2\pi)^3}(-\frac{i}{r})\int_0^{\infty}kdk(e^{ikr}-e^{-ikr})\int_{-\infty}^{\infty}\frac{e^{-i\omega t}}{\dfrac{\omega^2}{c^2}-k^2}d\omega,\\&=\frac{1}{(2\pi)^3}(-\frac{c^2i}{r})\int_0^{\infty}kdk(e^{ikr}-e^{-ikr})\int_{-\infty}^{\infty}d\omega\frac{e^{-i\omega t}}{\omega^2-c^2k^2}\end{aligned}$$

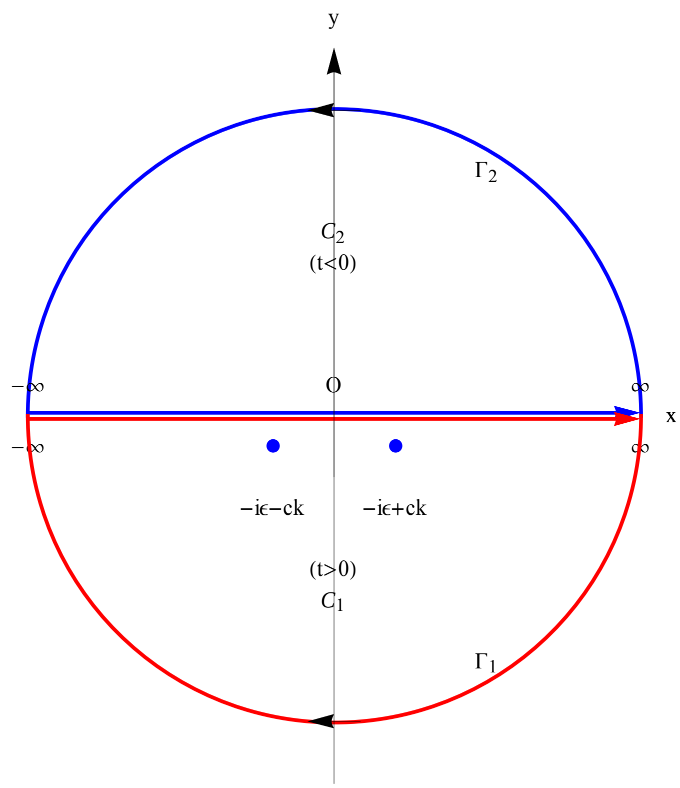
where

$$I = \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2 - c^2k^2}$$
$$= \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \frac{1}{2ck}\left(\frac{1}{\omega - ck} - \frac{1}{\omega + ck}\right)$$
$$= \oint dz e^{-izt} \frac{1}{2ck}\left(\frac{1}{z - ck} - \frac{1}{z + ck}\right)$$

**(i) Retarded Green function**

We calculte the intergral given by

$$I_1 = \oint dz e^{-izt} \frac{1}{2ck}\left(\frac{1}{z - ck + i\varepsilon} - \frac{1}{z + ck + i\varepsilon}\right)$$




For positive value of $t$, we need to choose the contour $C_1$ in the lower half plane. The complex exponential exp(-i$zt$) only decays at infinity if the imaginary of $z$ is negative. According to the Jordan's lemma, the integral along the path $\Gamma_1$ is zero. There are two simple poles inside the contour $C_1$. Since the path is taken with the clock-wise (negative) direction, we find that for $t$>0,

$$I_1 = \oint_{C_1} dz e^{-izt} \frac{1}{2ck}\left(\frac{1}{z-ck+i\varepsilon} - \frac{1}{z+ck+i\varepsilon}\right)$$
$$= \frac{-2\pi i}{2ck}[\mathrm{Re}\, s(z=ck-i\varepsilon) - \mathrm{Re}\, s(z=-ck-i\varepsilon)]$$
$$= \frac{\pi i}{ck}(e^{ickt} - e^{-ickt})$$

Then we have

$$G_{ret}(\mathbf{r},t) = -\frac{1}{(2\pi)^2}\frac{c}{2r}\int_0^\infty dk(e^{ikr} - e^{-ikr})(e^{-ickt} - e^{ickt})$$
$$= -\frac{1}{(2\pi)^2}\frac{c}{2r}\int_0^\infty dk[e^{ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)} + e^{-ik(r-ct)}]$$
$$= -\frac{1}{(2\pi)^2}\frac{c}{2r}\int_0^\infty dk[e^{ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)} + e^{-ik(r-ct)}]$$
$$= -\frac{1}{(2\pi)^2}\frac{c}{2r}\int_{-\infty}^\infty dk[e^{ik(r-ct)} - e^{ik(r+ct)}]$$
$$= -\frac{c}{4\pi r}[\delta(r-ct) - \delta(r+ct)]$$

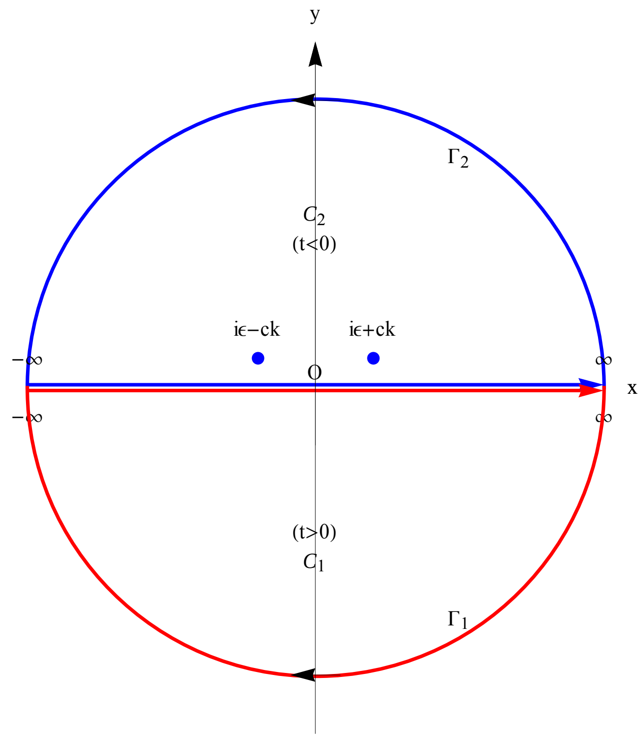
Since $r$>0 and $t$>0, we have

$$G_{ret}(\boldsymbol{r},t) = -\frac{c}{4\pi r}\pi[\delta(r-ct)].$$

For negative value of $t$, we need to choose the contour $C_2$ in the upper half plane. There is no poles inside the contour $C_2$. Then we find that for $t$<0,

$$G_{ret}(\boldsymbol{r},t) = 0$$

**(ii) Advanced Green function**




For positive value of $t$, we need to choose the contour $C_1$ in the lower half plane. There is no pole inside the contour $C_1$. So we find that for $t>0$,

$$I_2 = \oint_{C_1} dz e^{-izt} \frac{1}{2ck}\left(\frac{1}{z-ck-i\varepsilon} - \frac{1}{z+ck-i\varepsilon}\right) = 0$$

or

$$G_{adv}(\boldsymbol{r},t) = 0$$

For negative value of $t$, we need to choose the contour $C_2$ in the upper half plane. The complex exponential exp(-iz$t$) only decays at infinity if the imaginary of $z$ is positive. According to the Jordan's lemma, the integral along the path $\Gamma_2$ is zero. There are two simple poles inside the contour $C_2$. Since the path is taken with the clock-wise (positive) direction, we find that for $t<0$,

$$I_2 = \oint_{C_2} dz e^{-izt} \frac{1}{2ck}\left(\frac{1}{z - ck - i\varepsilon} - \frac{1}{z + ck - i\varepsilon}\right)$$

$$= \frac{2\pi i}{2ck}[\operatorname{Re} s(z = ck + i\varepsilon) - \operatorname{Re} s(z = -ck + i\varepsilon)]$$

$$= \frac{\pi i}{ck}(e^{-ickt} - e^{ickt})$$

$$G_{adv}(\boldsymbol{r},t) = \frac{1}{(2\pi)^2}\frac{c}{2r}\int_0^\infty dk(e^{ikr} - e^{-ikr})(e^{-ickt} - e^{ickt})$$

$$= \frac{1}{(2\pi)^2}\frac{c}{2r}\int_0^\infty dk[e^{ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)} + e^{-ik(r-ct)}]$$

$$= \frac{1}{(2\pi)^2}\frac{c}{2r}\int_0^\infty dk[e^{ik(r-ct)} - e^{ik(r+ct)} - e^{-ik(r+ct)} + e^{-ik(r-ct)}]$$

$$= \frac{1}{(2\pi)^2}\frac{c}{2r}\int_{-\infty}^\infty dk[e^{ik(r-ct)} - e^{ik(r+ct)}]$$

$$= \frac{c}{4\pi r}[\delta(r - ct) - \delta(r + ct)]$$

Since $r>0$ and $t<0$, we have

$$G_{adv}(\boldsymbol{r},t) = -\frac{c}{4\pi r}\pi[\delta(r + ct)].$$

((Note))

The retarded Green's function is represented by a spherical shell emitted at $t = 0$ and with increasing radius $r = ct$.

## 17 Green's function for the Klein-Gordon equation

We start with the Einstein's relation

$$E^2 = \boldsymbol{p}^2 c^2 + m^2 c^4$$

In quantum mechanics, we use the operators

$$E \to i\hbar\frac{\partial}{\partial t}, \quad \boldsymbol{p} \to \frac{\hbar}{i}\nabla$$

The Green's function is defined by

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2})G(\boldsymbol{r},t) = -\delta(t)\delta(\boldsymbol{r})$$

The Green's function is expressed by the inverse Fourier transform as

$$G(\boldsymbol{r},t) = \frac{1}{(2\pi)^{4/2}}\int e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}G(\boldsymbol{k},\omega)$$
.

From the Klein-Gordon equation, we have

$$\begin{aligned}(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2})G(\boldsymbol{r},t) &= \frac{1}{(2\pi)^{4/2}}\int d^3\boldsymbol{k}d\omega[e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}(-k^2 + \frac{\omega^2}{c^2} - \frac{m^2c^2}{\hbar^2})G(\boldsymbol{k},\omega)] \\ &= -\frac{1}{(2\pi)^4}\int d^3\boldsymbol{k}d\omega e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\end{aligned}$$

or

$$(-k^2 + \frac{\omega^2}{c^2} - \frac{m^2c^2}{\hbar^2})G(\boldsymbol{k},\omega) = -\frac{1}{(2\pi)^2}$$

or

$$G(\boldsymbol{k},\omega) = \frac{1}{(2\pi)^2}\frac{1}{k^2 + \frac{m^2c^2}{\hbar^2} - \frac{\omega^2}{c^2}}$$

Then

$$G(\boldsymbol{r},t) = \frac{1}{(2\pi)^4}\int d^3\boldsymbol{k}\int d\omega e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\frac{1}{k^2 + \frac{m^2c^2}{\hbar^2} - \frac{\omega^2}{c^2}}$$

$$= \frac{1}{(2\pi)^4}\int e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d^3\boldsymbol{k}\int d\omega \frac{1}{k^2 + \frac{m^2c^2}{\hbar^2} - \frac{\omega^2}{c^2}}$$

$$= -\frac{c^2}{(2\pi)^4}\int e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d^3\boldsymbol{k}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2 - \omega_0^2}$$

where

$$\omega_0 = (c^2k^2 + \frac{m^2c^4}{\hbar^2})^{1/2} = \frac{1}{\hbar}(\hbar^2c^2k^2 + m^2c^4) = \frac{1}{\hbar}E_0$$

We now calculate

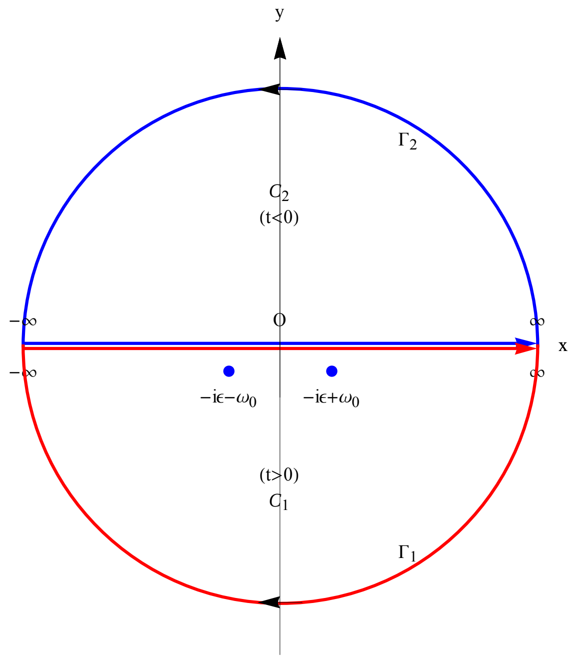
$$I = \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}}{\omega^2 - \omega_0^2}$$

$$= \frac{1}{2\omega_0}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}(\frac{1}{\omega - \omega_0} - \frac{1}{\omega + \omega_0})$$

(i) Retarded case

$$I_1 = \frac{1}{2\omega_0}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}(\frac{1}{\omega - \omega_0 + i\varepsilon} - \frac{1}{\omega + \omega_0 + i\varepsilon})$$

Two simple poles are located in the lower half plane.




For $t$>0

$$I_1 = \frac{1}{2\omega_0} \oint_{C_1} dz e^{-izt} \left(\frac{1}{z-\omega_0+i\varepsilon} - \frac{1}{z+\omega_0+i\varepsilon}\right)$$
$$= \frac{1}{2\omega_0}(-2\pi i)(e^{-it\omega_0} - e^{it\omega_0})$$
$$= -\frac{\pi i}{\omega_0}(e^{-it\omega_0} - e^{it\omega_0})$$

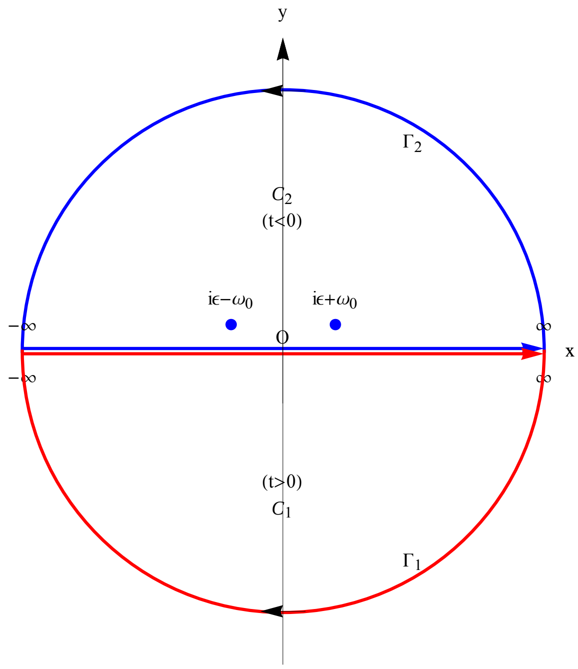
For $t$<0

$$I_1 = \frac{1}{2\omega_0} \oint_{C_2} dz e^{-izt} \left(\frac{1}{z-\omega_0+i\varepsilon} - \frac{1}{z+\omega_0+i\varepsilon}\right) = 0$$

(ii) Advanced case

$$I_2 = \frac{1}{2\omega_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left(\frac{1}{\omega-\omega_0-i\varepsilon} - \frac{1}{\omega+\omega_0-i\varepsilon}\right)$$

There are two simple poles in the upper half plane.




For $t$>0

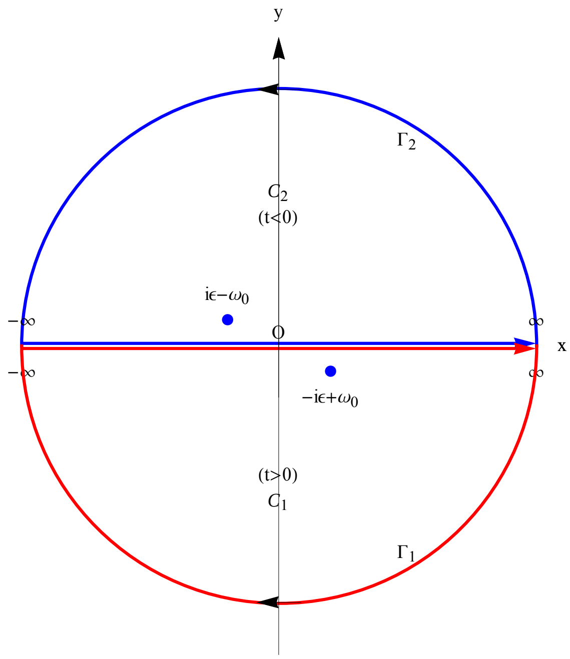
$$I_2 = \frac{1}{2\omega_0} \oint_{C_1} dz e^{-izt} \left(\frac{1}{z-\omega_0 - i\varepsilon} - \frac{1}{z+\omega_0 - i\varepsilon}\right) = 0$$

For $t$<0,

$$\begin{aligned} I_2 &= \frac{1}{2\omega_0} \oint_{C_2} dz e^{-izt} \left(\frac{1}{z-\omega_0 - i\varepsilon} - \frac{1}{z+\omega_0 - i\varepsilon}\right) \\ &= \frac{1}{2\omega_0} (2\pi i)(e^{-it\omega_0} - e^{it\omega_0}) \\ &= \frac{\pi i}{\omega_0} (e^{-it\omega_0} - e^{it\omega_0}) \end{aligned}$$

(iii)

$$I_3 = \frac{1}{2\omega_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left(\frac{1}{\omega-\omega_0 + i\varepsilon} - \frac{1}{\omega+\omega_0 - i\varepsilon}\right)$$




For $t$>0,

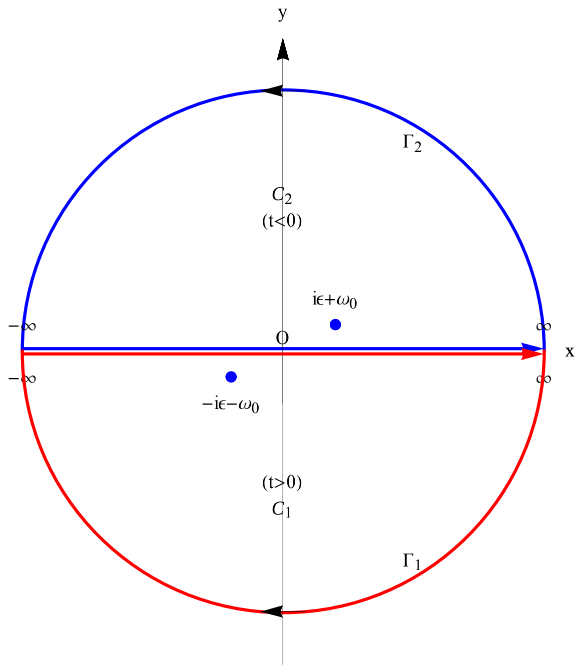
$$I_3 = \frac{1}{2\omega_0} \oint_{C_1} dz e^{-izt} \left(\frac{1}{z-\omega_0 - i\varepsilon} - \frac{1}{z+\omega_0 + i\varepsilon}\right)$$
$$= \frac{1}{2\omega_0}(-2\pi i)(-e^{-it\omega_0}) = \frac{\pi i}{\omega_0} e^{-it\omega_0}$$

For $t$<0,

$$I_3 = \frac{1}{2\omega_0} \oint_{C_2} dz e^{-izt} \left(\frac{1}{z-\omega_0 - i\varepsilon} - \frac{1}{z+\omega_0 + i\varepsilon}\right)$$
$$= \frac{1}{2\omega_0}(2\pi i)(e^{it\omega_0}) = \frac{\pi i}{\omega_0} e^{it\omega_0}$$

(iv) Feynman propagator

$$I_4 = \frac{1}{2\omega_0} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \left(\frac{1}{\omega-\omega_0 - i\varepsilon} - \frac{1}{\omega+\omega_0 + i\varepsilon}\right)$$




For $t$>0,

$$I_4 = \frac{1}{2\omega_0}\oint_{C_1} dz e^{-izt}\left(\frac{1}{z-\omega_0-i\varepsilon}-\frac{1}{z+\omega_0+i\varepsilon}\right)$$
$$= \frac{1}{2\omega_0}(-2\pi i)(-e^{it\omega_0}) = \frac{\pi i}{\omega_0}e^{it\omega_0}$$

For $t$<0,

$$I_4 = \frac{1}{2\omega_0}\oint_{C_2} dz e^{-izt}\left(\frac{1}{z-\omega_0-i\varepsilon}-\frac{1}{z+\omega_0+i\varepsilon}\right)$$
$$= \frac{1}{2\omega_0}(2\pi i)(e^{-it\omega_0}) = \frac{\pi i}{\omega_0}e^{-it\omega_0}$$

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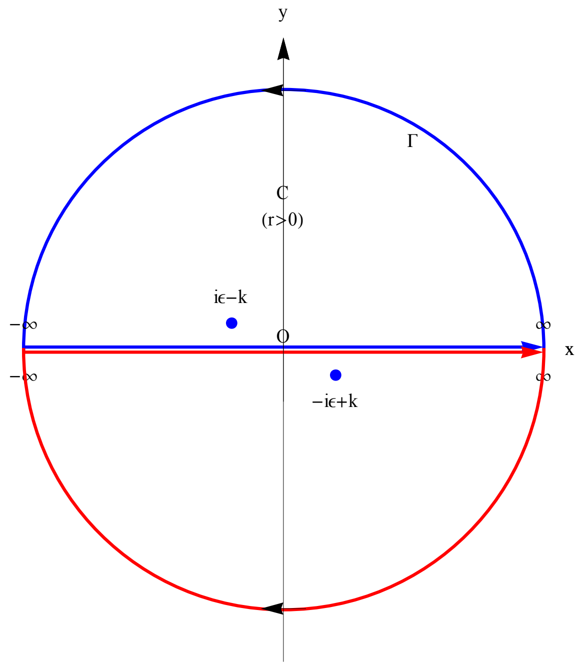


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